Vanishing Viscosity Solutions of Riemann Problems for Models of Polymer Flooding

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Abstract. We consider the solutions of Riemann problems for polymer flooding models. In a suitable Lagrangian coordinate the systems take a triangular form, where the equation for thermo-dynamics is decoupled from the hydro-dynamics, leading to the study of scalar conservation laws with discontinuous flux functions. We prove three equivalent admissibility conditions for shocks for scalar conservation laws with discontinuous flux. Furthermore, we show that a variation of minimum path of [10] proposed in [18] is the vanishing viscosity limit of a partially viscous model with viscosity only in the hydro-dynamics.

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1. Introduction

Consider the model for polymer flooding in two phase flow for secondary oil recovery [6, 18]

\[
\begin{aligned}
\frac{s_t}{t} + g(s, c)x &= 0, \\
\frac{(cs)_t}{t} + (cg(s, c))_x &= 0.
\end{aligned}
\]  

Here \(s\) is the saturation of the water phase and \(c\) is the fraction of the polymer dissolved in the water phase. The function \(g(s, c)\) denotes the fractional flow, which is the classical S-shaped Buckley-Levrett flux function [7].

A related model, which takes into consideration the adsorption effect of the porous media, takes the form

\[
\begin{aligned}
\frac{s_t}{t} + g(s, c)x &= 0, \\
\left(m(c) + cs\right)_t + (cg(s, c))_x &= 0,
\end{aligned}
\]  

where the term \(m(c)\) models the adsorption of polymer in the rock. Physical property of the porous media usually prompts the assumptions

\[m'(c) \geq 0 \quad \text{and} \quad m''(c) \leq 0.\]

The system in (1.1) can be viewed as a special case of (1.2) with \(m(c) = \text{constant}\).

The two systems (1.1) and (1.2) share some common features. When \(c\) is constant, the second equation is equivalent to the first equation. Thus, one family of integral
curves are all straight lines where \(c\) is constant. We call this the \(s\)-family. The other family, referred to as the \(c\)-family, will have waves that connect different straight lines of \(s\)-integral curves. For each given \(c\), there exists at least one \(s\) value where the two eigenvalues and two eigenvectors both coincide. Thus, there exist at least one curve \(s = S(c)\) in the domain where the system is parabolic degenerate. Along these degenerate curves, non-linear resonance occurs, and the total variation of the unknown \(s(t, x)\) could blow up in finite time [20].

Another very interesting feature shared by these two systems is that they partially decouple using a suitably defined Lagrangian coordinate, see [17, 21]. We define the Lagrangian coordinate \((\psi, \phi)\) as

\[
\phi_x = -s, \quad \phi_t = g, \quad \psi = x. \tag{1.3}
\]

We see that \(\phi\) is the potential function of the first equation in (1.1). Using \((\psi, \phi)\) as the independent variables, the system (1.1) becomes

\[
\begin{align*}
\frac{\partial}{\partial \psi} \left( \frac{1}{g(s, c)} \right) - \frac{\partial}{\partial \phi} \left( \frac{s}{g(s, c)} \right) &= 0, \\
\frac{\partial c}{\partial \psi} &= 0.
\end{align*}
\tag{1.4}
\]

For the system (1.2), the same coordinate change gives

\[
\begin{align*}
\frac{\partial}{\partial \psi} \left( \frac{1}{g(s, c)} \right) - \frac{\partial}{\partial \phi} \left( \frac{s}{g(s, c)} \right) &= 0, \\
\frac{\partial c}{\partial \psi} + \frac{\partial}{\partial \phi} m(c) &= 0.
\end{align*}
\tag{1.5}
\]

The equivalence between weak solutions of the systems in the two coordinates is proved in the seminar paper of Wagner [21]. We observe that the systems (1.5) and (1.4) are in triangular form, where the second equation is decoupled from the first one. Referring to physical principles, the first equation in (1.5) (and (1.4)) governs the hydro-dynamics, while the second equation describes the thermo-dynamics. This decoupled feature indicates that the thermo-dynamic process is independent of the hydro-dynamics.

The solutions of (1.5) and (1.4) could be obtained by first solving the second equation for \(c\), then plugging the solution of \(c\) into the first equation and solving for \(s\). The solution for \(c\), either being constant in “time” \(\psi\) for (1.4) or as the solution of a scalar conservation law in (1.5), might well contain discontinuities. This motivates the study of a scalar conservation law with discontinuous flux.

Therefore, we consider the solution of the Riemann problem for scalar conservation law with discontinuous flux in a general setting. We consider

\[
u_t + f(a(x), u)_x = 0, \tag{1.6}
\]
where

\[ a(x) = \begin{cases} a^- & (x < 0), \\
           a^+ & (x > 0), \end{cases} \tag{1.7} \]

associated with the initial Riemann data

\[ u(0, x) = \begin{cases} u^L & (x < 0), \\
           u^R & (x > 0). \end{cases} \tag{1.8} \]

We observe that the solutions to (1.6)-(1.8) can be obtained as limits of two combined approximations:

(i) One may approximate the jump function \( a(\cdot) \) by a sequence of smooth functions. For example, let \( \hat{a}(x) \) be a smooth function such that \( \lim_{x \to \pm \infty} \hat{a}(x) = a^{\pm} \). Given a decreasing sequence \( \epsilon_n \to 0 \), one can define

\[ a_n(x) = \hat{a}\left(\frac{x}{\epsilon_n}\right). \]

One may also take any other convergent sequence.

(ii) One can add a viscosity term \( \epsilon_n u_{xx} \) on the right hand side of (1.6).

From modeling considerations, it is natural to consider the solutions which are obtained as limits of approximations: \( u = \lim_{n \to \infty} u^{(n)} \), where \( u^{(n)} \) denotes a solution to the viscous conservation law with smooth flux

\[ u_t + f\left(a_n(x), u\right)_x = \epsilon_n u_{xx}. \tag{1.9} \]

In this setting, three main issues arise:

1. For a given initial Riemann data, is the limit solution \( u = \lim u^{(n)} \) uniquely defined, or does it depend on the relative rate at which the sequences \( \epsilon_n, \epsilon'_n \) approach zero?

2. If in general the solution depends on the ratio, what is the sufficient assumptions one can make such that all these limits are the same?

3. How can one determine the traces \( u^- = u(t, 0-) \) and \( u^+ = u(t, 0+) \) in this limit solution?

In a parallel paper [19], through several detailed counter examples, we show that the answer to the first question is the latter. The same paper also addresses the answer for the second question, showing that a suitable monotonicity condition on the flux \( c \mapsto g(s, c) \), among others, will ensure the uniqueness of the double-limit. The third question is addressed in this paper, and we consider the case \( \epsilon'_n = 0 \). Here we present
three equivalent admissible conditions, and a detailed construction of the solution of a Riemann problem, and the proof that it is the vanishing viscosity limit.

Scalar conservation laws with discontinuous flux functions have been an extremely active research field in the past few decades, where numerous results are available. We refer to a survey paper [1] and the references therein, and apologize that a comprehensive list of reference is without the scope of this paper. In this paper we seek vanishing viscosity solutions of Riemann problem, leading to certain entropy conditions. In the literature, various forms of entropy conditions have been proposed and studied. For some related works, see e.g. [2, 3, 4, 8, 12, 13, 14, 15].

