Differential Equations Homework 10

Due November 7

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. You'll be handing in your solutions to parts 1, 2, and 3 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. Continuing problems 5 and 6 on the previous homework, we now consider $y = x \sin 2x$.
   (a) Write down a homogeneous linear equation with constant coefficients so that $y = x \sin 2x$ is one of its solutions.
   (b) i. Apply the reasoning of problem 6 on the previous homework to answer whether the existence and uniqueness theorem allows $y = x \sin 2x$ to be a solution to a first-order homogeneous linear differential equation

$$y' + p_1(x)y = 0,$$

where $p_1(x)$ is continuous at $x = 0$.

ii. Does the existence and uniqueness theorem allow $y = x \sin 2x$ to be asolution to a second-order homogeneous linear differential equation

$$y'' + p_1(x)y' + p_2(x)y = 0,$$

where the coefficients $p_1(x)$ and $p_2(x)$ are continuous at $x = 0$?

iii. What stops us from using the trick from problem 6(f) on the previous homework to construct such an equation?

iv. Does the existence and uniqueness theorem allow $y = x \sin 2x$ to be a solution to a third-order homogeneous linear equation with, as above, coefficients $p_i(x)$ that are continuous at $x = 0$?

v. What about a fourth-order one?

(c) For the smallest order n where you determined it is possible to do so, write down an nth order homogeneous linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0,$$

where all of the coefficients are continuous at $x = 0$, so that $y = x \sin 2x$ is one of the solutions.

2. (a) Do problem 3.4.8 on page 181.

   (b) Do the same problem if instead the clock gains 5 minutes per day.

3. (a) Do problem 3.4.10 on page 182.

   (b) Do problem 3.4.11 on page 182.

Part 2

4. Consider the free undamped harmonic oscillator

$$m\ddot{x} + kx = 0.$$

The kinetic energy of the mass is $K = \frac{1}{2}mv^2$, where v is the velocity. The potential energy of the spring is $U = \frac{1}{2}kx^2$.

(a) Consider motion with an initial position of $x(0) = 0$ and an initial velocity of v_0. Compute the position of the mass $x(t)$ in terms of m, k, and v_0.

(b) Compute the potential energy $U(t)$ for this solution in terms of m, k, v_0, and t.

(c) Compute the kinetic energy $K(t)$ for this solution in terms of m, k, v_0, and t.

(d) Compute the total energy $E(t) = K(t) + U(t)$. The law of conservation of energy says that in a system with no damping, the total energy of the system remains the same for all time. Simplify your expression for the total energy until it no longer depends on t.

5. Now consider the free damped harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = 0.$$

Because the dashpot is absorbing energy, the total energy $E(t)$ of the system decreases over time. Verify that the rate of change of the energy $\dot{E}$, a negative number, is equal to the negative of the power absorbed by the dashpot, $F_{\text{dashpot}}v$.

It is a lot of work to do this computation by writing down a solution like we did in the previous problem. Instead, use the chain rule to write down the derivative $\frac{d}{dt}E$ in terms of m, k, x, $\dot{x}$, and $\ddot{x}$. Then, write the power absorbed by the dashpot in terms of c and $\dot{x}$. Use the differential equation to write down a relationship between these two quantities.

6. Do problem 3.4.23 on page 183.

Part 3

7. (a) Do problem 3.4.24 on page 183.
   (b) Do problem 3.4.25 on page 183.
   (c) Do problem 3.4.26 on page 183.
8. (a) Do probem 3.4.32 on page 183.
   (b) Do probem 3.4.33 on page 183.
   (c) Do probem 3.4.34 on page 184.
9. (a) Do problem 3.4.35 on page 184.
   (b) Do problem 3.4.36 on page 184.
   (c) Do problem 3.4.37 on page 184.
   (d) Do problem 3.4.38 on page 184. Use $\sin h \approx h \approx \sinh h$ for small h, so, consequently,

$$\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\sinh h}{h} = 1.$$