Hamel's Formalism for Infinite-Dimensional Mechanical Systems

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Motivation

In his 1752 paper, Euler showed that the equations of motion of a rigid body can be written in a much simpler way by expressing the angular velocity with respect to a basis that rotates along with the rigid body. In 1904, Hamel generalized these Euler-Poincaré equations to equations of motion where the velocity is expressed with respect to an arbitrary basis, not necessarily one arising from a Lie group action. We extend Hamel's equations to infinite-dimensional systems, providing a systematic way of deriving simpler equations of motion in the presence of nonholonomic constraints or symmetry, without unnecessary Lagrange multipliers. These results appear in the *Journal of Nonlinear Science* vol. 27 (2017), no. 1, 241–283.

Lagrangian mechanics

We view the evolution of a mechanical system as a trajectory $q(t)$ in some *configuration manifold* Q. We describe the system with a *Lagrangian* $L(q,\dot q)$, a real-valued function of position $q \in Q$ and velocity $\dot q \in T_qQ$ representing the difference between the potential and the kinetic energy of the system.

Hamilton's principle of stationary action

Given a parametrized curve $q(t)$ for $a \le t \le b$, the *action functional* of the curve is

$$\int_a^b L(q(t),\dot q(t))\,dt.$$

Hamilton's principle of stationary action states that the trajectories of a mechanical system will be critical points of the action functional. That is, a mehanical system will evolve along the trajectory $q(t)$ if

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_a^b L(q_\varepsilon(t),\dot q_\varepsilon(t)) = 0,$$

for any variation q_ε of curves through $q = q_0$ with fixed endpoints $q_\varepsilon(a) = q(a)$ and $q_\varepsilon(b) = q(b)$.

The Euler-Lagrange equations

In local coordinates, $q = \left(q^1,\dots,q^n\right)$ and

$$\dot q = \sum \dot q^i \tfrac{\partial}{\partial q^i}.$$

The principle of stationary action is equivalent to the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = \frac{\partial L}{\partial q^i}, \qquad i = 1,\dots,n.$$

Example: A spinning hockey puck

The configuration space is a position (x,y) in the plane, along with an angle θ for the puck's orientation. There is no potential energy in this system, so the Lagrangian is equal to the kinetic energy

$$L(\theta,x,y,\dot\theta,\dot x,\dot y) = \tfrac12 J\dot\theta^2 + \tfrac12 m\left(\dot x^2+\dot y^2\right),$$

where m is the mass of the hockey puck and J is its moment of inertia.

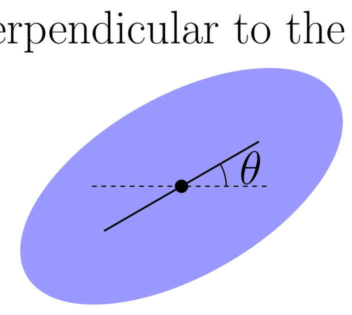
The Euler-Lagrange equations for this system are

$$J\ddot\theta = 0, \qquad m\ddot x = 0, \qquad m\ddot y = 0.$$

The hockey puck moves in a straight line with constant velocity, spinning at a constant rate.

Example: The Chaplygin sleigh

We introduce a constraint into the previous example. Instead of a hockey puck, we have a platform on top of a blade. The blade enforces the ideal constraint that the object only moves in the direction θ by supplying a normal force perpendicular to the blade.



We introduce a Lagrange multiplier λ to represent the magnitude of the normal force between the blade and the ice. The force acts in the orthogonal direction $-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}$, so the equations for the constrained system become

$$J\ddot\theta = 0, \qquad m\ddot x = -\lambda\sin\theta, \qquad m\ddot y = \lambda\cos\theta,$$

coupled with the constraint $-\dot x\sin\theta + \dot y\cos\theta = 0$ reflecting that the normal component of the velocity of the blade is zero. Using this constraint, we find that

$$\lambda = m(\dot x\cos\theta + \dot y\sin\theta)\dot\theta.$$

Hamel's formalism for finite-dimensional systems

In the Lagrangian formalism, we expressed the components of the velocity $\dot q$ with respect to the coordinate frame $\frac{\partial}{\partial q^i}$. In the presence of either constraints or symmetry, we obtain simpler equations of motion when we write the velocity in terms of a more natural local frame.

The Hamel equations

We express the velocity with respect to an arbitrary frame $u_i(q)$. Each u_i is a local vector field, and, at each point q, the $u_i(q)$ form a basis. Instead of writing $\dot q = \sum \dot q^i\frac{\partial}{\partial q^i}$, we split the velocity into components as

$$\dot q = \sum \xi^i u_i(q).$$

We then rewrite the Lagrangian as

$$l\left(q,\xi^1,\dots,\xi^n\right) = L\left(q,\sum\xi^i u_i(q)\right).$$

From Hamilton's principle of stationary action, we derive the *Hamel equations*

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = u_j[l] + \sum_{i,k} c_{ij}^k\frac{\partial l}{\partial \xi^k}\xi^i, \qquad j = 1,\dots,n,$$

where $u_j[l]$ denotes the directional derivative of $l(\cdot,\xi^i)\colon Q\to\mathbb{R}$ in the direction u_j, and the *structure functions* $c_{ij}^k(q)$ are defined via the commutators

$$[u_i,u_j](q) = c_{ij}^k(q)u_k(q).$$

Unlike the coordinate vector fields $\frac{\partial}{\partial q^i}$, the vector fields u_i will have nontrivial commutators, which gave us an extra term in the equations of motion.