In connection with the adsorption model (1.2) in the Lagrangian coordinate (1.5), we may consider the triangle system in a general setting

$$\begin{cases} u_t + f(a, u)_x = 0, \\ a_t + m(a)_x = 0, \end{cases}$$

(1.10)

associated with initial Riemann data

$$a(0, x) = \begin{cases} a^- (x < 0), \\ a^+ (x > 0), \end{cases} \quad u(0, x) = \begin{cases} u^L (x < 0), \\ u^R (x > 0). \end{cases}$$

(1.11)

For the triangle system (1.10), let \(a = a(t, x)\) be a solution to the second equation of conservation law, consisting of a single entropy-admissible shock with left and right states \(a^-\) and \(a^+\), respectively. By performing a linear transformation of the \(t-x\) variables, we can assume that the shock speed is zero, so that \(a(t, x) = a(x)\) as in (1.7) for all times \(t \geq 0\). Inserting this solution in the first equation one obtains (1.6). Thus, the results for (1.6) can be applied immediately to (1.10).

The rest of the paper is organized as follows. In section 2 we study the scalar conservation law with discontinuous flux, and prove partially three equivalent admissibility conditions. These results are used to construct a Riemann solver, stated in Theorem 3.1 in section 3, which is proved to be the vanishing viscosity limit. Finally, in section 4 we go back to the polymer flooding models, and show that the Riemann solver proposed in [19] is the vanishing viscosity limit where viscosity is only added to the hydro-dynamics.

2. Equivalent admissible conditions for the trace

We seek a weak solution of the Riemann problem (1.6)-(1.8) in the sense that

$$\int_0^\infty \int_{-\infty}^0 [u\phi_t + f(a^-, u)\phi_x] \, dx \, dt + \int_0^\infty \int_0^\infty [u\phi_t + f(a^+, u)\phi_x] \, dx \, dt = 0$$
for all smooth test function $\phi(t, x)$ with compact support. Here the mapping $u \mapsto f$ is continuous. For notation convenience, we denote the functions

$$f^-(u) = f(a^-, u), \quad f^+(u) = f(a^+, u).$$

It is well-known that solutions to such Riemann problems are self-similar, consisting of left-going waves, a stationary discontinuity at $x = 0$, and some right-going waves. We denote the solution by $u(t, x) = U(x/t)$, and let

$$u^- \doteq U(0-), \quad u^+ \doteq U(0+)$$

(2.1)
denote the left and right state of the stationary jump. The key step in the construction of the solution lies in the selection of the values $u^-, u^+$. Once they are selected, one can solve two Riemann problems for two scalar conservation laws and obtain the left-going and right-going waves. Therefore, the entropy weak solution of the Riemann problem

$$u_t + f^-(u)_x = 0, \quad u(0, x) = \begin{cases} u^L & \text{if } x < 0, \\ u^- & \text{if } x > 0, \end{cases}$$

(2.2)
must generate only waves with speed $\leq 0$. We denote by $W^-(u^L)$ the set of such $u$ values. At the same time, the entropy weak solution of the Riemann problem

$$u_t + f^+(u)_x = 0, \quad u(0, x) = \begin{cases} u^R & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases}$$

(2.3)
must contain only waves of speed $\geq 0$. We denote by $W^+(u^R)$ the set of such $u$ values.

The Rankine-Hugoniot jump condition for the stationary jump at $x = 0$ gives

$$f^-(u^-) = f^+(u^+).$$

(2.4)

We conclude that, the possible candidates for $u^-, u^+$ must satisfy

$$u^- \in W^-(u^L), \quad u^+ \in W^+(u^R), \quad f^-(u^-) = f^+(u^+).$$

(2.5)

In general, conditions in (2.5) yield multiple (or even infinitely many) choices for the trace $u^-, u^+$. Additional entropy conditions are needed to single out a unique solution.

We now seek conditions on trace $u^-, u^+$ such that the piecewise constant functions

$$u(t, x) = \hat{u}(x) \doteq \begin{cases} u^-, & x < 0, \\ u^+ & x > 0, \end{cases}$$

(2.6)
can be obtained as limit of a sequence of viscous approximations $u^\varepsilon$ of

$$u_t + f(a(x), u)_x = \varepsilon u_{xx}, \quad u(0, x) = \hat{u}(x),$$

(2.7)
with $a(x)$ in (1.7), as the viscosity coefficient $\varepsilon \to 0+$. 

\textit{Equivalent Admissible Conditions for the Trace}
Motivated by [9, 10], we introduce the monotone functions
\[ G^\#(u; u^+) = \begin{cases} \max \{ f^+(w); \ w \in [u^+, u]\}, & \text{if } u \geq u^+, \\ \min \{ f^+(w); \ w \in [u, u^+]\}, & \text{if } u \leq u^+, \end{cases} \]
and
\[ G^\flat(u; u^-) = \begin{cases} \min \{ f^-(w); \ w \in [u^-, u]\}, & \text{if } u \geq u^-, \\ \max \{ f^-(w); \ w \in [u, u^-]\}, & \text{if } u \leq u^-\}. \]

Here, \( u \mapsto G^\flat(u; u^-) \) is non-decreasing, and \( u \mapsto G^\#(u; u^+) \) is non-increasing. See Figure 1 for an illustration.

Figure 1. Illustrations of the functions \( u \mapsto G^\#(u; u^+) \) and \( u \mapsto G^\flat(u; u^-) \) for several cases of \( u^+ \) and \( u^- \).

To ensure the solvability of the Riemann problem with Riemann data \((u^L, u^R)\), we assume that the range of the two functions \( G^\flat(\cdot; u^L) \) and \( G^\#(\cdot; u^R) \) have non-empty intersection. To be precise, we assume that, for the given data \( u^L, u^R \), there exists some \( \tilde{u}^* \) such that
\[ G^\flat(\tilde{u}^*; u^L) = G^\#(\tilde{u}^*; u^R). \] (2.10)
Note that, although the point \( \tilde{u}^* \) in (2.10) might not be unique, the common value of the two fluxes \( G^\#(\cdot; u^R) \) is always uniquely determined, thanks to the monotonicity properties of the functions \( G^\flat(\cdot; u^-) \) and \( G^\#(\cdot; u^+) \).

Next Theorem states three equivalent admissible conditions for the jump at \( x = 0 \).

**Theorem 2.1.** (Equivalent admissibility conditions) Given \((u^-, u^+)\), let \( \hat{u} \) be the jump function in (2.6) and let \( a(x) \) be the jump function in (1.7). The following three conditions are equivalent.
(I) There exists a family of monotone viscous solutions $u'(t,x)$ of (2.7) such that
\[
\lim_{t \to 0^+} \|u'(t,\cdot) - \bar{u}(\cdot)\|_{L^1} = 0,
\] (2.11)
uniformly on every bounded time interval $[0,T]$.

(II) The Rankine-Hugoniot condition (2.4) holds, i.e.,
\[
f^-(u^-) = f^+(u^+) = \bar{f}
\] (2.12)
together with the following generalized Oleinik-type conditions [16]:

(i) If $u^- < u^+$, then there exists an intermediate state $u^* \in [u^-,u^+]$ such that
\[
\begin{align*}
    f^-(u) &\geq \bar{f} & \text{for } u \in [u^-,u^*], \\
    f^+(u) &\geq \bar{f} & \text{for } u \in [u^*,u^+].
\end{align*}
\] (2.13)

(ii) If $u^- > u^+$, then there exists an intermediate state $u^* \in [u^+,u^-]$ such that
\[
\begin{align*}
    f^+(u) &\leq \bar{f} & \text{for } u \in [u^+,u^*], \\
    f^-(u) &\leq \bar{f} & \text{for } u \in [u^*,u^-].
\end{align*}
\] (2.14)

(III) There exists a state $\bar{u}^*$, between $u^-$ and $u^+$, such that
\[
\bar{f} = f^-(u^-) = G^2(\bar{u}^-;u^-) = G^2(\bar{u}^*;u^*) = f^+(u^+) = \bar{f}^+.
\] (2.15)

**Remark 1.** Condition (II) is useful to check whether a path between $u^-, u^+$ is admissible, by only using the information of $f^-, f^+$ on the interval between $u^-$ and $u^+$. The behavior of the flux functions outside the interval between $u^-$ and $u^+$ is not important for the admissible condition. See Figure 2 for examples of admissible and non-admissible paths.

