Fibonacci numbers

(1202, Leonardo of Pisa, also known as Fibonacci)

An idealized rabbit population:
- At the beginning there is one newborn pair of rabbits;
- Every pair produces a new pair every month, starting at 2 months old;
- Rabbits never die.

(2) How many pairs of rabbits will be there in $n$ months?

\[ F_0 = 1, \quad F_1 = 1, \quad F_2 = 1 + 1 = 2, \quad F_3 = 2 + 1 = 3, \ldots \]

\[ F_n = F_{n-1} + F_{n-2} \quad \text{number of new pairs} = \text{number of pairs 2 months ago} \]

\[ = \text{number of pairs 1 month ago} \]

Fibonacci sequence: \(1, 1, 2, 3, 5, 8, 13, 21, \ldots\)

We will use the contraction principle to show that

\[ \lim_{n \to \infty} \frac{F_{n+1}}{F_n} \text{ exists and then find it} \]

Let \( a_n = \frac{F_{n+1}}{F_n} \) Then \( a_1 = 2 \) and \( a_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1} + F_n}{F_{n+1}} = 1 + \frac{1}{a_n} \).

Therefore, \( (a_n)_{n=1}^\infty \) is the orbit of 2 under the map \( f(x) = 1 + \frac{1}{x} \).

\( a_1 = 2, \quad a_2 = f(2), \quad a_3 = f^2(2) \ldots \)

If \( f \) is a contraction, then \( f^n(2) \) converges, i.e., \( (a_n) \) converges.

\( |f'(x)| = \frac{1}{x^2} \), so \( f \) is not a contraction on, say, \( \mathbb{R}_{[1, \infty)} \).

We cannot consider \( \mathbb{R}_{[2, \infty)} \) since \( f(2) = 1 + \frac{1}{2} = \frac{3}{2} \neq \mathbb{R}_{[2, \infty)} \).

Let us find a closed interval \( I \) s.t.

- \( 2 \in I \)
- \( f \) maps \( I \) to \( I \)
- \( f \) is a contraction on \( I \).

Since \( I \) must contain 2, it must also contain \( f(2) = \frac{3}{2} \).

\( f(\frac{3}{2}) = 1 + \frac{2}{3} = \frac{5}{3}, \quad \frac{3}{2} < \frac{5}{3} < 2. \)

Let \( I = \left[ \frac{3}{2}, 2 \right] \).

Since \( f \) is decreasing,

for any \( x \) s.t. \( \frac{3}{2} \leq x \leq 2, \)

\( \frac{3}{2} = f(2) < f(x) < f(\frac{3}{2}) = \frac{5}{3} < 2, \) and so \( f(I) \subset I \).

\( |f'(x)| = \frac{1}{x^2} \leq \left( \frac{3}{2} \right)^2 = \frac{9}{4} < 1 \) for all \( x \in I \), and so \( f \) is a contraction on \( I \).

It follows that \( f \) has a unique fixed point, \( a \), in \( I \) and \( \lim a_n = a \).

Find \( a \): \( f(a) = a + \frac{1}{a} = a \) and \( a \in I \).

Thus \( \frac{F_{n+1}}{F_n} \to \frac{1 + \sqrt{5}}{2} \) as \( n \to \infty \).
Weak contractions

Question: Is it true that if $X$ is a closed set in $\mathbb{R}$ and $f: X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ in $X$ ($\star$), then $f$ has a fixed point in $X$?

(?) Let $X = [1, \infty)$ and let $f(x) = x + \frac{1}{x}$. Then $X$ is closed; $f: X \to X$ since for any $x > 1$, $f(x) > x > 1$;

$|f'(x)| = |1 - \frac{1}{x^2}| < 1$ for all $x \in X$ $\Rightarrow$ ($\star$) holds.

But $f$ does not have a fixed pt. since $f(x) = x + \frac{1}{x} > x$ for all $x \in X$.

Question: What if $X = [a, b]$?

Thm. If $f: [a, b] \to [a, b]$ is continuous (no other assumptions!), then $f$ has at least one fixed pt. in $[a, b]$.

Proof. If $f(a) = a$ or $f(b) = b$ we are done.

Otherwise, consider $g(x) = f(x) - x$, also cont.

Then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$.

Hence by the Intermediate Value Thm there is $x_0 \in (a, b)$ s.t. $g(x_0) = 0$, i.e. $f(x_0) = x_0$. D.

Note. Such $f$ may have several fixed points.

Even if a fixed pt is unique, $f^n(x)$ does not necessarily tend to $x^*$.

Ex. $X = [-1, 1]$, $f(x) = -x$, $x^* = 0$.

Thm. If $f: [a, b] \to [a, b]$ satisfies ($\star$), then $f$ has a unique fixed pt.

Pf. Since $f$ satisfies ($\star$), it is continuous. Hence it has a fixed pt.

If $f(x) = x$ and $f(y) = y$ with $x \neq y$, then $d(x, y) = d(f(x), f(y)) < d(x, y)$, a contradiction. So a fixed pt. is unique. D.

Another proof of existence: Consider $h(x) = d(x, f(x))$.

Then $h$ is cont. on $[a, b]$ and hence attains min at some $x^* \in [a, b]$.

If $f(x^*) \neq x^*$, then $d(f(x^*), f(f(x^*))) < d(x^*, f(x^*))$, a contradiction. D.

Note. One can also show that $f^n(x) \to x^*$, but no estimate on the rate.

Ex. $f(x) = \sin x$ on $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$f'(0) = 1$, $|f'(x)| < 1$ for $x \neq 0$.

$f$ satisfies ($\star$). Fixed pt $0$. 