MATH 497 B  Hw 10  Solutions

Topological entropy

1. Find the topological entropy for each of the maps below. Justify your answer.
   (a) An expanding map of the circle of degree $m \geq 2$.
   (b) A translation $T_{\alpha, \beta}(x, y) = (x + \alpha, y + \beta)$ (mod 1) on the torus $\mathbb{T}^2$.
   (c) The map $f : [0, \pi] \rightarrow [0, \pi]$ given by $f(x) = \frac{1}{3} \sin(x)^2$.

Solutions.
   (a) The entropy of an expanding map of degree $m$ is $\log m$, since this map is topologically conjugate to $E_m$, and $h(E_m) = \log m$.
   (b) This map is an isometry, and hence has 0 entropy.
   (c) Note that $|f'(x)| = |\frac{2}{3} \sin x \cos x| < \frac{2}{3} < 1$, so $f$ is a contraction, and hence has 0 entropy.

2. Let $X$ and $Y$ be compact metric spaces and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be continuous maps. Show that if $g$ is a factor of $f$, then $h(g) \leq h(f)$.

Recall that $g$ is a factor of $f$ if there exists a continuous surjective map $H : X \rightarrow Y$ such that $H \circ f = g \circ H$.  Hint: use an argument similar to the one we used in class to show that the entropy is the same for two metrics generating the same topology.

Solutions. Since $h(g, \varepsilon_1) \geq h(g, \varepsilon_2)$ for $0 < \varepsilon_1 < \varepsilon_2$, we know that $h(g) = \sup_{\varepsilon>0} h(g, \varepsilon)$. Thus it suffices to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $h(g, \varepsilon) \leq h(f, \delta)$. Indeed, it follows that $h(g, \varepsilon) \leq h(f)$ for every $\varepsilon > 0$ and hence $h(g) \leq h(f)$.

Suppose that we are given $\varepsilon > 0$. Then since $X$ is compact, $H$ is uniformly continuous, so there exists $\delta > 0$ such that if $d^X(x, y) < \delta$ then $d^Y(H(x), H(y)) < \varepsilon$. Furthermore, for any $i \in \mathbb{N}_0$, $d^X(f_i(x), f_i(y)) < \delta$ implies that $d^Y(H(f_i(x)), H(f_i(y))) = d^Y(g_i(H(x)), g_i(H(y))) < \varepsilon$.

Thus, for any $n \geq 1$, $d^X_n(x, y) < \delta$ implies $d^Y_n(H(x), H(y)) < \varepsilon$ for all $x, y \in X$. It follows that if $X = \bigcup_{i=1}^K B_{d^X}(x_i, \delta)$, then $Y = \bigcup_{i=1}^K B_{d^Y}(H(x_i), \varepsilon)$, so $S(g, \varepsilon, n) \leq S(f, \delta, n)$. Taking log, dividing by $n$ and taking the limits as $n \rightarrow \infty$, we get $h(g, \varepsilon) \leq h(f, \delta)$, and the result follows.

3. Let $X$ be a compact metric space and let $f, g : X \rightarrow X$ be continuous maps.
   (a) Show that $h(f^2) = 2 h(f)$, where $f^2 = f \circ f$.
   (b) Give an example of $f$ and $g$ such that $h(f \circ g) < h(f) + h(g)$.

Solutions.
   (a) We prove this with two inequalities: $h(f^2) \leq 2h(f)$ and $h(f^2) \geq 2h(f)$.