![Figure 2](image-url)

Figure 2. Left and center: two cases where the jump $u^-, u^+$ is admissible. Right: the jump $u^-, u^+$ is not admissible.
Proof. We will first prove the implication \((II) \implies (I)\) and the equivalence \((II) \iff (III)\). We leave the last implication \((I) \implies (III)\) after the proof of Theorem 3.1. The proof will take several steps.

1. We first prove the implication \((II) \implies (I)\). To fix the ideas, assume that \(u^- < u^+\), while the other case being entirely similar. We have three cases:

   a) We can choose \(u^*\) such that \(u^- < u^* < u^+\). In this case we must have both \((f^+)'(u^+) \leq 0\) and \((f^-)'(u^-) \geq 0\). See for example Figure 2 center plot.

   b) We can only choose \(u^* = u^-\). In this case we must have \((f^+)'(u^+) \leq 0\).

   c) We can only choose \(u^* = u^+\). In this case we must have \((f^-)'(u^-) \geq 0\). See for example Figure 2 left plot.

In this step we only deal with the easier case where all the inequality signs are strict in (2.13)-(2.14). To be precise, we make the following additional assumptions.

Case (a): If \(u^- < u^* < u^+\), then we further assume that

\[
\begin{align*}
  f^-(u) > \bar{f} & \quad \text{for } u \in [u^-, u^*], \\
  f^+(u) > \bar{f} & \quad \text{for } u \in [u^*, u^+],
\end{align*}
\]

and

\[
\begin{align*}
  (f^-)'(u^-) > 0, \\
  (f^+)'(u^+) < 0.
\end{align*}
\]

(2.16)

Case (b): If \(u^* = u^-\) only, then we also assume

\[
\begin{align*}
  f^+(u) > \bar{f} & \quad \text{for } u \in [u^-, u^+],
\end{align*}
\]

and

\[
(f^+)'(u^+) < 0.
\]

(2.17)

Case (c): If \(u^* = u^+\) only, then we assume in addition

\[
\begin{align*}
  f^-(u) > \bar{f} & \quad \text{for } u \in [u^-, u^+],
\end{align*}
\]

and

\[
(f^-)'(u^-) > 0.
\]

(2.18)

Under these stricter assumptions, we now show that there exists a family of traveling wave solutions such that condition (I) holds.

Let \(\epsilon = 1\), and let \(U^1(x)\) be a stationary traveling wave profile of (2.7) with unit viscosity, with the boundary conditions

\[
\lim_{x \to -\infty} U^1(x) = u^-, \quad \lim_{x \to +\infty} U^1(x) = u^+.
\]

If it exists, the viscous traveling wave \(U^1(x)\) must satisfy the ODE:

\[
f(a(x), U^1)_{xx} = U^1_{xx}.
\]

Integrating it once in \(x\), and using the boundary condition at \(x = -\infty\), we get

\[
U^1_x = f(a(x), U^1) - f(a^-, u^-) = f(a(x), U^1) - \bar{f}.
\]

(2.19)

Similarly, using the boundary condition at \(x = +\infty\), we get

\[
U^1_x = f(a(x), U^1) - f(a^+, u^+) = f(a(x), U^1) - \bar{f}.
\]

(2.20)
Combining (2.19)-(2.20), we consider the following initial value problem

\[
U^1(x) = \begin{cases} 
    f^-(U^1(x)) - \bar{f}, & \text{if } x < 0, \\
    f^+(U^1(x)) - \bar{f}, & \text{if } x > 0,
\end{cases}
\quad U^1(0) = u^*.
\]

(2.21)

Under the assumptions in (2.16)-(2.18), we have \( U^1 \geq 0 \), so the ODE (2.21) has a unique monotone solution which satisfies

\[
\lim_{x \to -\infty} U^1(x) = u^-, \quad \lim_{x \to +\infty} U^1(x) = u^+.
\]

The solution is strictly increasing for case (a), strictly increasing on \( x > 0 \) for case (b), and strictly increasing on \( x < 0 \) for case (c). Furthermore, we have

\[
\|U^1 - \hat{u}\|_{L^1} = \int_{-\infty}^{0} |U^1(x) - u^-| \, dx + \int_{0}^{+\infty} |U^1(x) - u^+| \, dx < \infty.
\]

(2.22)

Here, the differences \( |U(x) - u^\pm| \) are integrable thanks to the stricter assumptions (2.16)-(2.18) which ensure that the limits as \( x \to \pm\infty \) in (2.22) are approached at least at an exponential rate.

We observe that the functions \( u(t,x) = U^1(x) \) provide a traveling wave solution to the viscous system (2.7) with \( \epsilon = 1 \). In turn, for every \( \epsilon > 0 \), the rescaled function

\[
u'(t,x) \equiv U^1(x/\epsilon)
\]

(2.23)

gives a traveling wave solution to (2.7). The variable rescaling implies

\[
\|v'(t, \cdot) - \hat{u}(\cdot)\|_{L^1} = \epsilon \|U^1 - \hat{u}\|_{L^1},
\]

thus the norm \( \|v'(t, \cdot) - \hat{u}(\cdot)\|_{L^1} \) approaches 0 as \( \epsilon \to 0 \). This proves the implication (II) \( \Rightarrow \) (I), under the stricter assumptions (2.16)-(2.18).

2. However, if the stricter assumptions (2.16)-(2.18) are removed, viscous traveling wave profiles might not exist or converge to the shock in \( L^1 \). For a counter example, consider

\[
u_t + f(x,u)_x = \epsilon u_{xx}, \quad \text{with} \quad f(x,u) = \begin{cases} 
    f^-(u) = u^2 & \text{if } x \leq 0, \\
    f^+(u) = u^2 - 1 & \text{if } x > 0.
\end{cases}
\]

The function

\[
U^\epsilon(t,x) = \begin{cases} 
    (1 - x/\epsilon)^{-1} & \text{if } x \leq 0, \\
    1 & \text{if } x > 0,
\end{cases}
\]

is a stationary traveling wave which converges pointwise to the stationary shock

\[
U(x) = \begin{cases} 
    0 & \text{if } x < 0, \\
    1 & \text{if } x \geq 0,
\end{cases}
\]
but \( \| U^\epsilon(t, \cdot) - U(\cdot) \|_{L^1} = \infty \) for every \( \epsilon > 0 \).

3. By slightly modifying the construction in step 1, we now show that (2.11) remains valid even without the stricter assumptions (2.16)-(2.18). We discuss the three cases separately.

