Differential Equations (Math 217) Final

December 16, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to the two questions to separate piles.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. There are two questions, so you will do this two times.

- The hyperbolic trigonometric functions are defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

- You can compute other logarithms using $\ln(mn) = \ln m + \ln n$ or by interpolating between two known values.

$$\ln 2 \approx 0.69,$$
$$\ln 3 \approx 1.10,$$
$$\ln 5 \approx 1.61,$$
$$\ln 7 \approx 1.95,$$
$$\ln 11 \approx 2.40.$$

- Other approximations:
  - $\pi \approx 3$.
  - $g \approx 10\,\mathrm{m/s^2}$.

- If n is positive, to determine whether $n < \sqrt{x}$, determine whether $n^2 < x$.

- For an RLC circuit,

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

1. Compute the eigenvalues λ_1 and λ_2 of the matrix

$$\begin{pmatrix} -1 & -2 \\ -6 & 3 \end{pmatrix}.$$

Then compute $|\lambda_1 - \lambda_2|$, and use that as your answer.

A. 1.
B. 2.
C. 3.
D. 4.
E. 5.
F. 6.
G. 7.
H. 8.
I. 9.
J. 10.
K. 11.

2. Compute the eigenvalues λ_1, λ_2, and λ_3 of the matrix

$$\begin{pmatrix} 0 & 0 & 2 \\ 4 & 0 & 0 \\ 2 & 2 & -4 \end{pmatrix}.$$

Then compute $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, and use that as your answer.

A. -5.
B. -4.
C. -3.
D. -2.
E. -1.
F. 0.
G. 1.
H. 2.
I. 3.
J. 4.
K. 5.

3. One of the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 0 & 2 \\ 4 & 0 & 0 \\ 2 & 2 & -4 \end{pmatrix}$$

is $\lambda = -4$. Compute an eigenvector $\mathbf{v}$ associated to this eigenvalue. Write it in the form

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Then compute

$$\frac{w^2}{uv},$$

and use that as your answer.

A. -5.

B. -4.

C. -3.

D. -2.

E. -1.

F. 0.

G. 1.

H. 2.

I. 3.

J. 4.

K. 5.

4. One of the eigenvalues of the matrix

$$\begin{pmatrix} 1 & -5 \\ 13 & 3 \end{pmatrix}$$

is $\lambda = 2 + 8i$. Compute an eigenvector $\mathbf{v}$ associated to this eigenvalue. Write it in the form

$$\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}.$$

Compute the real part of

$$\frac{v}{w},$$

and use that as your answer.

A. $\frac{1}{2}$.
B. $\frac{1}{3}$.
C. $\frac{1}{5}$.
D. $\frac{1}{8}$.
E. $\frac{1}{13}$.
F. 0.
G. $-\frac{1}{2}$.
H. $-\frac{1}{3}$.
I. $-\frac{1}{5}$.
J. $-\frac{1}{8}$.
K. $-\frac{1}{13}$.

5. The matrix

$$A = \begin{pmatrix} 6 & 3 \\ -6 & -5 \end{pmatrix}$$

has eigenvectors $\left(\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)$.

Let $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ be the solution to the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Viewing the positive x axis as to the right and the positive y axis as upward, what is the direction of travel (velocity) of this solution at $t = 7$?

A. Directly to the right.

B. Up and to the right.

C. Directly up.

D. Up to and the left.

E. Directly left.

F. Down and to the left.

G. Directly down.

H. Down and to the right.

6. The matrix

$$A = \begin{pmatrix} 3 & 5 \\ -5 & -5 \end{pmatrix}$$

has eigenvalues $-1 \pm 3i$. An eigenvector corresponding to $-1+3i$ is $\left(\begin{smallmatrix} -5 \\ 4-3i \end{smallmatrix}\right)$.

Let $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ be the solution to the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Viewing the positive x axis as to the right and the positive y axis as upward, what is the direction of travel (velocity) of this solution at $t = \frac{4\pi}{3}$?

A. Directly to the right.

B. Up and to the right.

C. Directly up.

D. Up to and the left.

E. Directly left.

F. Down and to the left.

G. Directly down.

H. Down and to the right.

7. The matrix
$$A = \begin{pmatrix} -3 & 7 \\ 4 & 5 \end{pmatrix}$$
has eigenvalues $1 + 2\sqrt{11}$ and $1 - 2\sqrt{11}$, with corresponding eigenvectors
$$\begin{pmatrix} -2 + \sqrt{11} \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} -2 - \sqrt{11} \\ 2 \end{pmatrix}.$$

Let $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ be the solution to the initial value problem
$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Viewing the positive x axis as to the right and the positive y axis as upward, what is the direction of travel (velocity) of this solution at $t = 0$?

