Monotonic maps of an interval and differential equations

1. Let $I = [a, b]$ and let $f : I \to I$ be a non-decreasing continuous function.
   Suppose that for some $x_0 \in (a, b)$, $f(x_0) = x_0$ and $|f'(x_0)| > 1$.

   Show that $x_0$ is a repelling fixed point for $f$, i.e. that there is $\delta > 0$ such that for each $x \neq x_0$ in $I$ with $d(x, x_0) < \delta$
   there exists $N$ such that $d(f^n(x), x_0) > \delta$ for all $n \geq N$.

   **Solution.** Choose any $\lambda \in (1, f'(x_0))$, and note that since $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) > \lambda$,
   there exists some $\delta > 0$ such that if $|x - x_0| < \delta$, then:

   \[
   \frac{f(x) - f(x_0)}{x - x_0} \geq \lambda \implies |f(x) - x_0| \geq \lambda |x - x_0|
   \]

   By induction, we see that if $\{f^k(x) : 0 \leq k \leq n - 1\} \subset B_\delta(x_0) = \{x : |x - x_0| < \delta\}$, then
   $|f^n(x) - x_0| \geq \lambda^n |x - x_0|$. But since the right hand size tends to $\infty$, we know that eventually,
   $|f^n(x) - x_0| > \delta$. Let $N$ be the smallest such number.

   We now claim that if $n \geq N$, then $f^n(x) \notin B_\delta(x_0)$. This follows easily from the non-decreasing property of the map. If $f : I \to I$ is non-decreasing and $x \in I$, then the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is either non-decreasing or non-increasing. Indeed, if $x < f(x)$ then
   $f^n(x) = f^n(f(x)) = f^n(f(x))$ and it follows by induction that $f^n(x) \leq f^{n+1}(x)$ for all $n \geq 0$.
   Similarly, if $f(x) \leq x$, then $f^{n+1}(x) \leq f^n(x)$ for all $n \geq 0$.

   Suppose that $x < x_0$. In this case, $f^n(x) \leq f^n(x_0) = x_0$ and hence $d(x, x_0) = x_0 - x$
   and $d(f(x), x_0) = x_0 - f(x)$. But we know that $x_0 - f(x) \geq \lambda(x_0 - x) \geq x_0 - x$, so
   $f(x) \leq x$. It follows that $f^{n+1}(x) \leq f^n(x)$ for all $n \geq 0$. Hence if $n \geq N$, $f^n(x) \leq f^N(x)$
   and $d(f^n(x), x_0) \geq d(f^N(x), x_0) > \delta$.

   An argument for $x > x_0$ is completely symmetric.

2. Let $I = [a, b]$ and let $f : I \to I$ be a continuous function.

   Prove that the set of fixed points of $f$ is closed.

   **Solution.** Let $\text{Fix}(f)$ denote the set of fixed points of $f$. Suppose $\{x_n\} \subset \text{Fix}(f)$ is any
   sequence which converges in $I$, say to a point $x$. We wish to show that $x \in \text{Fix}(f)$. But by the continuity of $f$:

   \[
   f(x) = f \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = x.
   \]

   Hence $x$ is also fixed by $f$. □
3. (a) Let \( a, b \in \mathbb{R}, a < b \). Give a formula for an increasing continuous function \( f : [a, b] \to [a, b] \) such that \( f(a) = a \), \( f(b) = b \), and \( f(x) \neq x \) for \( x \in (a, b) \).

(b) Let \( E \) be a closed non-empty set in \( \mathbb{R} \). Construct an increasing continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that \( E \) is the set of fixed points of \( f \).

**Solutions.**

(a) Consider the function \( f(x) = a + (x - a)^2/(b - a) \). One can easily check that \( f(a) = a \) and \( f(b) = b \). Furthermore, \( f'(x) = 2(x - a)/(b - a) > 0 \) on \([a, b]\), so \( f \) is increasing on \([a, b]\) and \( f([a, b]) = [a, b] \). Finally, \( f(x) = x \) has at most two solutions since it reduces to a quadratic equation, and we have already found them. So this function is an example of that described in the statement of the problem.

Another example would be \( f \) such that \( f(a) = a \), \( f(b) = b \), \( f(a + b/2) = a + b/(b - a) \), \( f \) is linear on \([a, a + b/2]\) and on \([a + b/2, b]\). It is easy to write an explicit formula for \( f \).

(b) For each \( x \in E \), we set \( f(x) = x \). Now we define \( f \) on the complement of \( E \).

Since \( E \) is closed, \( \mathbb{R} \setminus E = U \) is open. Then we know that \( U \) is a finite or countable union of disjoint open intervals: \( I_i = (\alpha_i, \beta_i) \), \( \alpha_i, \beta_i \in \mathbb{R} \), and possibly one or two open rays \((a, \infty)\) and \((-\infty, b)\). For each \( i \) we choose a continuous increasing function \( f_i : [\alpha_i, \beta_i] \to [\alpha_i, \beta_i] \) such that \( \text{Fix}(f_i) = \{\alpha_i, \beta_i\} \). Such a function exists by part (a). For \( x \in (\alpha_i, \beta_i) \), we set \( f(x) = f_i(x) \).