Case (a), with \( u^- < u^* < u^+ \). For any \( \delta > 0 \), consider the modified flux function

\[
    f_\delta(a, u) = f(a, u) + \delta(u^+ - u)(u - u^-),
\]

so

\[
    \begin{align*}
        f^-_\delta(u) & = f^-(u) + \delta(u^+ - u)(u - u^-), \\
        f^+_\delta(u) & = f^+(u) + \delta(u^+ - u)(u - u^-).
    \end{align*}
\]

Note that \( f^-_\delta(u^-) = f^+_\delta(u^+) = \tilde{f} \). Moreover, the functions \( f^-_\delta(u) \) and \( f^+_\delta(u) \) satisfy the stricter inequalities in (2.16). Hence the ODE

\[
    U'_x(x) = \begin{cases} 
        f^-_\delta(U(x)) - \tilde{f}, & \text{if } x < 0, \\
        f^+_\delta(U(x)) - \tilde{f}, & \text{if } x > 0,
    \end{cases} \quad U(0) = u^*
\]

has a unique solution, denoted by \( U_\delta(\cdot) \), which is strictly increasing and satisfies

\[
    \begin{align*}
        \lim_{x \to -\infty} U_\delta(x) & = u^-, \\
        \lim_{x \to +\infty} U_\delta(x) & = u^+,
    \end{align*}
\]

\[
    \| U_\delta - \tilde{u} \|_{L^1(\mathbb{R})} = \int_{-\infty}^{0} |U_\delta(x) - u^-| \, dx + \int_{0}^{+\infty} |U_\delta(x) - u^+| \, dx < \infty.
\]

We now have that, for every \( \delta, \epsilon > 0 \), the function

\[
    u^{\epsilon, \delta}(t, x) = U_\delta(x/\epsilon)
\]

provides a traveling profile solution to the Cauchy problem

\[
    u_t + \left( f_\delta(a, u) \right)_x = \epsilon u_{xx}, \quad u(0, x) = U_\delta(x/\epsilon).
\]

Next, we observe that for every \( \epsilon > 0 \) the evolution equation

\[
    u_t + f(a, u)_x = \epsilon u_{xx}
\]

generates a contractive semigroup w.r.t. the \( L^1 \) distance. Denote by \( t \mapsto u^\epsilon(t) = S^\epsilon_t \tilde{u} \) the solution to (2.30) with initial data \( u(0) = \tilde{u} \). If \( t \mapsto u(t) \) is any approximate solution, with the same initial data \( u(0) = \tilde{u} \), then for every \( \tau > 0 \) we have the error estimate

\[
    \| u(t) - u^\epsilon(t) \|_{L^1(\mathbb{R})} \leq \int_{0}^{\tau} \left( \lim_{h \to 0^+} \frac{1}{h} \| u(t + h) - S^\epsilon_h u(t) \| \right) \, dt.
\]
Regarding $u(t, x) \equiv u^\varepsilon(\cdot, x)$ as an approximation of (2.30), defining $u^\varepsilon(t, x)$ as the solution to (2.30) with initial data $u^\varepsilon(0, x) = u^{\varepsilon}(0, x)$, the formula (2.31) leads to the following error estimate

$$
\int |u^\varepsilon(\tau, x) - u^{\varepsilon}(\tau, x)| \, dx \\
\leq \int_0^\tau \int \left| \left[ f_a(u^\varepsilon(t, x)) - f_a(u^{\varepsilon}(t, x), \cdot) \right] \right| \, dx \, dt \\
= \int_0^\tau \int \delta \left| \left( u^+ - u^{\varepsilon}(t, x) \right) \left( u^{\varepsilon}(t, x) - u^- \right) \right| \, dx \, dt \\
\leq 2\delta \left( u^+ - u^- \right) \int_0^\tau \int |u^{\varepsilon}(t, x)| \, dx \, dt \\
= 2\delta \left( u^+ - u^- \right) \cdot \tau \left( u^+ - u^- \right). 
$$

(2.32)

Combining (2.32) with (2.27), we have, for every $\tau > 0$,

$$
\int |u^\varepsilon(\tau, x) - \tilde{u}(x)| \, dx \\
\leq \int |u^\varepsilon(\tau, x) - u^{\varepsilon}(\tau, x)| \, dx + \int |u^{\varepsilon}(\tau, x) - \tilde{u}(x)| \, dx \\
\leq 2\tau\delta \left( u^+ - u^- \right)^2 + \epsilon \|U_\delta - \tilde{u}\|_{L^1(\mathbb{R})}. 
$$

(2.33)

Finally, we choose $\delta = \delta(\epsilon)$ such that

$$
\lim_{\epsilon \to 0} \delta(\epsilon) = 0, \quad \lim_{\epsilon \to 0} \left\|U_\delta(\cdot) - \tilde{u}\right\|_{L^1(\mathbb{R})} = 0. 
$$

(2.34)

This yields a family of solutions $u^\varepsilon(\cdot, \cdot) = S^\varepsilon U_\delta(\cdot)$ of (2.7), for which (2.11) holds.

Case (b), where we must choose $u^* = u^-$. The approach here is very similar to that of Case (a). For any $\delta > 0$, we define the modified flux functions

$$
f_\delta(a, u) = f(a, u) + \delta \left( a^+ - a^- \right), \\
f_\delta^+(a, u) = f^+(a, u) + \delta \left( a^+ - a^- \right). 
$$

Let $U_\delta$ be the solution to the ODE

\[
\begin{cases}
U'(x) = U(x) \cdot \left[ f_\delta^+(U(x)) - f \right], & (x > 0), \\
U(x) = u^-, & (x \leq 0).
\end{cases}
\]

With this modified flux, the stricter assumptions in (2.17) hold, and we also have for every $\tau > 0$

$$
\int |u^\varepsilon(\tau, x) - u^{\varepsilon}(\tau, x)| \, dx \\
\leq \delta \int_0^\tau \int |u^+ - u^{\varepsilon}(t, y)| \, dx \, dt \leq \delta \tau \left( u^+ - u^- \right). 
$$
This leads to the estimate

\[
\int |u^\varepsilon(t, x) - \tilde{u}(x)| \, dx \leq \delta \tau (u^+ - u^-) + \varepsilon \|U_\delta - \tilde{u}\|_{L^1(\mathbb{R})}.
\]

The rest follows.

Case (c), where we must choose \( u^* = u^+ \), is completely similar to Case (b).

4. The equivalence (III) \( \iff \) (II) is straightforward. Indeed, assume \( u^- < u^+ \), then

\[
G^\circ(u^-; u^+) = f^- (u^-) \quad \text{iff} \quad f^- (w) \geq f^- (u^-) \quad \forall w \in [u^-, u^+],
\]

\[
G^\circ(u^+; u^+) = f^+ (u^+) \quad \text{iff} \quad f^+ (w) \geq f^+ (u^+) \quad \forall w \in [u^+, u^+] .
\]

Hence (III) \( \iff \) (II). A completely similar argument shows the equivalency for the case \( u^- > u^+ \).

This proves the most part of Theorem 2.1, leaving only the implication (I) \( \Rightarrow \) (III), which will be established after proving Theorem 3.1.

Remark 2. Condition (III) can be used to construct the unique solution for the Riemann problem. After having constructed the functions \( G^\circ, G^\circ \), one can take the unique minimum path to determine \( u^-, u^+ \). See Figure 3 for an example. See also [10].