   (b) Consider $X = [0, 1]$, $f(x) = 2x$, and $g(x) = -x$.
First, we show that \( h(f^2) \leq 2h(f) \). If \( A \) is an \( \varepsilon \)-spanning set for the metric \( d_{2n} \), then for each point \( x \in X \) there is point \( a \in A \) such that \( d(f^i(x), f^i(a)) < \varepsilon \) for \( 0 \leq i \leq 2n-1 \). In particular, \( d(f^{2j}(x), f^{2j}(a)) < \varepsilon \) for \( 0 \leq j \leq n-1 \), and thus \( A \) is an \( \varepsilon \)-spanning set of the metric \( d_n^2 \). Thus, \( S(f^2, \varepsilon, n) \leq S(f, \varepsilon, 2n) \) and
\[
h(f^2, \varepsilon) = \lim_{n \to \infty} \frac{\log S(f^2, \varepsilon, n)}{n} \leq \lim_{n \to \infty} 2 \frac{\log S(f, \varepsilon, 2n)}{2n} = 2h(f, \varepsilon).
\]
Taking the limit as \( \varepsilon \to 0 \) yields the desired result.

\[(*) \text{ Note that in general } \lim_{n \to \infty} a_{2n} \text{ can be smaller than } \lim_{n \to \infty} a_n. \]

In this case, however, we have equality since \( S(f, \varepsilon, 2n-1) \leq S(f, \varepsilon, 2n) \), and hence
\[
\lim_{n \to \infty} \log S(f, \varepsilon, 2n-1)/(2n-1) = \lim_{n \to \infty} \log S(f, \varepsilon, 2n)/(2n) \leq \lim_{n \to \infty} \log S(f, \varepsilon, 2n)/(2n).
\]

We now prove the opposite inequality. Since \( f \) is continuous and \( X \) is compact, for any \( \varepsilon > 0 \) there exists \( \delta < \varepsilon \) such that \( d(x, y) < \delta \) implies that \( d(f(x), f(y)) < \varepsilon \). In particular, if \( d(f^{2i}(x), f^{2i}(y)) < \delta \) for \( 0 \leq i \leq n-1 \), then \( d(f^{2i+1}(x), f^{2i+1}(y)) < \varepsilon \).

It follows that if a set \( A \) is \( \delta \)-spanning for the metric \( d_n^2 \), then \( A \) is \( \varepsilon \)-spanning for the metric \( d_{2n} \). So \( S(f, \varepsilon, 2n) \leq S(f^2, \delta, n) \). Arguing as before, we get that \( 2h(f, \varepsilon) \leq h(f^2, \delta) \), and hence that \( 2h(f) \leq h(f^2) \) as we let \( \varepsilon \) (and hence \( \delta \)) tend to 0.

(b) Let \( f \) be any invertible map with positive entropy (eg, a hyperbolic toral automorphism). Then \( 0 = h(\text{Id}) = h(f \circ f^{-1}) < h(f) + h(f^{-1}) = 2h(f) \).

\[
4. \text{ Show that the topological entropy of the full two-sided shift } \sigma : \Omega_m \to \Omega_m, \ m \geq 2, \text{ equals } \log m.
\]

Use the metric \( d_\lambda(\omega, \omega') = \sum_{i=-\infty}^{\infty} \frac{\delta(\omega_i, \omega'_i)}{\lambda^{|i|}} \) with \( \lambda > 3 \).

**Solution.** We use the definition of topological entropy with \( \varepsilon \)-spanning sets. Take \( \varepsilon = \lambda^{-k} \). Then for any sequence \( \omega \), the ball of radius \( \varepsilon \) centered at \( \omega \) is \( B(\omega, \varepsilon) = C_{\omega^{-k}...\omega^{-1}\omega_1...\omega_k} \). We use the \( \sim \) to emphasize the location of the 0 index. It follows that that:
\[
B_{d_\lambda}(\omega, \varepsilon) = \bigcap_{i=0}^{n-1} \sigma^{-n}(B_{d_\lambda}(\sigma^n \omega, \varepsilon)) = \bigcap_{i=0}^{n-1} \sigma^{-n}(C_{\omega^{-k-1}...\omega_{-i}^{-1}\omega_i\omega_{i+1}...\omega_{k+i}})
= C_{\omega^{-k}...\omega_0^{-1}\omega_1...\omega_{k+n-1}}
\]

