Topological mixing, transitivity, and sensitive dependence

1. (a) Suppose that \( X \) and \( Y \) are compact metric spaces and \( f : X \to X \) and \( g : Y \to Y \) are continuous maps. Show that if \( f \) is topologically mixing and \( g \) is a factor of \( f \), then \( g \) is also topologically mixing.

(b) Use (a) to show that \( E_m : S^1 \to S^1, \ m \geq 2, \) is topologically mixing.

(c) Give another proof of topological mixing for \( E_m \).

Solutions.

(a) Let \( U \) and \( V \) be nonempty open subsets of \( Y \). We prove the claim directly, by showing that there exists an \( N \) such that if \( n \geq N \), \( g^n(U) \cap V \neq \emptyset \).

If \( h : X \to Y \) is the semiconjugacy between \( f \) and \( g \), we know that \( h \) is continuous and surjective, and hence \( U' = h^{-1}(U) \) and \( V' = h^{-1}(V) \) are open, nonempty subsets of \( X \). Since \( f \) is topologically mixing, there exists \( N \in \mathbb{N} \) such that if \( n \geq N \), \( f^n(U') \cap V' \neq \emptyset \). But note that \( h(U') = U \) and \( h(V') = V \), and the image of the intersection of two sets is contained in the intersection of their images, so if \( n \geq N \):

\[
g^n(U) \cap V = g^n(h(U')) \cap h(V') = h(f^n(U')) \cap h(V') \supseteq h(f^n(U') \cap V') \neq \emptyset
\]

and hence \( g^n(U) \cap V \neq \emptyset \).

(b) Note that \( E_m \) is a factor of \( \sigma : \Omega^+_m \to \Omega^+_m \), which is topologically mixing, so by part (a), it is topologically mixing as well.

(c) Let \( U \subset S^1 \) be a nonempty open set. Then \( U \) contains an interval of the form \([k/2^N, (k + 1)/2^N]\) for some \( N \in \mathbb{N} \) and \( 0 \leq k \leq 2^N - 1 \). Then \( f^N(U) = S^1 \), so if \( V \subset S^1 \) is a nonempty open set and \( n \geq N \), \( f^n(U) \cap V = V \neq \emptyset \).

\[ \square \]

2. Let \( X \) be a compact metric space containing at least two points and let \( f : X \to X \) be a contraction. Show that \( f \) is not topologically mixing.

Solution. Since \( f \) is a contraction, it has a unique fixed point \( x_* \). Since \( X \) has at least two points, we choose any \( x_0 \neq x_* \) and let \( \delta = d(x_0, x_*)/2 \), \( U = B_\delta(x_*) \) and \( V = B_\delta(x_0) \). Clearly, \( U \cap V = \emptyset \). Let \( n \in \mathbb{N} \). Since \( f \) is a \( \lambda \)-contraction for some \( 0 \leq \lambda < 1 \), for any \( y \in U \), \( d(f^n y, x_*) \leq \lambda^n \delta < \delta \). It follows that \( f^n(U) \subseteq U \) and hence \( f^n(U) \cap V = \emptyset \).

Thus, for every \( n \in \mathbb{N} \), \( f^n(U) \cap V = \emptyset \), which implies that \( f \) is not topologically mixing.

\[ \square \]
3. Let $X$ be a compact metric space without isolated points and let $f : X \to X$ be a transitive homeomorphism. Can there be
(a) Two disjoint non-empty closed invariant sets?
(b) A non-empty invariant open set $\neq X$?
(c) A non-empty invariant open set that is not dense in $X$? 
\textit{Justify your answers.}

\textbf{Solutions.}  
(1) Yes. Consider a hyperbolic toral automorphism $f$. Then $f$ has many periodic orbits, choose two of them. Each orbit is closed and invariant, and because we choose two distinct orbits, we get two nonempty closed invariant sets.

(2) Yes. Consider again a hyperbolic toral automorphism, and choose a periodic orbit $\mathcal{O}(x)$. Then $X \setminus \mathcal{O}(x)$ is open and invariant. An explicit example is $\mathbb{T}^2 \setminus \{0\}$.

(3) No. Let $U$ be a nonempty open invariant set. We will show that for every open ball $B$, $U \cap B \neq \emptyset$, which implies that $U$ is dense in $X$. Since $f$ is transitive, there exists $n$ such that $f^n(U) \cap B \neq \emptyset$. But by invariance, $f^n(U) = U$ for every $n$, so $U \cap B \neq \emptyset$.

\hfill $\Box$

4. Let $X$ be a compact metric space containing at least two points and let $f : X \to X$ be a topologically mixing continuous map. Show that any number $\Delta$ less than half the diameter of $X$ is a sensitivity constant.

\textbf{Solution.} Let $0 < \Delta < \text{diam}(X)/2$, $x \in X$ and $\varepsilon > 0$. Since $X$ is compact, the diameter is achieved, so there exists $x_1$ and $x_2$ with $d(x_1, x_2) = \text{diam}(X) > 2\Delta$.

Let $\delta < \text{diam}(X)/2 - \Delta$. Denote $B_\delta(x_1)$ by $V_1$ and $B_\delta(x_2)$ by $V_2$. Since $f$ is topologically mixing, there exists $N_1$ and $N_2$ such that if $n \geq N_i$, then $f^n(B_\varepsilon(x)) \cap V_i \neq \emptyset$. Then if $n \geq \max\{N_1, N_2\}$, we can find for $z_1, z_2 \in B_\varepsilon(x)$ such that $f^n(z_1) \in V_1$ and $f^n(z_2) \in V_2$. Now,

$$d(f^n(z_1), f^n(z_2)) \geq d(x_1, x_2) - d(f^n(z_1), x_1) - d(f^n(z_2), x_2) \geq \text{diam}(X) - 2\delta \geq 2\Delta.$$ 

Since $d(f^n(z_1), f^n(x)) + d(f^n(z_2), f^n(x)) \geq d(f^n(z_1), f^n(z_2))$, it follows that

$$d(f^n(z_1), f^n(x)) + d(f^n(z_2), f^n(x)) \geq 2\Delta,$$

and hence for some $i = 1, 2$, $d(f^n(z_i), f^n(x)) \geq \Delta$. 

\hfill $\Box$

\textbf{Extra credit problem}

5*. Let $X$ be a compact metric space, let $f : X \to X$ be a homeomorphism, and let $Y \subseteq X$ be a non-empty closed invariant set. Prove or give a counterexample:

(a) If $f : X \to X$ is transitive then the restriction of $f$ to $Y$, $f|_Y : Y \to Y$,

is also transitive.

(b) If $f : X \to X$ is topologically mixing then the restriction of $f$ to $Y$ is also topologically mixing.

\textbf{Solution.} We produce a counterexample for both claims. Let $Y$ be the union of two disjoint periodic orbits of a hyperbolic toral automorphism. Then the restriction of $f$ to $Y$ is not transitive. Since mixing implies transitivity, this example is also not mixing. $\Box$