3. The Riemann Solver by Vanishing Viscosity

The partial result (III) \( \iff \) (II) \( \Rightarrow \) (I) in Theorem 2.1 motivates a Riemann solver for (1.6)-(1.7) with Riemann data (1.8), as commented in Remark 2. This Riemann solver, described in the next Theorem, generates solutions which are the vanishing viscosity limit of

\[
u_t + f(a(x), u)_x = \varepsilon u_{xx}, \quad u(0, x) = \begin{cases} u^L, & \text{if } x < 0, \\ u^R, & \text{if } x > 0, \end{cases}
\]

as \( \varepsilon \to 0 \).
Theorem 3.1. (Vanishing viscosity solution to the Riemann problem) Given a left and right states \((u^L, u^R)\), let \(G^\sharp (u^*; u^L)\), \(G^\flat (u^*; u^R)\) be defined as in \((2.8)-(2.9)\), and let \(\tilde{f}\) be the unique value such that
\[
\tilde{f} = G^\sharp (u^*; u^L) = G^\flat (u^*; u^R)
\]
for some \(u^*\).

We define the trace \(u^-, u^+\) of \(u\) along \(x = 0\) as follows:
\[
u^- = \arg \min \left\{ \left\| u - u^L \right\| ; f^-(u) = \tilde{f} \right\}, \tag{3.3}
\]
\[
u^+ = \arg \min \left\{ \left\| u - u^R \right\| ; f^+(u) = \tilde{f} \right\}. \tag{3.4}
\]
We call the path \((u^-, u^+)\) the minimum path connecting fluxes \(f^-, f^+\) with data \((u^L, u^R)\).

Then, the vanishing viscosity solution \(u(t, x)\) of \((1.6)-(1.8)\) is obtained by piecing together the solutions to
\[
u^+ \rightleftharpoons f^+(u)_x = 0, \quad u(0, x) = \begin{cases} u^L, & \text{if } x < 0, \\ u^-, & \text{if } x > 0, \end{cases} \tag{3.5}
\]
for \(x < 0\), and the solution to
\[
u^- \rightleftharpoons f^-(u)_x = 0, \quad u(0, x) = \begin{cases} u^+, & \text{if } x < 0, \\ u^R, & \text{if } x > 0, \end{cases} \tag{3.6}
\]
for \(x > 0\). In particular, for every \(t > 0\) we have
\[
\lim_{x \to 0^-} u(t, x) = u^-, \quad \lim_{x \to 0^+} u(t, x) = u^+, \tag{3.7}
\]
and
\[
\lim_{t \to 0} \| u^\epsilon(t, \cdot) - u(t, \cdot) \|_{L^1(\mathbb{R})} = 0 \tag{3.8}
\]
uniformly on every bounded time interval \([0, T]\), where \(u^\epsilon\) is a solution to the viscous equation (3.1).

Proof. The proof takes several steps.

(1). By the definitions of \(u^-\) and \(u^+\) in (3.3)-(3.4), the followings hold.

(i) The entropy-admissible solution \(v(t, x) = U^-(x/t)\) to the Riemann problem (3.5) contains only waves of speed smaller than 0. Indeed
\[
\lim_{x \to 0^-} U^-(x) = u^-, \quad \text{and} \quad U^-(x) = u^- \quad \forall x \geq 0. \tag{3.9}
\]
(ii) The entropy-admissible solution \( u(t, x) \) to the Riemann problem (3.6) contains only waves of speed larger than 0. Indeed

\[
\lim_{x \to 0^+} U^+(x) = u^+, \quad \text{and} \quad U^-(x) = u^- \quad \forall x \leq 0.
\]

(iii) The left and right states of \( u \) where \( a(x) \) has a jump, denoted as \( u^- \) and \( u^+ \), satisfy the condition (III) in Theorem 2.1.

Therefore there exist three families of viscous approximations \( v^\varepsilon, w^\varepsilon, z^\varepsilon \), satisfying

\[
\begin{align*}
(v^\varepsilon_t + f^-(v^\varepsilon))_x &= \varepsilon v^\varepsilon_{xx}, \\
(w^\varepsilon_t + f^+(w^\varepsilon))_x &= \varepsilon w^\varepsilon_{xx}, \\
(z^\varepsilon_t + f(a(x), z^\varepsilon))_x &= \varepsilon z^\varepsilon_{xx},
\end{align*}
\]

where \( a(x) \) is given in (1.7). Moreover, as \( \varepsilon \to 0 \) one has

\[
\|v^\varepsilon(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \to 0, \quad \|w^\varepsilon(t, \cdot) - w(t, \cdot)\|_{L^1(\mathbb{R})} \to 0,
\]

and

\[
\int_{-\infty}^{0} |z^\varepsilon(t, x) - u^-| \, dx + \int_{0}^{\infty} |z^\varepsilon(t, x) - u^+| \, dx \to 0
\]

uniformly as \( t \) ranges on bounded intervals. The functions \( v^\varepsilon \) and \( w^\varepsilon \) can be uniquely determined by imposing the initial data

\[
v^\varepsilon(0, x) = \begin{cases} u^L, & (x < 0), \\ u^-, & (x > 0) \end{cases}, \quad w^\varepsilon(0, x) = \begin{cases} u^+, & (x < 0), \\ u^R, & (x > 0) \end{cases}
\]

while \( z^\varepsilon \) is obtained by the construction in Step 3 of the proof for Theorem 2.1.

We note that all function \( v^\varepsilon, w^\varepsilon \) and \( z^\varepsilon \) are monotone, either non-increasing or non-decreasing w.r.t. the variable \( x \). Thanks to this fact, we conclude that the \( L^1 \) convergence in (3.12) implies pointwise convergence, at every point \( (t, x) \) where \( v, w \) are continuous.

(2). For any given \( T, \delta > 0 \), define the domains

\[
\Omega_\delta = \{(t, x) ; \quad t \in [\delta^{1/4}, T + 1], \quad |x| \leq \delta^{1/2}\},
\]

\[
\Omega'_\delta = \{(t, x) ; \quad t \in [\delta^{1/4}, T + 1], \quad |x| \leq 2\delta^{1/2}\}.
\]

We recall that, by (3.9)-(3.10), the functions \( v \) and \( w \) are continuous at all points \((t, 0)\) with \( t > 0 \). For any \( \delta > 0 \), we can thus find \( \varepsilon = \varepsilon(\delta) > 0 \) small enough such that

\[
\sup_{(t, x)\in \Omega_\delta} |v^\varepsilon(t, x) - u^-| \leq \delta + \sup_{(t, x)\in \Omega'_\delta} |v(t, x) - u^-|,
\]

\[
\sup_{(t, x)\in \Omega_\delta} |w^\varepsilon(t, x) - u^+| \leq \delta + \sup_{(t, x)\in \Omega'_\delta} |w(t, x) - u^+|,
\]
and

\[
\sup \left\{ \left| z'(t, x) - u^- \right| : t \in [0, T], \ x \leq -\delta \right\} \leq \delta, \tag{3.18}
\]

\[
\sup \left\{ \left| z'(t, x) - u^+ \right| : t \in [0, T], \ x \geq \delta \right\} \leq \delta. \tag{3.19}
\]

Without loss of generality we can assume that the map \( \delta \mapsto \epsilon(\delta) \) is continuous, strictly increasing, and satisfies \( \epsilon(\delta) \in [0, \delta^2] \). Its inverse \( \epsilon \mapsto \delta(\epsilon) \) is thus well defined and satisfies

\[
\delta(\epsilon) \geq \sqrt{\epsilon}, \quad \lim_{\epsilon \to 0} \delta(\epsilon) = 0. \tag{3.20}
\]

(3). To construct a family of vanishing viscosity solutions to the Riemann problem (3.1), we need to patch together the three solutions \( v, w, z \), on different domains.

