Recurrence

- A metric space, \( f: X \to X \) - a continuous map.

**Def.** A point \( x \in X \) is called **recurrent** if for any \( \epsilon > 0 \) there is \( n \in \mathbb{N} \) s.t. \( d(f^n(x), x) < \epsilon \).

Equivalently, if \( x = \lim f^n(x) \) for some subsequence \( (n_k) \).

(If \( f \) is invertible, such a point is called **positively recurrent**.)

**Ex.**
- A fixed point is recurrent.
- A periodic point is recurrent.
- \( R^1: S^1 \to S^1 \). Every point is recurrent.
- \( f: [a, b] \to [a, b] \) cont. increasing. Only fixed points are recurrent.
- For any other point \( x \), \( f^n(x) \) converges to a fixed point, and hence there is no subsequence converging to \( x \).
- \( f: [0, \infty) \to [0, \infty) \), \( a > 0 \), \( f(x) = x + a \).
  - There are no recurrent points.
- \( f: (0, 1] \to (0, 1] \), \( f(x) = \frac{x}{2} \). No recurrent points.

**Note:** \( S^1 \) and \( [a, b] \) are compact, \( [0, \infty) \) and \( (0, 1] \) are not.

If \( X \) is not compact, there may be no recurrent points.

**Fact.** Let \( X \) be a closed bounded set in \( \mathbb{R}^k \) or, more generally, let \( X \) be a compact metric space, and let \( f: X \to X \) be continuous.

Then \( X \) contains a minimal closed set for \( f \) (not necessarily unique).

That is, there is a non-empty closed set \( Y \subseteq X \) s.t.

- \( Y \) has no closed \( f \)-invariant subsets except for \( Y \) and \( \emptyset \).

**Corollary.** If \( X \) is compact and \( f: X \to X \) is continuous,

then there is a recurrent point.

**Proof.** Let \( Y \subseteq X \) be a minimal set. Then for every \( y \in Y \),

the orbit of \( y \) is dense in \( Y \) and so \( y \) is recurrent. \( \square \)

- Let \( X \) be compact and \( f: X \to X \) continuous. Then there is a recurrent point. Are there just a few of them?
- Is "almost every" \( x \in X \) recurrent?

**Note:** \( R^1: S^1 \to S^1 \) preserves length contractions and continuous maps of \( [a, b] \) do not.
Consider \( \mathbb{R}^k \). We can define the usual \( k \)-dimensional volume for "boxes" in \( \mathbb{R}^k \):

\[
\mu([a_1, b_1] \times \ldots \times [a_k, b_k]) = (b_1 - a_1) \ldots (b_k - a_k)
\]

It can be extended to a large class of sets, called measurable sets:

- All open sets and all closed sets are measurable.
- If \( A \) and \( B \) are measurable, then so are \( A \setminus B, A \cup B, \text{ and } A \cap B \).
- If \( A_1, A_2, \ldots \) are measurable, then so are \( \bigcap_{n=1}^{\infty} A_n \) and \( \bigcup_{n=1}^{\infty} A_n \).

**Def.** A set \( A \subseteq \mathbb{R}^k \) is open if for every \( a \in A \) there exist \( r > 0 \) s.t. \( B(a, r) \subseteq A \).

\[
B(a, r) = \{ x \in \mathbb{R}^k : d(a, x) < r \}
\]

**Note:** All sets we can think of are measurable.

The \( k \)-dimensional Lebesgue measure \( \mu = \mu_k \) is the extension of the \( k \)-dimensional volume to measurable sets. For each measurable set \( A \), \( \mu(A) \in [0, \infty] \), and if \( A_1, A_2, A_3, \ldots \) are pairwise disjoint, then

\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) \quad (**)
\]

**Ex.** For any point \( x \), \( \mu(\{x\}) = 0 \). Hence for any countable set

\[
Y = \{ y_1, y_2, y_3, \ldots \} \quad \mu(Y) = \sum_{n=1}^{\infty} \mu(\{y_n\}) = 0.
\]

**Note.** For any measurable \( A, B \),

- if \( A \cap B = \emptyset \), then \( \mu(A \cup B) = \mu(A) + \mu(B) \). (Take \( A_1 = A, A_2 = B, A_3 = \emptyset, \ldots \))
- if \( A \subseteq B \), then \( \mu(A) \leq \mu(B) \) since \( B = A \cup (B \setminus A) \)

Hence if \( A \) is open, then \( A \) contains an open ball, which has positive measure, and hence \( \mu(A) > 0 \).

**Ex.** The Cantor set \( C \) has Lebesgue measure \( 0 \). (HW).