Monotonic maps of an interval and differential equations

1. Let \( I = [a, b] \) and let \( f : I \to I \) be a non-decreasing continuous function. Suppose that for some \( x_0 \in (a, b) \), \( f(x_0) = x_0 \) and \( |f'(x_0)| > 1 \).

   Show that \( x_0 \) is a repelling fixed point for \( f \), i.e. that there is \( \delta > 0 \) such that for each \( x \neq x_0 \) in \( I \) with \( d(x, x_0) < \delta \) there exists \( N \) such that \( d(f^n(x), x_0) > \delta \) for all \( n \geq N \).

2. Let \( I = [a, b] \) and let \( f : I \to I \) be a continuous function.

   Prove that the set of fixed points of \( f \) is closed.

3. (a) Let \( a, b \in \mathbb{R} \), \( a < b \). Give a formula for an increasing continuous function \( f : [a, b] \to [a, b] \) such that \( f(a) = a \), \( f(b) = b \), and \( f(x) \neq x \) for \( x \in (a, b) \).

   (b) Let \( E \) be a closed non-empty set in \( \mathbb{R} \). Construct an increasing continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that \( E \) is the set of fixed points of \( f \).

   *Hint: What can you say about the complement of \( E \)?

   Just describe the construction here. The proof of continuity is problem 6*.

4. Let \( I = [a, b] \) and let \( f : I \to I \) be a continuous non-increasing function.

   (a) Describe the set of fixed points of \( f \).

   (b) What are the possible prime periods for periodic points of \( f \)?

   *Hint: What can you say about \( f^2 \)?

   Justify your answers.

5. Give an example of a differential equation of the form \( \dot{x} = g(x) \) for which a solution diverges to infinity in finite time.

   Write down an equation and its solution. No explanations are required.

Extra credit problems:

6*. Prove that the function you constructed in Problem 3(b) is indeed continuous.

7*. Consider a function \( g : \mathbb{R} \to \mathbb{R} \) such that for some \( M > m > 0 \), \( -M \leq g'(x) \leq -m \) for all \( x \). Show that the differential equation \( dx/dt = g(x) \) has a unique constant solution \( x(t) = c \) and that all other solutions satisfy \( \lim_{t \to \infty} x(t) = c \).