To remove the discontinuity of \( v', w' \) at the point \((0, 0)\), we perform a time shift and define

\[
\begin{align*}
\tilde{v}'(t, x) &= v'(t + \delta(\epsilon), x), \\
\tilde{w}'(t, x) &= w'(t + \delta(\epsilon), x).
\end{align*} \tag{3.21}
\]

Next, let \( \varphi : \mathbb{R} \mapsto [0, 1] \) be a smooth, non-decreasing function such that

\[
\varphi(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ 1 & \text{if } y \geq 1. \end{cases} \tag{3.22}
\]

For any \( \epsilon > 0 \), set \( \delta = \delta(\epsilon) \) and define the interpolated function

\[
\tilde{u}'^\delta(t, x) = \left[ \varphi \left( \frac{\sqrt{\delta} + x}{\delta} \right) + \varphi \left( \frac{\sqrt{\delta} - x}{\delta} \right) - 1 \right] z'(t, x) + \left[ 1 - \varphi \left( \frac{\sqrt{\delta} + x}{\delta} \right) \right] \tilde{v}'(t, x) + \left[ 1 - \varphi \left( \frac{\sqrt{\delta} - x}{\delta} \right) \right] \tilde{w}'(t, x). \tag{3.23}
\]

For \( \delta < 1 \), the interpolated function satisfies

\[
\tilde{u}'^\delta(\epsilon)(t, x) = \begin{cases} \tilde{v}'(t, x), & \text{if } x \leq -\sqrt{\delta}, \\ z'(t, x), & \text{if } -\sqrt{\delta} + \delta \leq x \leq \sqrt{\delta} - \delta, \\ \tilde{w}'(t, x), & \text{if } x \geq \sqrt{\delta}. \end{cases}
\]

In the region \(-\sqrt{\delta} \leq x \leq -\sqrt{\delta} + \delta \), it interpolates between \( \tilde{v}' \) and \( z' \), while in the region \( \sqrt{\delta} - \delta \leq x \leq \sqrt{\delta} \), it interpolates between \( z' \) and \( \tilde{w}' \).

We call \( u = u(t, x) \) the solution to the Riemann problem (1.6)-(1.8) obtained by the Riemann solver described in this Theorem, i.e., by piecing together the two solutions at (3.5)-(3.6). Then, for any choice of \( \delta = \delta(\epsilon) \) such that \( \lim_{\epsilon \to 0^+} \delta(\epsilon) = 0 \), we have with \( u'^{\delta(\epsilon)} = \tilde{u}'^{\delta(\epsilon)} \)

\[
\lim_{\epsilon \to 0^+} \left\| u'^{\delta(\epsilon)}(t, \cdot) - u(t, \cdot) \right\|_{L^1(\mathbb{R})} = 0 \tag{3.24}
\]
uniformly for \( t \) in bounded set.

(4) Since each \( u^{\varepsilon, \delta(e)} \) is only an approximate solution to the viscous Cauchy problem
\[
u_t + f(a(x), u)_x = \varepsilon u_{xx}, \quad \text{with} \quad u(0, x) = u^{\varepsilon, \delta(e)}(0, x),
\]
it remains to prove that the exact solution \( u^\varepsilon \) of (3.25) satisfies
\[
\lim_{\varepsilon \to 0^+} \left\| u^\varepsilon(t, \cdot) - u^{\varepsilon, \delta(e)}(t, \cdot) \right\|_{L^1(\mathbb{R})} = 0
\]
uniformly for \( t \) in bounded sets.

Indeed, observe that, for any \( \tau > 0 \) the error estimate (2.31) yields
\[
\left\| u^\varepsilon(\tau, \cdot) - u^{\varepsilon, \delta(e)}(\tau, \cdot) \right\|_{L^1(\mathbb{R})} \leq \int_0^\tau \int_{-\infty}^{+\infty} \left| u^{\varepsilon, \delta(e)}(t, x) - \varepsilon u^{\varepsilon, \delta(e)}_{xx} \right| \, dx \, dt + \int_0^\tau \int_{-\infty}^{+\infty} \left| u^{\varepsilon, \delta(e)}(t, x) - \varepsilon u^{\varepsilon, \delta(e)}_{xx} \right| \, dx \, dt.
\]
By construction, the integrand on the right hand side of (3.27) is nonzero only along the two strips where the interpolation takes place, namely for
\[
x \in \left[ -\sqrt{\delta}, -\sqrt{\delta} + \delta \right] \cup \left[ \sqrt{\delta} - \delta, \sqrt{\delta} \right].
\]
Fix \( t > 0 \), we consider the second interval
\[
I_\delta \doteq \left[ \sqrt{\delta} - \delta, \sqrt{\delta} \right]
\]
where \( u^{\varepsilon, \delta(e)} \) interpolates between \( z^\varepsilon \) and \( \tilde{u}^\varepsilon \). We define
\[
\varphi^\varepsilon(x) \doteq \varphi\left( \frac{\sqrt{\delta} - x}{\delta} \right).
\]
Recalling (3.23), for \( x \in I_\delta \) we have
\[
u^{\varepsilon, \delta(e)}(t, x) = \varphi^\varepsilon(x) z^\varepsilon(t, x) + \left(1 - \varphi^\varepsilon(x)\right) \tilde{u}^\varepsilon(t, x).
\]
We compute
\[
u^{\varepsilon, \delta(e)}(t, x) + f(a^\varepsilon, u^{\varepsilon, \delta(e)})_x - \varepsilon u^{\varepsilon, \delta(e)}_{xx}
\]
\[
= \left[ \varphi^\varepsilon z^\varepsilon + (1 - \varphi^\varepsilon) \tilde{u}^\varepsilon \right]
\]
\[
+ \left( f(a^\varepsilon, u^{\varepsilon, \delta(e)}) \right)_x + \left( \left( \varphi^\varepsilon z^\varepsilon \right)_x + \left( (1 - \varphi^\varepsilon) \tilde{u}^\varepsilon \right)_x \right)
\]
\[
- \varepsilon \left[ \left( \varphi^\varepsilon z^\varepsilon \right)_{xx} + \left( (1 - \varphi^\varepsilon) \tilde{u}^\varepsilon \right)_{xx} \right]
\]
\[
= A_1 + A_2 + A_3.
\]
where

\[
A_1 = \varphi^\delta z^\epsilon_t + (f^+)'(u^{\delta,\epsilon(t)}) \cdot \varphi^\delta z^\epsilon_{xx},
\]

\[
A_2 = (1 - \varphi^\delta) \tilde{u}^\epsilon_t + (f^+)'(u^{\delta,\epsilon(t)}) \cdot (1 - \varphi^\delta) \tilde{u}^\epsilon_x - \varepsilon (1 - \varphi^\delta) \tilde{u}^\epsilon_x,
\]

\[
A_3 = (f^+)'(u^{\delta,\epsilon(t)}) \varphi^\delta (z^\epsilon - \tilde{u}^\epsilon) - \varepsilon \varphi^\delta (z^\epsilon - \tilde{u}^\epsilon) - 2\varepsilon \varphi^\delta (z^\epsilon - \tilde{u}^\epsilon).
\]

We have the estimates

\[
|A_1| = \varphi^\delta \left| (f^+)'(u^{\delta,\epsilon(t)}) - (f^+)'(z^\epsilon) \right| |z^\epsilon_x|
\leq \varphi^\delta (1 - \varphi^\delta) \cdot \left\| (f^+)' \right\|_{L^\infty} \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)} \cdot |z^\epsilon_x|,
\]