Example: The Chaplygin sleigh

We wrote the velocity $\dot q$ as $\dot\theta\frac{\partial}{\partial\theta}+\dot x\frac{\partial}{\partial x}+\dot y\frac{\partial}{\partial y}$. Because the motion is constrained to be in the direction θ, it is more natural to consider the components of the velocity that are parallel and orthogonal to the direction θ. Thus, we rewrite the velocity as

$$\dot q = \omega\tfrac{\partial}{\partial\theta} + v\left(\cos\theta\tfrac{\partial}{\partial x}+\sin\theta\tfrac{\partial}{\partial y}\right) + w\left(-\sin\theta\tfrac{\partial}{\partial x}+\cos\theta\tfrac{\partial}{\partial y}\right).$$

In terms of these components, the Lagrangian is

$$l(q,\omega,v,w) = \tfrac12 J\dot\theta^2 + \tfrac12 m\left(v^2+w^2\right).$$

The unconstrained Hamel equations for this system are

$$J\dot\omega = 0, \qquad m\dot v = mw\omega, \qquad m\dot w = -mv\omega.$$

The constraint is just $w = 0$, and, as before, we enforce it by supplying a normal force λ. The constrained equations for the Chaplygin sleigh become

$$J\dot\omega = 0, \qquad m\dot v = mw\omega, \qquad m\dot w = -mv\omega + \lambda,$$

coupled with the constraint $w = 0$. Since $\dot w = 0$, it is easy to solve for $\lambda = mv\omega$, and so we reduce these equations to

$$J\dot\omega = 0, \qquad m\dot v = 0,$$

again coupled with the constraint $w = 0$. These simpler equations are equivalent to the ones we obtained in the Lagrangian formalism.

Hamel's formalism for infinite-dimensional systems

In the finite-dimensional setting, instead of providing a local frame $u_i(q)$, we could have equivalently provided a *local trivialization*, namely, a linear isomorphism $\Psi_q\colon\mathbb{R}^n\to T_qQ$ sending $\xi = \left(\xi^1,\dots,\xi^n\right)$ to $\sum\xi^i u_i(q)$ for each point q in a local neighborhood. The map Ψ_q sends the velocity components $(\xi^1,\dots,\xi^n)$ to the velocity vector $\dot q$ that they represent in T_qQ.

$$\dot q = \Psi_q(\xi).$$

Whereas the concept of a local frame does not extend to infinite dimensions, the concept of a local trivialization does; we merely need to replace $\mathbb{R}^n$ by an appropriate infinite-dimensional vector space W with a bounded linear isomorphism $\Psi_q\colon W\to T_qQ$ sending ξ to $\dot q$.

The Hamel equations

We can rewrite the Lagrangian as

$$l(q,\xi) = L(q,\Psi_q(\xi)).$$

For fixed ξ and η, we have vector fields $\Psi_q(\xi)$ and $\Psi_q(\eta)$, and we can compute their commutator $[\Psi(\xi),\Psi(\eta)]$. At each point q, we can pull back this bracket to a Lie bracket $[\xi,\eta]_q$ on W via

$$\Psi_q([\xi,\eta]_q) = [\Psi(\xi),\Psi(\eta)]\,(q)$$

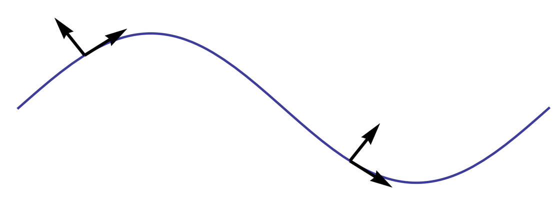
The Hamel equations on the left generalize to

$$\frac{d}{dt}\left\langle\frac{\partial l}{\partial\xi},\eta\right\rangle_W = \left\langle\frac{\partial l}{\partial q},\Psi_q(\eta)\right\rangle_{T_qQ} + \left\langle\frac{\partial l}{\partial\xi},[\xi,\eta]_q\right\rangle_W$$

for all $\eta\in W$.

Example: An inextensible string

Consider a free inextensible string in the plane parametrized by $0\le s\le L$. Its position is described by coordinate functions $x(s)$ and $y(s)$. We can represent the velocity of each point on the string using its components $\dot x(s)$ and $\dot y(s)$, but it is more natural to write the velocity in terms of the component tangent to the string and the component normal to the string, which we denote as $\xi^t(s)$ and $\xi^n(s)$, respectively.



Using our formalism for this infinite-dimensional system, we can write the equation of motion in terms of $\xi^t(s)$ and $\xi^n(s)$ directly. It is helpful to use complex numbers, writing $\xi = \xi^t + i\xi^n$. The equations of motion are

$$\dot\xi = \xi\bar\xi_s + \tau_s + i\kappa(\tau - \xi\bar\xi),$$

where κ is the signed curvature of the string and τ is a Lagrange multiplier representing the tension of the string and enforcing its incompressibility.

Imagining the string as a flexible blade on ice, we impose the constraint that $\xi^n = 0$. Using our formalism for this constrained system, our equations of motion become

$$\dot\xi = \xi\bar\xi_s + \tau_s.$$

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