The circle.

let $m \in \mathbb{N}, m \geq 3$. For $x \in S^1$, $E_m(x) = mx \mod 1$

Consider $m = 3$

Recall: $S^1 = \mathbb{R}/\mathbb{Z}$. The map $E_3 : S^1 \to S^1$ is well-defined since $x \sim y \in \mathbb{Z}$ implies $3x \sim 3y \in \mathbb{Z}$

Each $x \in S^1$ has a unique representative $3x$ (fractional part) in $[0, 1)$

so we can view $E_3$ as a map on $[0, 1)$ given by $x \mapsto 3x$

To visualize, stretch the circle by a factor of 3 and wrap it around itself 3 times.

OR sketch the graph of the map.

$\left( x \mapsto \frac{3x}{3} \right)$ on $[0, 1)$

Keep in mind that 0 and 1 are identified.

- The map $E_3$ is not one-to-one for example $0, \frac{1}{3}, \frac{2}{3}$ are mapped to 0

In fact, every $y$ has exactly 3 pre-images: $\frac{y}{3}, \frac{y+1}{3}, \frac{y+2}{3}$. So the map is 3-to-1.

Since $E_3$ is not one-to-one, it is not invertible.

Clearly, the map is onto.

- If $d(x, y)$ on the circle is small, then $d(E_3(x), E_3(y)) = 3d(x, y)$.

This holds as long as $f(x), f(y)$ remain $\leq \frac{1}{2}$ apart, i.e., whenever $d(x, y) \leq \frac{1}{6}$.

Thus $E_3$ is an expanding map.

1. Does $E_3$ preserve length, that is, is it true that for any interval $I$ on the circle, the length of $I$ equals the total length of its pre-image, $(E_3)^{-1}(I)$? Yes! $(E_3)^{-1}$ consists of 3 intervals of length $\frac{1}{3}|I|$.  

2. What are the fixed points of $E_3$?

$E_3(x) = x \quad x = 3x \mod 1, \quad 2x = 0 \mod 1 \quad x = 0$ or $\frac{1}{2} \mod 1$.

Check: $E_3(0) = 0, \quad E_3\left( \frac{1}{2} \right) = 3 \cdot \frac{1}{2} = \frac{3}{2}$ mod 1.

Let us find all periodic points of period $n \in \mathbb{N}$.

$x$ is $n$-periodic $\iff E^n_3(x) = x \mod 1 \iff$

$\iff 3^n x = x + k$ for some $k \in \mathbb{Z} \iff x = \frac{k}{3^n - 1}$

Such $x$ is in $[0, 1)$ if and only if $0 \leq k < 3^n - 1$.  

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Thus, $E_3$ has exactly $3^n - 1$ periodic pts. of period $n$.

For example, for $n = 2$ we get $\frac{K}{3^2 - 1} = \frac{K}{8}$, where $K = 0, 1, \ldots, 7$.

Check for $\frac{5}{8}$: $E_3^2(\frac{5}{8}) = 3 \cdot \frac{5}{8} = 4 \frac{5}{8} = \frac{5}{8}$ mod 1.

The numbers of the form $\frac{K}{3^2 - 1}$, where $0 \leq K < 3^n - 1$ are dense in $[0, 1)$.

So the set of periodic pts of $E_3$ is dense in $S^1$ — arbitrarily close to any given point there is a periodic point.

But of course not all points are periodic under $E_3$.

Is there a point with dense orbit?

To study $E_3$ further, we will write numbers in $[0, 1]$ in base 3.

Let us review base 10 first.

$x = 0.123\ldots$ in base 10

means that $x$ is in interval $2$,

more specifically, in interval $[1/3, 1/10]$.

$x = 0.\alpha_1 \alpha_2 \alpha_3 \ldots$ means that $x = \sum_{n=1}^{\infty} \alpha_n / 10^n$.

Multiplication by 10 and taking the remainder shifts the sequence to the left: $10x = 2\alpha_1 \alpha_2 \alpha_3 \ldots$.

Does every number in $[0, 1)$ have a unique expansion in base 10?

No! $0.999\ldots = 1.000\ldots$.

In general, $0.d_1 d_2 \ldots d_k 999\ldots = 0.d_1 d_2 \ldots (d_k+1) 00\ldots$.

For numbers not of this form the expansion is unique.

Note: The numbers with non-unique expansion correspond to endpoints of the decimal intervals.