(3.29)

and

\[
|A_2| = (1 - \varphi^\delta) \left| (f^+)'(u^{\delta,\epsilon(t)}) - (f^+)'(\tilde{u}^\epsilon) \right| |\tilde{u}^\epsilon_x|
\leq \varphi^\delta (1 - \varphi^\delta) \cdot \left\| (f^+)' \right\|_{L^\infty} \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)} \cdot |\tilde{u}^\epsilon_x|.
\]

(3.30)

The last term can be estimated as

\[
|A_3| \leq \frac{1}{\delta} \left\| \varphi' \right\|_{L^\infty} \cdot \left\| (f^+)' \right\|_{L^\infty} \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)} + \frac{\varepsilon}{\delta^2} \left\| \varphi'' \right\|_{L^\infty} \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)} + \frac{2\varepsilon}{\delta} \left\| \varphi' \right\|_{L^\infty} \left\{ \| z^\epsilon_x \| + |\tilde{u}^\epsilon_x| \right\}.
\]

(3.31)

Combining the estimates (3.29)–(3.31) we obtain

\[
\int_{I_t} \left| u^{\delta,\epsilon(t)} + f(a^+, u^{\delta,\epsilon(t)}) - \varepsilon u^{\delta,\epsilon(t)} \right| dx
\leq \left\| (f^+)' \right\|_{L^\infty} \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)} \left( \| z^\epsilon_x \|_{L^1(I_t)} + \| \tilde{u}^\epsilon_x \|_{L^1(I_t)} \right)
+ \left( \left\| \varphi' \right\|_{L^\infty} \cdot \left\| (f^+)' \right\|_{L^\infty} + \frac{\varepsilon}{\delta^2} \left\| \varphi'' \right\|_{L^\infty} \right) \cdot \| z^\epsilon - \tilde{u}^\epsilon \|_{L^\infty(I_t)}
+ \frac{2\varepsilon}{\delta} \left\| \varphi' \right\|_{L^\infty} \left( \| z^\epsilon_x \|_{L^1(I_t)} + \| \tilde{u}^\epsilon_x \|_{L^1(I_t)} \right).
\]

(3.32)

Since \( \tilde{u}^\epsilon \) and \( z^\epsilon \) are all monotone functions of \( x \), for every fixed time \( t \geq 0 \) their total variation is computed simply by

\[
\| \tilde{u}^\epsilon_x \|_{L^1(R)} = \| u^R - u^+ \|, \quad \| z^\epsilon_x \|_{L^1(R)} = \| u^+ - u^- \|.
\]

Thanks to the estimates (3.16)–(3.19) we have

\[
\| z^\epsilon(t, \cdot) - \tilde{u}^\epsilon(t, \cdot) \|_{L^\infty(I_{0^+})} \to 0
\]
as \( \varepsilon \to 0 \), uniformly for \( t \in [0, T] \). Moreover, (3.20) implies \( \varepsilon / \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). We thus conclude that the right hand side of (3.32) approaches zero as \( \varepsilon \to 0 \), uniformly for \( t \in [0, T] \).

Of course, an entirely similar estimate is valid for the integral over the interval

\[
[-\sqrt{\delta} - \sqrt{\delta} + \delta].
\]

From (3.27) we thus conclude

\[
\lim_{\varepsilon \to 0} \| u'(t) - u(\tau) \|_{L^1(\mathbb{R})} \leq \lim_{\varepsilon \to 0} \| u'(\tau) - \tilde{u}'\delta(\varepsilon)(\tau) \|_{L^1(\mathbb{R})} + \lim_{\varepsilon \to 0} \| \tilde{u}'\delta(\varepsilon)(\tau) - u(\tau) \|_{L^1(\mathbb{R})} = 0,
\]

proving that the solution \( u \) of the Riemann problem (1.6)-(1.8) described in this Theorem is indeed a limit of vanishing viscosity approximations of the viscous model (3.1). This completes the proof of Theorem 3.1.

We now go back to Theorem 2.1, and complete the last part of the proof.

**Proof.** (of Theorem 2.1) We now prove the implication (I) \( \Rightarrow \) (III). Assuming condition (I), i.e., there exists a family of monotone viscous solution \( u'(t, x) \) of (2.7) such that (2.11) holds. A standard argument shows that the Rankine-Hugoniot condition (2.12) must hold. To prove (2.13)-(2.14), we argue with contradiction. Suppose that (III) fails. Then, by Theorem 3.1, we can construct a family of viscous solution \( \tilde{u}'(t, x) \) that converges in \( L^1 \) to the solution \( \tilde{u}(t, x) \) of the Riemann solver stated in Theorem 3.1.

The solution \( \tilde{u}(t, x) \) consists of a stationary jump at \( x = 0 \), and at least one left-going or right-going wave. Clearly, \( \| \tilde{u}(\cdot) - \tilde{u}(t, \cdot) \|_{L^1} = 0 \) for \( t > 0 \). Since the equation (2.7) generates a contractive semigroup, we must have for \( 0 < t < T \)

\[
\lim_{\varepsilon \to 0^+} \| u'(t, \cdot) - \tilde{u}'(t, \cdot) \|_{L^1} = \| \tilde{u}(\cdot) - \tilde{u}(t, \cdot) \|_{L^1} = 0,
\]

reaching a contradiction. This completes the proof for Theorem 2.1.

**4. Riemann problem for polymer flooding models**

We now go back to the polymer flooding models, and consider (1.2) while treating (1.1) as a special case with \( m(c) \) = constant. We consider (1.2) with the Riemann data

\[
s(0, x) = \begin{cases} s^L, & (x < 0) \\ s^R, & (x > 0) \end{cases}, \quad c(0, x) = \begin{cases} c^-, & (x < 0) \\ c^+, & (x > 0) \end{cases}.
\]

Consider the Lagrangian coordinate \( (\psi, \phi) \), defined in (1.3), and the associated polymer flooding system (1.5). In this coordinate, since \( m'' < 0 \), the solution for \( c \)
contains a single admissible jump, traveling with speed
\[ \sigma = \frac{m(c^-) - m(c^+)}{c^- - c^+}. \]

We consider the model where viscosity is only added for the hydrodynamics
\[
\begin{align*}
\frac{\partial}{\partial \psi} \left( \frac{1}{g(s,c)} \right) - \frac{\partial}{\partial \phi} \left( \frac{s + \sigma}{g(s,c)} \right) &= \varepsilon \frac{\partial^2}{\partial \phi^2} \left( \frac{1}{g(s,c)} \right), \\
\frac{\partial c}{\partial \psi} + \frac{\partial}{\partial \phi} m(c) &= 0.
\end{align*}
\]
(4.2)

With a simple coordinate shift \( \tilde{\phi} = \phi - \sigma \psi \), the \( c \)-jump will be stationary at \( \tilde{\phi} = 0 \), and the first equation in (4.2) becomes
\[
\begin{align*}
\frac{\partial}{\partial \psi} \left( \frac{1}{g(s,c)} \right) - \frac{\partial}{\partial \phi} \left( \frac{s + \sigma}{g(s,c)} \right) &= \varepsilon \frac{\partial^2}{\partial \phi^2} \left( \frac{1}{g(s,c)} \right),
\end{align*}
\]
(4.3)

We now assume \( s^L > 0, s^R > 0 \). For notational convenience, we denote the conserved quantity and the flux in (4.3) as
\[ W(s,c) = \frac{1}{g(s,c)}, \quad H : W \mapsto \frac{s + \sigma}{g(s,c)}. \]