There are exactly \( m^{2k+n} \) distinct cylinders \( C_{\omega^{-k}...\omega_0^{-1}\omega_1...\omega_{k+n-1}} \) in \( \Omega_m \). Clearly, each infinite sequence is contained in one of these cylinders / balls, and hence their centers form an \( \varepsilon \)-spanning set in \( d_{2n} \), so \( S(\sigma, \varepsilon, n) \leq m^{2k+n} \). On the other hand, if \( A \) is an \( \varepsilon \)-spanning set then there should be a point of \( A \) within \( \varepsilon \) of the center of each ball, but the balls are disjoint, so \( A \) should contain at least \( m^{2k+n} \) points. Thus \( S(\sigma, \varepsilon, n) \geq m^{2k+n} \), and we conclude that \( S(\sigma, \varepsilon, n) = m^{2k+n} \). Hence for \( \varepsilon = \lambda^{-k} \),
\[
h(\sigma, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log m^{2k+n} = \lim_{n \to \infty} \log m \frac{(2k + n)}{n} = \log m.
\]

Since the \( h(\sigma, \lambda^{-k}) = \log m \) for every \( k \in \mathbb{N} \) and \( \lim_{\varepsilon \to 0} h(\sigma, \varepsilon) \) exists, we conclude that \( h(\sigma) = \log m \).
**Extra credit problem**

5*. Consider the space of all two-sided sequences of numbers in the interval $[0, 1]$:

$$\Omega_{[0,1]} = \{ (\omega_i)_{i=-\infty}^{\infty} \mid \omega_i \in [0, 1] \}$$

with the metric

$$d_\lambda(\omega, \omega') = \sum_{i=-\infty}^{\infty} \frac{|\omega_i - \omega'_i|}{\lambda^{|i|}}.$$

Show that the topological entropy of the shift $\sigma : \Omega_{[0,1]} \to \Omega_{[0,1]}$ is infinite.

**Solution.** We use the following property: if $f : X \to X$ is a continuous map of a compact metric space and $Y \subset X$ is a closed invariant subset, then $h(f) \geq h(f|_Y)$.

For $m \geq 2$, we consider the set $\Omega_m$ with the metric as in Problem 4 and the map

$$H_m : \Omega_m \to \Omega_{[0,1]} \text{ defined by } (H_m(\omega))_i = \omega_i/(m-1).$$

Then $H_m$ is continuous, since it is distance-decreasing. Furthermore, $H_m$ is injective, and hence a homeomorphism onto its image (since $\Omega_m$ is compact). $H_m$ also conjugates the two shifts, since the map is defined coordinate-wise in the same way. Thus, $\sigma|_{H_m(\Omega_m)}$ is topologically conjugate to the shift on $\Omega_m$, and hence has entropy $\log m$. So for every $m \geq 2$:

$$h(\sigma) \geq \log m$$

and hence $h(\sigma) = \infty$.

**Another solution.** Fix $k \in \mathbb{N}$. It follows easily from the definition of the metric that for two sequences $\omega$ and $\omega'$ in $\Omega_{[0,1]}$, if $|\omega_i - \omega'_i| \geq 1/k$ for some $0 \leq i \leq n - 1$, then $d_{\lambda,n}(\omega, \omega') \geq 1/k$.

Since there are $k + 1$ points in $[0, 1]$ with pairwise distances $\geq 1/k$, there are at least $(k + 1)^n$ sequences in $\Omega_{[0,1]}$ with pairwise $d_{\lambda,n}$ distances at least $1/k$. For example, the sequences $\omega$ such that $\omega_i \in \{0, 1/k, 2/k, \ldots, (k-1)/k, 1\}$ for $0 \leq i \leq n - 1$ and $\omega_i = 0$ for all other $i$ form a $1/k$-separated set.

It follows that $N(\sigma, 1/k, n) \geq (k + 1)^n$, and hence

$$h(\sigma, 1/k) \geq \lim_{n \to \infty} \frac{\log N(\sigma, 1/k, n)}{n} \geq \lim_{n \to \infty} \frac{\log(k + 1)^n}{n} = \log(k + 1).$$

Thus $h(\sigma) = \lim_{\varepsilon \to 0} h(\sigma, \varepsilon) = \lim_{k \to \infty} h(\sigma, 1/k) = \infty.$

□