If \( \sup E = a < \infty \), the open ray \((a, \infty)\) is one of the disjoint open intervals in \( U \) and we define \( f_{[a, \infty)}(x) = (x + a)/2 \) on \([a, \infty)\). The function \( f_{[a, \infty)} \) is increasing and \( \text{Fix}(f_{[a, \infty)}) = a \). For \( x \in (a, \infty) \), we set \( f(x) = f_{[a, \infty)}(x) \).

Similarly, if \( \inf E = b > -\infty \), we define \( f_{(-\infty, b]}(x) = (x + b)/2 \) on \((-\infty, b] \) and set \( f(x) = f_{(-\infty, b]}(x) \) for \( x < b \).

Then it is clear that \( \text{Fix}(f) = E \) and \( f \) is increasing. The proof of continuity is delayed until Problem 6.
4. Let $I = [a, b]$ and let $f : I \rightarrow I$ be a continuous *non-increasing* function.
   (a) Describe the set of fixed points of $f$.
   (b) What are the possible prime periods for periodic points of $f$?

*Solutions.*

(a) We claim that $f$ can have exactly 1 fixed point. Indeed, consider the function
   $$g : [a, b] \rightarrow \mathbb{R} \text{ defined by } g(x) = f(x) - x.$$  
   Then $g(x)$ is decreasing, since if $x < y$, then $f(x) \geq f(y)$ so $f(x) - x > f(y) - y$. 
   Furthermore, $g(a) = f(a) - a \geq a - a = 0$. Similarly, $g(b) = f(b) - b \leq b - b = 0$. Thus $g$ has exactly one zero, and $f$ has exactly one fixed point.

(b) Let $c$ be the unique fixed point of $f$. We claim that $f$ maps $[a, c]$ into $[c, b]$ (and vice-versa). Indeed, if $x \leq c$, then $f(x) \geq f(c) = c$. But then $f^2$ is a function from $[a, c]$ to $[a, c]$, since $f^2([a, c]) \subset f([c, b]) \subset [a, c]$ (and similarly for $[c, b]$). Hence all prime periods other than that of $c$ must be even. The map $f^2$ is nondecreasing since if $x < y$, then $f(x) \geq f(y)$ and $f^2(x) \leq f^2(y)$. So it suffices to see the prime periods of $f^2$. But $f^2$ is a nondecreasing map of an interval, and all its periodic points are fixed points. So we get that:

   $f$ has a unique fixed point, and all other periodic points have prime period 2.

5. Give an example of a differential equation of the form $\dot{x} = g(x)$ for which a solution diverges to infinity in finite time.

*Solution.* Consider the equation $\frac{dx}{dt} = x^2$. A solution is:

   $$x(t) = \frac{1}{1 - t}$$

Indeed:

   $$\frac{dx}{dt} = -\left(\frac{-1}{(1-t)^2}\right) = \left(\frac{1}{1-t}\right)^2 = x(t)^2$$

*Remark:* The equation $\frac{dx}{dt} = x^2$ can be easily solved by separating the variables.
Remark. Note the similarity of the conclusions to the Contraction Mapping Theorem. This is because by the Intermediate Value Theorem, which contradicts the uniqueness of the solution with \( |x_0| = \alpha \) for all \( t \geq 0 \). Therefore, the ODE has a unique constant solution \( x(t) = c \) and that all other solutions satisfy \( \lim_{t \to \infty} x(t) = c \).

Remark. Consider a function \( g : \mathbb{R} \to \mathbb{R} \) such that for some \( M > m > 0 \), \( -M \leq g'(x) \leq -m \) for all \( x \). Show that the differential equation \( dx/dt = g(x) \) has a unique constant solution \( x(t) = c \) and that all other solutions satisfy \( \lim_{t \to \infty} x(t) = c \).

Solution. First, note that since \( g' \in [-M,-m] \), we know that \( g \) is Lipschitz continuous on \( \mathbb{R} \) (with constant \( M \)) and hence for any \( a, b \in \mathbb{R} \) the ODE has a unique solution satisfying \( x(a) = b \). Furthermore, since \( g \) is strictly decreasing, it has a unique zero, say \( g(x_0) = 0 \). That is, the ODE has a unique constant solution \( x(t) = x_0 \).

If \( x > x_0 \), then \( g(x) < 0 \) and if \( x < x_0 \), \( g(x) > 0 \). Hence if a solution begins above \( x_0 \), then it must begin by decreasing. It must continue decreasing unless at some time \( t \), \( x(t) \leq x_0 \).

Remark. Note the similarity of the conclusions to the Contraction Mapping Theorem. This is exactly the analogous condition for flows (Autonomous ODEs). Indeed, one can show that if \( g' \in [-M,-m] \) then:

\[
|x(t) - x_0| \leq |x(0) - x_0| e^{-mt}
\]

so we have exponential convergence. Also, if we define \( g(y) = x_y(1) \), where \( x_y \) is the solution of the ODE is such \( x_y(0) = y \), then \( g \) will be a contraction mapping. Try computing what happens for \( g(x) = -k(x - x_0) \)!