A. Directly to the right.

B. Up and to the right.

C. Directly up.

D. Up to and the left.

E. Directly left.

F. Down and to the left.

G. Directly down.

H. Down and to the right.

8. The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$$

has eigenvectors $\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)$.

Let $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ be the solution to the initial value problem

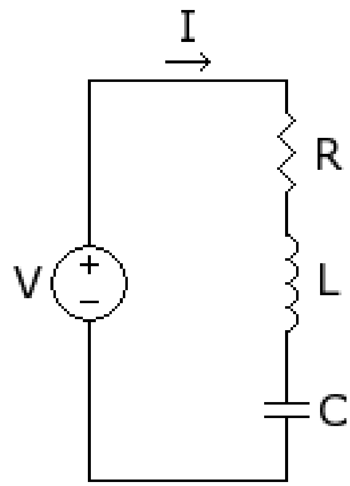
$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

What is the smallest value of the distance of $\mathbf{x}(t)$ to the origin for $t \geq 0$?

Hint: The distance to the origin is $\sqrt{x(t)^2 + y(t)^2}$, but, when finding the minimum, it's easier to minimize $x(t)^2 + y(t)^2$. Keep in mind that the problem is only interested in $0 \leq t < \infty$.

A. Between 0 and 1.
B. Between 1 and 2.
C. Between 2 and 3.
D. Between 3 and 4.
E. Between 4 and 5.
F. Between 5 and 6.
G. Between 6 and 7.
H. Between 7 and 8.
I. Between 8 and 9.
J. Between 9 and 10.
K. Between 10 and 11.

9. In the following circuit, the resistor has a resistance of $10\,\Omega$, the inductor has an inductance of $50\,\mathrm{H}$, and the capacitor has a capacitance of $0.005\,\mathrm{F}$.



The voltage source provides an input voltage of $(100\,\mathrm{V})\sin\omega t$, where t is in seconds. The input voltage frequency ω is tuned so as to cause the largest possible amplitude of the steady periodic response of the current I, that is, to achieve electrical resonance. What is the amplitude of the steady periodic response of the current at electrical resonance?

A. 1 Amp.
B. 2 Amp.
C. 4 Amp.
D. 5 Amp.
E. 10 Amp.
F. 20 Amp.
G. 25 Amp.
H. 50 Amp.
I. 100 Amp.
J. 200 Amp.
K. 400 Amp.

10. Consider two tanks that are part of a larger fermentation system. Tank 1 has a volume of 10 gal. Tank 2 has a volume of 40 gal. Liquid with sugar concentration 2 lbs/gal enters Tank 1 at a rate of 3 gal/hr. The well-mixed liquid in Tank 1 flows into Tank 2 at a rate of 4 gal/hr. Meanwhile, the well-mixed liquid in Tank 2 flows into Tank 1 at a rate of 1 gal/hr. Additionally, the well-mixed liquid in Tank 2 flows out to the larger fermentation system at a rate of 3 gal/hr. Initially, the concentration of sugar in Tank 1 is 2 lbs/gal, and the concentration of sugar in Tank 2 is 4 lbs/gal.

    Let x be the amount of sugar in Tank 1, and let y the amount of sugar in Tank 2. Set up a differential equation for x and y of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{v},$$

    where A is a matrix and $\mathbf{v}$ is a vector, with units in terms of gallons, pounds, and hours. Compute the determinant of A.

    A. $-\frac{1}{20}$.
    B. $-\frac{1}{25}$.
    C. $-\frac{3}{100}$.
    D. $-\frac{1}{50}$.
    E. $-\frac{1}{100}$.
    F. 0.
    G. $\frac{1}{100}$.
    H. $\frac{1}{50}$.
    I. $\frac{3}{100}$.
    J. $\frac{1}{25}$.
    K. $\frac{1}{20}$.

11. Find all of the equilibria of the following system.

$$\dot{x} = (x+5)(y^2-16),$$
$$\dot{y} = (x-3)^2 - y.$$

Add up the x-coordinates of all of the equilibria, and use that as your answer.

A. -5.
B. -4.
C. -3.
D. -2.
E. -1.
F. 0.
G. 1.
H. 2.
I. 3.
J. 4.
K. 5.

12. Let $f(t) = 2e^{-2t} - 2e^{-5t}$. Let $F(s)$ be the Laplace transform of $f(t)$. Compute $F(4)$.

A. 1.

B. $\frac{1}{2}$.

C. $\frac{1}{3}$.

D. $\frac{1}{4}$.

E. $\frac{1}{5}$.

F. $\frac{1}{6}$.

G. $\frac{1}{7}$.

H. $\frac{1}{8}$.

I. $\frac{1}{9}$.

J. $\frac{1}{10}$.

K. $\frac{1}{11}$.

13. Let

$$f(t) = \begin{cases} 1 & \text{if } 1 \le t < 2, \\ e^{-3t} & \text{if } 3 \le t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(s)$ be the Laplace transform of $f(t)$. Evaluate $F(3)$.

