Each ticket will contain a theoretical question from the list below and a problem. Questions on these topics may also be asked after you answer the questions on the ticket.

- **Circle rotations**
  - Rational rotations: periods of points
  - Irrational rotations:
    - Minimality: statement and proof
    - Equidistribution: statement
    - Distribution of first digits of powers: statement and connection to circle rotations

- **Contractions**
  - Definition
  - Contractions of an interval: statements and proofs
  - Contraction Principle: statement and proof
  - Contractions in $\mathbb{R}^k$: statement

- **Periodic and fixed points**
  - Definitions;
  - Attracting fixed points: definition,
    - a sufficient condition for a fixed point of an interval map: statement and proof

- **Non-decreasing continuous interval maps:**
  - Fixed points: statement and proof
  - Orbits of points: statement and proof

- **Autonomous differential equations, $\dot{x} = g(x)$**
  - Lipschitz functions: definition and examples
  - Behavior of the solutions: statement and proof
  - Existence and uniqueness of solutions: statement and proof,
    - an example of non-uniqueness

- **Linear maps of the plane**
  - “Models”
  - Invariant curves and orbits of points for each “model”
  - Conjugacy of real $2 \times 2$ matrices to the models
  - Topological conjugacy of dynamical systems
  - Linear differential equations in the plane and matrix exponential
• Homeomorphisms
  – Definition
  – Continuity of the inverse for a continuous bijection: two proofs

• Recurrence
  – Measurable sets and Lebesgue measure – an overview
  – Measure-preserving map: definition
  – Poincare Recurrence Theorem: statement and proof
  – Poincare Recurrence – a topological version: statement and proof

• Topological transitivity, minimality, and topological mixing
  – Definitions and examples
  – Minimality and closed invariant sets: statement and proof
  – Criteria for topological transitivity: four equivalent statements and proof of their equivalence
  – Meaning of topological mixing

• Times-\(m\) map of the circle
  – Fixed and periodic points
  – Writing numbers in base \(m\)
  – Constructing a point with dense orbit
  – Topological mixing

• Sequence spaces \(\Omega_m\) and \(\Omega_{m+}\), and the shift map
  – Definitions
  – Metrics on \(\Omega_m\) and \(\Omega_{m+}\), and open balls in these metrics
  – The spaces are compact and do not have isolated points, the shift is continuous
  – Fixed points and periodic points
  – Topological mixing with proof
  – Symbolic dynamical system: definition and examples

• Expanding maps of the circle
  – Definition of an expanding map
  – Definitions of lift and degree for a circle map
  – For expanding circle maps of degree 2:
    – Fixed points
    – Coding and semiconjugacy with the shift
    – Definition of semiconjugacy for two dynamical systems
    – Topological conjugacy for expanding circle maps of degree 2 (or \(m\)) and implications for periodic points, transitivity, etc.
• **Linear maps of the torus** $\mathbb{T}^2$
  
  - The torus
  - Invertible linear maps (automorphisms) of the torus
  - Hyperbolic toral automorphisms
    - An example, and the action on the fundamental domain (unit square)
    - The eigenvalues are irrational and eigendirections have irrational slopes
    - A point is periodic if and only if its coordinates are rational, with *proof*
    - The number of periodic points of period $n$: statement and an outline of a proof; a formula for the case if eigenvalues $\lambda > 1$ and $1/\lambda$
    - Topological mixing, with *proof*

• **Topological entropy**
  
  - Definition and three quantities that can be used in the definition ($N, S, D$)
  - Topological entropy of isometries and contractions is zero, with *proof*
  - Entropy of $E_m$ is $\log m$, with *proof*
  - If two metrics generate the same topology, the entropy is the same, with *proof*
  - Topological entropy is an invariant of topological conjugacy, with *proof*
  - Properties of topological entropy
  - Topological entropy of a hyperbolic toral automorphism

• **Chaos and Sensitive dependence on the initial conditions**
  
  - Definitions
  - Examples
  - Chaotic maps exhibit sensitive dependence
  - Topological mixing implies sensitive dependence, with *proof*