Differential Equations Homework 13

Due November 30

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. You'll be handing in your solutions to parts 1, 2, and 3 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. Do problem 4.1.17 on page 236.
2. Do problem 4.1.18 on page 236.
3. Do problem 4.1.27 on page 236.

Part 2

4. Do problem 4.1.32 on page 236.
5. (a) Do problem 4.1.33 on page 237.
   (b) Do problem 4.1.34 on page 237.

6. (a) Do problem 5.1.24 on page 280.
   (b) There are four straight solutions to this differential equation, namely $\mathbf{x}_1$, $\mathbf{x}_2$, and their negatives $-\mathbf{x}_1$ and $-\mathbf{x}_2$. Draw them by hand, labeling which one is which, and drawing an arrow to specify the direction of motion. Then, using either of the two phase portrait drawing tools to guide you, add rough sketches of four other solutions to your diagram, again drawing an arrow to specify the direction of motion.

Part 3

7. In class, we saw a video, `https://www.youtube.com/watch?v=YyOUJUOUvso`, where two pendula were connected by a soft spring. We also computed the differential equations for the angular displacements θ_1 and θ_2 of the pendula from equilibrium. We have

$$mL\ddot{\theta}_1 + (mg + kL)\theta_1 - kL\theta_2 = 0,$$
$$mL\ddot{\theta}_2 + (mg + kL)\theta_2 - kL\theta_1 = 0,$$

where m is the mass of the pendula, L is their length, g is the acceleration due to gravity, and k is the spring constant.

   (a) Let $u = \theta_1 + \theta_2$. Add the above two equations to write a differential equation for u. Your equation should be in terms of u, $\ddot{u}$, g, and L, but not θ_1 or θ_2.
   (b) Solve your differential equation for u. Your general solution should be in terms of time t, the constants g and L, and two new constants, which you should call A and B. Your answer should be a sinusoid. Call its angular frequency ω_0.
   (c) Let $v = \theta_1 - \theta_2$. Subtract the above two equations to write a differential equation for v.
   (d) Solve your differential equation for v. Your general solution should be in terms of t, g, l, k, and m, and two new constants, which you should call C and D. Your answer should be a sinusoid. Call its angular frequency ω_1.
   (e) Verify that

$$\theta_1 = \frac{u+v}{2}, \qquad \theta_2 = \frac{u-v}{2}.$$

   (f) In the first part of the video, the pendula are displaced an equal amount to the right, and released from rest. Thus, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, and $\theta_1(0) = \theta_2(0) = \phi$ for some small initial angle ϕ.
      i. Determine the corresponding initial conditions $u(0)$, $\dot{u}(0)$, $v(0)$, and $\dot{v}(0)$ in terms of ϕ.
      ii. Determine the solutions u and v to this initial value problem. Your answer should be in terms of time t and any of the constants m, g, k, L, ϕ, ω_0, and ω_1.
      iii. Compute the solutions θ_1 and θ_2 to this initial value problem.
      iv. Compute the period of the oscillation. Your answer should be familiar, and depend only on g and L. Why doesn't the spring constant k matter?
   (g) In the second part of the video, the pendula are displaced towards each other by an equal amount, and released from rest. Thus, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, and $\theta_1(0) = \phi$, but $\theta_2(0) = -\phi$.

i. Compute the solutions θ_1 and θ_2 to this initial value problem as above.

ii. Compute the period of the oscillation. Your answer should be in terms of g, L, k, and m. It is larger or smaller than the period of the oscillation in the previous part? Double check that your answer matches what you see in the video.

(h) In the third part of the video, the right pendulum is displaced to the right while the left pendulum is held at the equilibrium position, and then both pendula are released from rest. Thus, again, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, but this time $\theta_1(0) = 0$ and $\theta_2(0) = \phi$.

i. Compute the solutions θ_1 and θ_2 to this initial value problem. You can solve the intial value problem like before, or you can apply the principle of superposition to quickly find a solution that works. Either way, I recommend using the constants ω_0 and ω_1 in your work to save yourself some writing.

ii. In the video, you can see that the left pendulum periodically returns to being almost at rest at close to its equilibrium position. Review beats on pages 199–200, and use the trigonometric identity on that page to determine how long it takes the left pendulum to return to being almost at rest at close to its equilibrium position. Call this time T.

iii. The derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$. Thus,

$$\sqrt{x+h} - \sqrt{x} \approx \frac{h}{2\sqrt{x}}$$

when h is small. Assuming the spring constant k is small, use this approximation to simplify your expression for T in the previous problem. Your answer should be in terms of k, m, g, and L.

iv. Use your simplified formula for T to determine what happens to the time it takes the left pendulum to return to being almost at rest at close to its equilibrium position in each of the following three scenarios.

1. We make the masses four times heavier.
2. We connect three additional identical springs between the masses, making the spring constant four times larger.
3. We make the pendula four times longer.

For each scenario, determine whether the time for the left pendulum to return to almost rest at almost equilibrium will become

A. the same as before.
B. twice as long.
C. twice as fast.
D. four times longer.
E. four times faster.

The answer to the last one may be counterintuitive, since if we increase the length of the pendula, they will swing back and forth more slowly.