A. $\frac{1}{15}e^{-3} - \frac{1}{15}e^{-6} + \frac{1}{18}e^{-9} - \frac{1}{18}6e^{-12}$.
B. $\frac{1}{21}e^{-3} - \frac{1}{21}e^{-9} + \frac{1}{9}e^{-18} - \frac{1}{9}e^{-21}$.
C. $\frac{1}{9}e^{-3} - \frac{1}{9}e^{-6} + \frac{1}{3}e^{-9} - \frac{1}{3}e^{-15}$.
D. $\frac{1}{3}e^{-9} - \frac{1}{3}e^{-18} + \frac{1}{6}e^{-21} - \frac{1}{6}e^{-24}$.
E. $\frac{1}{3}e^{-12} - \frac{1}{3}e^{-15} + \frac{1}{18}e^{-18} - \frac{1}{18}e^{-24}$.
F. $\frac{1}{9}e^{-6} - \frac{1}{9}e^{-9} + \frac{1}{6}e^{-12} - \frac{1}{6}e^{-18}$.
G. $\frac{1}{12}e^{-3} - \frac{1}{12}e^{-6} + \frac{1}{6}e^{-9} - \frac{1}{6}e^{-18}$.
H. $\frac{1}{6}e^{-6} - \frac{1}{6}e^{-15} + \frac{1}{15}e^{-18} - \frac{1}{15}e^{-21}$.
I. $\frac{1}{3}e^{-3} - \frac{1}{3}e^{-6} + \frac{1}{6}e^{-18} - \frac{1}{6}e^{-24}$.
J. $\frac{1}{6}e^{-6} - \frac{1}{6}e^{-9} + \frac{1}{15}e^{-15} - \frac{1}{15}e^{-24}$.
K. $\frac{1}{12}e^{-3} - \frac{1}{12}e^{-15} + \frac{1}{9}e^{-21} - \frac{1}{9}e^{-24}$.

14. Consider the initial value problem

$$(2x+8)y' = y, \qquad\qquad y(0) = 2.$$

Let

$$y = \sum_{n=0}^{\infty} c_n x^n$$

be a power series solution of this equation. Compute c_3.

A. $\frac{1}{2}$.
B. $\frac{1}{4}$.
C. $\frac{1}{8}$.
D. $\frac{1}{16}$.
E. $\frac{1}{32}$.
F. $\frac{1}{64}$.
G. $\frac{1}{128}$.
H. $\frac{1}{256}$.
I. $\frac{1}{512}$.
J. $\frac{1}{1024}$.
K. $\frac{1}{2048}$.

16. These are the instructions for the second free response question. Do not answer it on this page.

(a) The eigenvalues of the matrix

$$A = \begin{pmatrix} -6 & 2 \\ -2 & -1 \end{pmatrix}.$$

are -2 and -5, with corresponding eigenvectors $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. Let $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, and consider the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

On the left plot of the xy-plane, draw a phase portrait containing parametric plots of the solution curves specified below.

- Putting aside the initial condition for now, draw lines through the origin corresponding to the eigenvectors.
  - For any solution along each of those lines, draw an arrow for its direction of motion.
  - If the solutions along the line do not move, instead draw dots along the line.
- Make a rough (qualitative) plot of the solution the initial value problem above.
  - Include an arrow for the direction of motion.
  - Label the location of $\mathbf{x}$ at $t = 0$.
  - Make sure your solution is going in the correct direction at $t = 0$.
  - Make sure your solution is approaching the correct direction as $t \to \infty$ and $t \to -\infty$.
- Do not draw any additional solutions, or, if you do, clearly label which one is the solution to the initial value problem above.

On the right plot of $x(t)$ versus t, for $t \geq 0$:

- Plot a rough (qualitative) graph of the x-coordinate of the solution to the initial value problem above.
  - Make sure your graph has the right value and derivative at $t = 0$.
  - Make sure it's clear from your graph whether or not $x(t)$ ever changes sign.

– Make sure your graph has the right behavior as $t \to \infty$.

Finally, write the name of the type of equilibrium at the origin, and then specify whether it is unstable or stable, and if it is stable, make sure to mention whether or not it is asymptotically stable.

(b) Do the same problem, except with

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -5 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The matrix has a repeated eigenvalue of -3, with an eigenvector of $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$.

