Recurrence, Transitivity, and Minimality

In the problems 1-4 and 6*, \( X \) is a non-empty compact set in \( \mathbb{R}^k \) (or a compact metric space). In problems 2 and 3, we do not assume that \( f \) is invertible.

1. Show that if \( f : X \to X \) is a contraction, then no point in \( X \), except for the fixed point, is positively recurrent.

2. Let \( f : X \to X \) be a continuous map. Prove that if a point \( x \in X \) is positively recurrent, then \( f(x) \) is also positively recurrent.
   
   Note: It follows that the set of recurrent points is invariant under \( f \), i.e. 
   \[ f(R_+(f)) \subseteq R_+(f). \]

3. Suppose that \( X \) does not contain isolated points and \( f : X \to X \) is a continuous map. Show that if the positive semi-orbit of \( x \in X \) is dense in \( X \), then \( x \) is positively recurrent.

4. Suppose that \( X \) contains infinitely many points, \( f : X \to X \) is a homeomorphism, and \( f : X \to X \) is minimal. What can you say about the periodic points of \( f \)?
   Justify your answer.

5. Let \( \mu \) be the \( k \)-dimensional Lebesgue measure, let \( X \subseteq \mathbb{R}^k \) be a set of finite measure, let \( f : X \to X \) be a measure-preserving map, and let \( A \subseteq X \) be a set of positive measure.
   Prove that there exists \( n \in \mathbb{N} \) such that \( \mu(f^{-n}(A) \cap A) > 0 \). Give two proofs:
   (a) Do not use the recurrence theorem proven in class in your arguments.
   (b) Deduce this result from the recurrence theorem proven in class.

Extra credit problem

6*. Suppose that \( X \) contains infinitely many points. Is it possible for a contraction \( f \) on \( X \) to be transitive? Is it possible for \( f \) to be minimal in the sense that:
   if \( A \) is a closed subset of \( X \) such that \( f(A) \subseteq A \), then \( A = X \) or \( A = \emptyset \)?
   Justify your answers.