Periodic points

Let \( X \) be a set and let \( f \) be a map from \( X \) to \( X \).

Def: A point \( p \in X \) is called periodic if \( f^n(p) = p \) for some \( n \in \mathbb{N} \).

A number \( n \in \mathbb{N} \) s.t. \( f^n(p) = p \) is called a period of \( p \).

The smallest such \( n \) is called the prime period of \( p \).

Note: If \( n \) is a period, then \( kn \) is also a period for any \( k \in \mathbb{N} \).

Note: Any fixed pt is a periodic pt with prime period 1.

Ex: Let \( f : X \to X \) be a contraction with a fixed pt \( x^* \).

Then \( f \) has no periodic pts except for \( x^* \) since for each \( x \in X \) and \( n \in \mathbb{N} \), \( d(f^n(x), x^*) < d(x, x^*) \) and so \( f^n(x) \neq x \).

Ex: \( f : \mathbb{R} \to \mathbb{R}, \ f(x) = -x \).

Fixed pt: 0, every other pt. is periodic with prime period 2.

Ex: \( f : \mathbb{R} \to \mathbb{R}, \ f(x) = -x^3 \).

Fixed pt: 0, \( 1, -1 \) - periodic w. prime period 2, no other.

The circle \( S^1 \)

\( S^1 = \mathbb{R}/\mathbb{Z} = \) the set of real numbers with \( x \) and \( y \) identified if they differ by an integer, i.e., have the same fractional part. We write \( x \equiv y \mod 1 \).

More formally, define an equivalence relation on \( \mathbb{R} \):

\( x \sim y \Leftrightarrow x - y \in \mathbb{Z} \). \( \mathbb{R}/\mathbb{Z} \) is the set of equivalence classes.

\( \mathbb{Z} = \{ \mathbb{Z} \} \subset \mathbb{R}/\mathbb{Z} \). Ex. \( [0, 1] = \mathbb{Z} \).

Each class has a unique representative in \([0, 1)\).

Adding equiv. classes corresponds to adding their representatives \mod 1.

Ex: \([\frac{2}{3}] + [\frac{1}{4}] = [\frac{2}{3} + \frac{1}{4}] = [\frac{7}{12}] = [\frac{7}{12} - \frac{6}{12}] = \frac{1}{12} \mod 1 \).

We will consider numbers in \([0, 1)\) with addition \mod 1.

To visualize: Wrap the real line around a circle of circumference 1, which we can think of as \([0, 1)\) with 0 and 1 identified.

Then adding 2 corresponds to the counterclockwise rotation by 2\(\pi\).

Note that angles 2\(\pi\) and \(\beta\) with \(\beta \in \mathbb{Z}\) with \(\beta, \beta \in \mathbb{Z}\) correspond to the same point on the circle.
Circle rotations

Let \( a \in \mathbb{R} \). Denote by \( R_{a} \) the rotation by \( a \), i.e. \( R_{a}(x) = x + a \mod 1 \). Then \( R_{a} \) is invertible with \( (R_{a})^{-1} = R_{-a} \) and \( (R_{a})^{n}(x) = x + na \mod 1 \).

EXAMPLE: Let \( a = \frac{1}{3} \) and \( x = \frac{2}{3} \).

\[
R_{a}(\frac{2}{3}) = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1 \mod 1
\]
\[
(R_{a})^{2}(\frac{2}{3}) = \frac{3}{3} + \frac{1}{3} = \frac{4}{3} = \frac{1}{3} \mod 1
\]
\[
(R_{a})^{3}(\frac{2}{3}) = \frac{3}{3} + 3 \cdot \frac{1}{3} = \frac{3}{3} + 1 = 2 \mod 1
\]

So \( \frac{2}{3} \) is periodic with prime period 3. Same for any \( x \).

Rational \( a \)

Proposition: Let \( a = \frac{p}{q} \), where \( p \) and \( q \) are relatively prime integers. Then for the map \( R_{a} \) on \( S^{1} \), every \( x \) is periodic with prime period \( q \).

Proof: We need to show that for each \( x \in S^{1} \), \( (R_{a})^{k}(x) = x \) and \( (R_{a})^{k}(x) \neq x \) for \( k = 1, 2, \ldots, q - 1 \).

Let \( x \in S^{1} \). \( (R_{a})^{k}(x) = (x + q \cdot \frac{p}{q}) = x + p = x \mod 1 \).

Suppose that \( (R_{a})^{k}(x) = x \) for some \( k \), \( 1 \leq k \leq q - 1 \).

Then \( x + k \cdot \frac{p}{q} = x \mod 1 \). Hence \( x + k \cdot \frac{p}{q} = x + m \cdot \frac{p}{q} \) for some \( m \in \mathbb{Z} \), and so \( pk = qm \). Since \( p \) and \( q \) are relatively prime, it follows that \( q \) divides \( k \), which is impossible as \( 1 \leq k \leq q - 1 \). \( \square \)

Irrational \( a \)

Prop: If \( a \) is irrational, then \( R_{a} \) has no periodic points. (HW)