15. Determine the *real* eigenvalues and the corresponding eigenfunctions, if any, of the following boundary value problem

$$y'' + y' + \lambda(y' + y) = 0, \quad y'(0) = 0, \quad y(1) = 0.$$

where prime denotes the differentiation with respect to x.

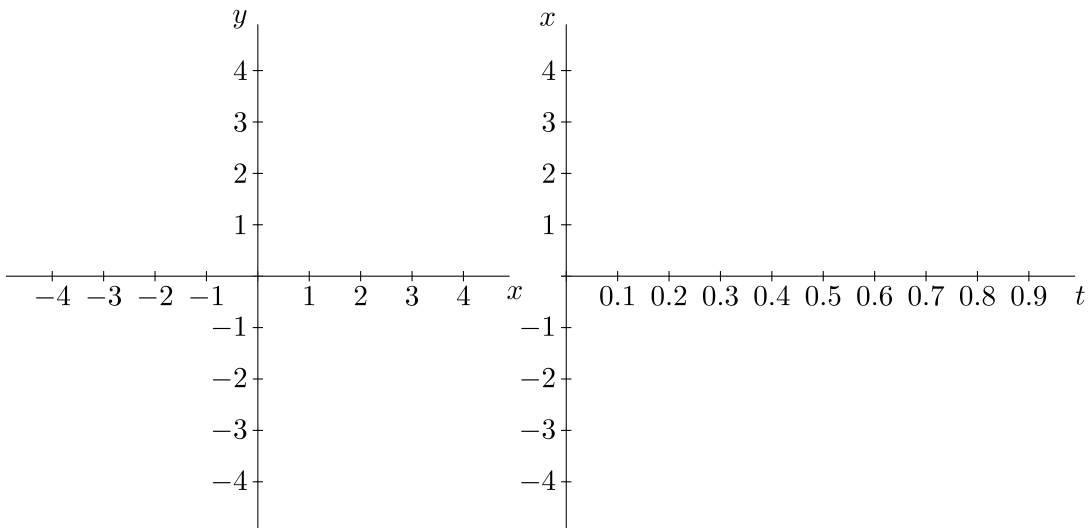
Math 217 **Name:** ____________________

Final **Student ID:** ____________________

December 16, 2016 **Circle one: 9am 11am**

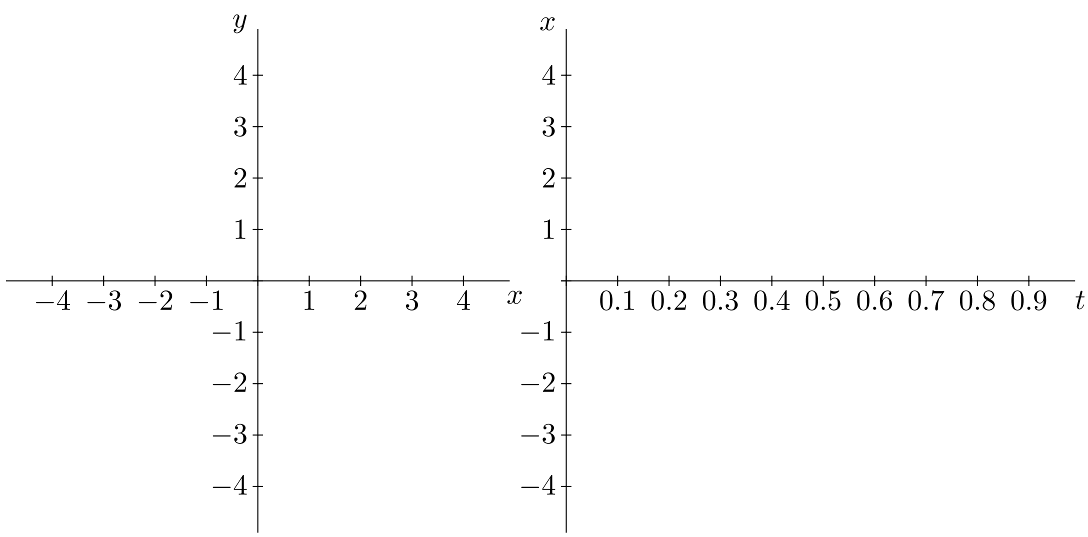
16. The instructions for this problem are at the end of the multiple choice section.

(a) $A = \begin{pmatrix} -6 & 2 \\ -2 & -1 \end{pmatrix}$, eigenvalues -2 and -5, corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.



Type of equilibrium: ____________________. Stability: ____________________.

(b) $A = \begin{pmatrix} -1 & -4 \\ 1 & -5 \end{pmatrix}$, eigenvalue -3, eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.



Type of equilibrium: ____________________. Stability: ____________________.