Recurrence theorems

Let $X \subseteq \mathbb{R}^k$ be measurable, for example, an open set or a closed set.

A map $f: X \to X$ is measure-preserving if for any measurable set $A \subseteq X$, $\mu(f^{-1}(A)) = A$.

Examples:
- $f: S^1 \to S^1$,
- $g: S^1 \to S^1$.

Let $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + b$ is measure-preserving, but there is no recurrent points, so to obtain recurrence we also require that $X$ has finite measure.

The latter holds for bounded sets in $\mathbb{R}^k$.

Pointwise Recurrence Thm

Let $X \subseteq \mathbb{R}^k$ be a set of finite measure (for example, a open or closed bounded set),

let $f: X \to X$ be a measure-preserving map, and let $A \subseteq X$ be a measurable set (for example, a ball or a box).

Then almost every point of $A$ returns to $A$, i.e.

for almost every $a \in A$, there is $n_0 \in \mathbb{N}$ s.t. $f^n(a) \in A$.

Almost every means every except for a set of points of measure 0.

Proof:

Let $Y$ be the set of points of $A$ that never return to $A$, i.e.

$Y = \{ a \in A : f^n(a) \notin A \text{ for all } n \in \mathbb{N} \}$

$= \{ a \in A : a \notin f^{-n}(A) \text{ for all } n \in \mathbb{N} \} = A \setminus \bigcap_{n=1}^{\infty} f^{-n}(A)$

So $Y$ is measurable.

We observe that for any $k \in \mathbb{N}$, $f^{-k}(Y) \cap Y = \emptyset$.

Indeed, if $x \in f^{-k}(Y) \cap Y$, then $x \in A$ and $f^k(x) \in A$,

which is impossible by def. of $Y$.

It follows that for any $m \neq n$ in $\mathbb{N}$, $f^{-n}(Y) \cap f^{-m}(Y) = \emptyset$.

Thus $f^{-n}(Y)$, $n \in \mathbb{N}$ are disjoint measurable sets and

$\mu(f^{-n}(Y)) = \mu(Y)$ for each $n$.

The sum of their measures = measure of their union is $\mu(Y) < \infty$.

It follows that $\mu(Y) = 0$. (Otherwise the sum is infinite) $\square$

Corollary: Under the assumptions of the Thm,

almost every $a \in A$ returns to $A$ infinitely many times, i.e.

there is a subsequence $(n_k)$ s.t. $f^{n_k}(a) \in A$ for all $k \in \mathbb{N}$.

Proof: A point $a \in A$ returns to $A$ only finitely many times or never

$\iff f^n(a) \in Y$ for some $n \in \mathbb{N} \cup \{0\} \iff a \in f^{-n}(Y)$ for some $n \in \mathbb{N} \cup \{0\}$.

Thus $a \notin \bigcap_{n=0}^{\infty} f^{-n}(Y)$. Since for each $n$, $\mu(f^{-n}(Y)) = \mu(Y) = 0$,

$\mu\left( \bigcup_{n=0}^{\infty} f^{-n}(Y) \right) \leq \sum_{n=0}^{\infty} \mu(f^{-n}(Y)) = 0.$ $\square$
Recall: \( x \in X \) is recurrent if for any \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) s.t.
\[
d(f^n(x), x) < \varepsilon.
\]

**Theorem**

Let \( X \subseteq \mathbb{R}^d \) be either a bounded open set or the closure of a bounded open set (bounded \( \implies \mu(X) < \infty \)), and let \( f : X \to X \) be a measure-preserving map.

Then almost every point in \( X \) is recurrent.

**Proof**

Let \( Q \) be the set of all points in \( X \) with rational coordinates. Then \( Q \) is countable and dense in \( X \).

Consider the collection of open balls \( B(q, \frac{1}{n}) \), where \( q \in Q \) and \( n \in \mathbb{N} \). This collection is countable: we can list them as \( B_1, B_2, \ldots \).

For each \( k \), let \( \bar{B}_k = B_k \cap X \).

We observe that \( x \) is recurrent \( \iff \) \( x \) returns to each \( \bar{B}_k \) that contains it.

So it suffices to show that the set of points that do not have this property has measure 0.

For each \( k \), let \( Y_k = \{ x \in \bar{B}_k : x \text{ does not return to } \bar{B}_k \} \).

Then by the previous theorem \( \mu(Y_k) = 0 \).

Hence \( \mu(\bigcup_{k=1}^{\infty} Y_k) = 0 \).

It follows that almost every \( x \in X \) is recurrent. \( \Box \).

**Corollary**

Under the assumptions of the Thm, the set of recurrent points is dense in \( X \).

Otherwise, the set of non-recurrent pt. contains an open ball in \( X \) and hence has positive measure.

**Note:** The theorem applies to \( E_3 : S^1 \to S^1 \). Indeed, \( S^1 \) is \([0,1] \) with endpoints identified and \( E_3 \) is measure-preserving, so almost every \( x \in S^1 \) is recurrent.

Observe that every \( x = \frac{k}{3^n} \), \( 1 \leq k \leq 3^n - 1 \) is non-recurrent.