In particular, the fluxes \( H \) at the left and right of the \( c \)-jump are denoted as
\[ H^-(s,c) \mapsto -\frac{s + \sigma}{g(s,c)}, \quad H^+(s,c) \mapsto -\frac{s + \sigma}{g(s,c)}. \]

Since \( s \mapsto g(s,c) \) is strictly increasing for any given \( c \), the mappings \( H^-, H^+ \) are well-defined. We can now rewrite (4.3) as
\[
W_{\psi} + H(W, \phi)_{\phi} = \varepsilon W_{\phi}, \quad H(W, \phi) = \begin{cases} 
H^-(W), & \text{if } \phi < 0, \\
H^+(W), & \text{if } \phi > 0,
\end{cases}
\]
(4.4)

which is associated with the Riemann data
\[ W(0, \phi) = \begin{cases} 
W^L = W(s^L, c^-), & \text{if } \phi < 0, \\
W^R = W(s^R, c^+), & \text{if } \phi > 0.
\end{cases} \]

By Theorem 3.1, as \( \varepsilon \to 0 \), the vanishing viscosity solution of (4.4) can be obtained by patching up solutions of two Riemann problems, where the traces
\[ W^- = W(\psi, 0), \quad W^+ = W(\psi, 0+) \]
are determined as the minimal path connecting the fluxes \( H^-, H^+ \) with data \( W^-, W^+ \).

We go back to the original system (1.2) in Eulerian coordinate. A Riemann solver was proposed in [18] for the case \( m(c) = \text{constant} \). One can easily extend this solver for the more general case with \( m'' > 0 \), which we present in our final Theorem. Furthermore, this Riemann solver is equivalent to the one in Lagrangian coordinate described above.
Theorem 4.1. Consider the model (1.2) with Riemann data (4.1) and \( s^L > 0, s^R > 0 \). Let \( (s(t, x), c(t, x)) \) be the solution of the following Riemann solver.

Define the flux functions

\[
\begin{align*}
    h(s, c) &= \frac{g(s, c)}{s + \sigma}, \\
    h^-(s) &= \frac{g^-(s)}{s + \sigma}, \\
    h^+(s) &= \frac{g^+(s)}{s + \sigma}.
\end{align*}
\]

Let \( (s^-, s^+) \) be the minimum path connecting the fluxes \( h^-, h^+ \) with data \( (s^L, s^R) \) as defined in Theorem 3.1, and let

\[
\sigma = \frac{g^+(s^+)}{s^+ + \sigma} = \frac{g^-(s^-)}{s^- + \sigma}.
\]

Then, \( c(t, x) \) contains a single jump traveling with speed \( \sigma \), and \( s(t, x) \) is obtained by piecing together the solutions to

\[
\begin{align*}
    s_t + g^-(s) &= 0, & s(0, x) &= \begin{cases} s^L, & \text{if } x < 0, \\
    s^-, & \text{if } x > 0, \end{cases}
\end{align*}
\]

for \( x < \sigma t \), and the solution to

\[
\begin{align*}
    s_t + g^+(s) &= 0, & s(0, x) &= \begin{cases} s^+, & \text{if } x < 0, \\
    s^R, & \text{if } x > 0, \end{cases}
\end{align*}
\]

for \( x > \sigma t \).

Then, this Riemann solver solution is the vanishing viscosity limit of (4.2), as \( \epsilon \to 0^+ \), where the convergence is in \( L^1 \)-norm in \( x \), and uniformly for \( t \) in bounded sets.

Proof. It suffices to show that condition (II) in Theorem 2.1 is equivalent in these two Riemann solvers.

By Rankine-Hugoniot condition, we have

\[
\begin{align*}
    H^-(W^-) &= H^+(W^+) = \tilde{H}, \\
    h^-(s^-) &= h^+(s^+) = \tilde{h}, \\
    \tilde{H} &= -\frac{1}{\tilde{h}}.
\end{align*}
\]

Consider the case \( s^- < s^+ \). The above condition gives

\[
\frac{s^- + \sigma}{g^-(s^-)} = \frac{s^+ + \sigma}{g^+(s^+)} \quad \Rightarrow \quad \frac{1}{g^-(s^-)} > \frac{1}{g^+(s^+)}. \quad \Rightarrow \quad W^- > W^+.
\]

Then, for the system in the Eulerian coordinate, (2.13) in condition (II) holds, i.e., there exists an \( s^a \) between \( s^-, s^+ \), such that

\[
\begin{align*}
    \begin{cases}
        h^+(s) \geq \tilde{h} & \text{for } s \in [s^-, s^a], \\
        h^-(s) \geq \tilde{h} & \text{for } s \in [s^a, s^+].
    \end{cases}
\end{align*}
\]
For the system in the Lagrangian coordinate, (2.14) in condition (II) holds, i.e., there exists a $W^*$ between $W^+, W^-$ such that

\[
\begin{align*}
H^+(W) &\leq \bar{H} \quad \text{for } W \in [W^+, W^*], \\
H^-(W) &\leq \bar{H} \quad \text{for } W \in [W^*, W^-].
\end{align*}
\] (4.6)

Observe that, for any fixed $c$, $s \mapsto W$ is strictly decreasing. Thus, it suffices to show that the derivatives

\[
\frac{\partial h}{\partial s}(\tilde{s}, c), \quad \frac{\partial H}{\partial W} (\tilde{W}, c), \quad \text{where } \tilde{W} = W(\tilde{s}, c)
\] (4.7)

have opposite signs.

Indeed, by the definition of $H, h$, for a given $c$, both mappings $W \mapsto H$ and $s \mapsto h$ have a unique maximum. Let $\tilde{s}$ be the point where $s \mapsto h(s, c)$ reaches the maximum values. Then, the maximum value of the mapping $W \mapsto H(W, c)$ is reached at $\tilde{W} = W(\tilde{s}, c)$. Furthermore, since $s \mapsto W(s, c)$ is strictly decreasing, the derivatives in (4.7) have opposite signs. See Figure 4 for an illustration. This completes the proof for Theorem 4.1.

Figure 4. Plots of the mapping $s \mapsto h(s, c)$ and $W \mapsto H(W, c)$. The colors show the correspondence of the two graphs where $W = W(s, c)$ for the same $s$ values.

**Remark 3.** The Riemann solver in Theorem 4.1, adapted to the polymer flooding model by setting $m(c) = \text{constant}$, is used in [18] in a front tracking approximation, which generates unique entropy solutions for (1.1), even with the effect of gravitation. However, for the gravitation model of (1.1), the flux $g$ is changed into

\[
\mathcal{G}(s, c) = g(s, c) \left(1 - K_g \dot{\lambda}(s, c)\right),
\]

where the term $K_g \dot{\lambda}(s, c)$ represents the effect of the gravitation force, see [6, 18]. Here, $\mathcal{G}(s, c)$ could be 0 for some $(s, c)$ where $s > 0, c > 0$. At that point, the Lagrangian coordinate $(\psi, \phi)$ in (1.3) is no longer valid. However, using another Lagrangian coordinate $(\tau, \xi)$ defined as (see [21])

\[
\begin{align*}
\xi_x &= s, \quad \xi_t = -\mathcal{G}, \quad \tau = t.
\end{align*}
\]
the system \((1.1)\) becomes
\[
\begin{cases}
\left( \frac{1}{s} \right)_t - \left( \frac{\mathcal{G}(s, c)}{s} \right)_x = 0, \\
c_t = 0.
\end{cases}
\]

Theorem 4.1 can be applied here with very little modifications.

References


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