Recurrence, Transitivity, and Minimality

In the problems 1-4 and 6∗, \( X \) is a non-empty compact set in \( \mathbb{R}^k \) (or a compact metric space). In problems 2 and 3, we do not assume that \( f \) is invertible.

1. Show that if \( f : X \to X \) is a contraction, then no point in \( X \), except for the fixed point, is positively recurrent.

Solution. Let \( x_* \) be the unique fixed point guaranteed by the Contraction Mapping Principle. Choose some \( x \in X \) such that \( x \neq x_* \) and \( \varepsilon = d(x, x_*)/2 \). The contraction mapping principle tells us that \( f^n(x) \to x_* \) as \( n \to \infty \). In particular, we can find \( N \) such that if \( n \geq N \), then \( d(f^n(x), x_*) < \varepsilon \). But this implies that \( d(f^n(x), x) = d(x, x_*) - d(f^n(x), x_*) = \varepsilon \). Thus there are only finitely many \( n \) such that \( d(f^n(x), x) < \varepsilon \) and hence \( x \) is not positively recurrent. □

2. Let \( f : X \to X \) be a continuous map. Prove that if a point \( x \in X \) is positively recurrent, then \( f(x) \) is also positively recurrent.

Solution. Suppose that \( x \) is positively recurrent. Then there exists a sequence \( n_k \) such that \( n_k \to \infty \) and \( f^{n_k}(x) \to x \). But by continuity, we then get that
\[
f^{n_k}(f(x)) = f(f^{n_k}(x)) \to f(x),
\]
so \( f(x) \) is also positively recurrent. □

3. Suppose that \( X \) does not contain isolated points and \( f : X \to X \) is a continuous map. Show that if the positive semiorbit of \( x \in X \) is dense in \( X \), then \( x \) is positively recurrent.

Solution. Let \( \varepsilon > 0 \) and choose \( y \in B_{\varepsilon/2}(x) \) that is not \( x \) (we can do so because \( x \) is not isolated). Let \( \varepsilon' = d(x, y) < \varepsilon/2 \). Since the orbit of \( x \) is dense, there exists \( n \in \mathbb{N} \) such that \( d(f^n(x), y) < \varepsilon' \). But then:
\[
d(f^n(x), x) \leq d(f^n(x), y) + d(y, x) < \varepsilon' + \varepsilon/2 < \varepsilon.
\]
Thus, \( x \) is positively recurrent. □

4. Suppose that \( X \) contains infinitely many points, \( f : X \to X \) is a homeomorphism, and \( f : X \to X \) is minimal. What can you say about the periodic points of \( f \)? Justify your answer.

Solution. We claim that \( f \) has no periodic points. If \( x \) were a periodic point, then the orbit of \( x \) would be a non-empty invariant set, which is closed because it is finite. By minimality, the orbit would then be the entire space \( X \), which is a contradiction to the existence of infinitely many points. □
5. Let $\mu$ be the $k$-dimensional Lebesgue measure, let $X \subseteq \mathbb{R}^k$ be a set of finite measure, let $f : X \to X$ be a measure-preserving map, and let $A \subseteq X$ be a set of positive measure. Prove that there exists $n \in \mathbb{N}$ such that $\mu(f^{-n}(A) \cap A) > 0$. Give two proofs:
(a) Do not use the recurrence theorem proven in class.
(b) Deduce this result from the recurrence theorem proven in class.

Solutions.
(a) We prove the statement by contradiction. Suppose that $\mu(f^{-n}(A) \cap A) = 0$ for all $n \in \mathbb{N}$. It follows that $\mu(f^{-m}(A) \cap f^{-m}(A)) = 0$ for all $m, n \in \mathbb{N}$ such that $m \neq n$. Indeed, without loss of generality, $m < n$, and since $f$ is measure-preserving we have
$$
\mu(f^{-n}(A) \cap f^{-m}(A)) = \mu(f^{-m}(f^{-(n-m)}(A) \cap A)) = \mu(f^{-(n-m)}(A) \cap A) = 0.
$$

To obtain pairwise disjoint sets, we remove the intersections: for each $n \in \mathbb{N}$ we consider
$$
B_n = f^{-n}(A) \setminus \left( \bigcup_{m \neq n} f^{-n}(A) \cap f^{-m}(A) \right).
$$

The measure of the union in the above formula is at most the sum of the measures of the sets $f^{-n}(A) \cap f^{-m}(A)$, $m \neq n$, and these measures are 0. Hence the measure of the union is 0, and $\mu(B_n) = \mu(f^{-n}(A)) = \mu(A) > 0$. Since the sets $B_n$ are pairwise disjoint, it follows that
$$
\mu \left( \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(A) = \infty.
$$

But $\bigcup_{n=1}^{\infty} B_n \subseteq X$ and $\mu(X) < \infty$, a contradiction. Thus $\mu(f^{-n}(A) \cap A) > 0$ for some $n$.

(b) Let $C_n = f^{-n}(A) \cap A$ and $C = \cup_{n \in \mathbb{N}} C_n$.
Since $C_n = \{x \in A \mid f^n(x) \in A\}$, $C = \{x \in A \mid f^n(x) \in A \text{ for some } n \in \mathbb{N}\}$.

Clearly, $C$ contains the set $S = \{x \in A \mid f^n(x) \in A \text{ for infinitely many } n \in \mathbb{N}\}$.

Poincare Recurrence Theorem implies that $\mu(S) = \mu(A)$, and hence $\mu(C) = \mu(A) > 0$.
If $\mu(C_n) = 0$ for all $n \in \mathbb{N}$, then $\mu(C) \leq \sum_{n=1}^{\infty} \mu(C_n) = 0$. Thus, for some $n$, $\mu(C_n) > 0$.

Extra credit problem

6*. Suppose that $X$ contains infinitely many points. Is it possible for a contraction $f$ on $X$ to be transitive? Is it possible for $f$ to be minimal in the sense that:
if $A$ is a closed subset of $X$ such that $f(A) \subseteq A$, then $A = X$ or $A = \emptyset$?

Justify your answers.

Solution. It is possible for a contraction $f$ on $X$ to be transitive. We give an example:

Consider the set $X = \{0\} \cup \{1/2^n \mid n \in \mathbb{N}\} \subseteq [0, 1]$ and the map $f : X \to X$ given by $f(x) = x/2$. The set $X$ is compact since it is a closed subset of $[0, 1]$, and it is easy to see that $f$ is a contraction. The orbit of $1/2$ is dense in $X$, indeed $f^n(1/2) = 1/2^{n+1}$ and hence every $x \neq 0$ belongs to the orbit and $0 = \lim_{n \to \infty} f^n(1/2)$. Thus $f$ is transitive.

A contraction on a compact set $X$ containing more than one point cannot be minimal. Let $x_*$ be the fixed point for $f$ given by the Contraction Mapping Principle. Then $\{x_*\}$ is a closed non-empty invariant set, and $\{x_*\} \neq X$. 

□