An Interesting Smooth Unimodal/Bimodal/Quadrimodal Distribution of Unknown Use

September 17, 2015

Zheye Yuan

I gained the motivation for the following work from the construction of the catenary. That is, a catenary can be decomposed in to a sum of two exponentials, so I thought, what would I get if I decomposed a quadratic. This is not the only way to decompose a quadratic. I feel this distribution is of limited utility but will present it as a curiosity as well as consolation for my efforts and time invested.

$$\begin{aligned} x^2 &= q(x) + q(-x) \\ \lim_{x\to-\infty} q(x) &= 0 \\ \lim_{x\to\infty} q(x) &= \infty \end{aligned}$$

In fact we demand more stricly that the following holds, which imply the last two conditions above.

$$\lim_{x\to\infty} \frac{q(x)}{x^2} = 1$$

I do not demand $q(x)$ to be monotone in x. Consider the following.

$$\frac{e^x}{e^x + e^{-x}}$$

It goes to 1 as x goes to infinity and it goes to 0 as x goes to negative infinity. We use this as weights on x^2 and 0 to obtain the following $q(x)$.

$$q(x) = \frac{x^2 e^x}{e^x + e^{-x}}$$

We check the conditions.

$$\begin{aligned} q(x) + q(-x) &= \frac{x^2 e^x}{e^x + e^{-x}} + \frac{x^2 e^{-x}}{e^{-x} + e^x} \\ &= \frac{x^2(e^x + e^{-x})}{e^x + e^{-x}} \\ &= x^2 \\ \lim_{x\to\infty} \frac{q(x)}{x^2} &= \lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}} \\ &= 1 \end{aligned}$$

We use this $q(x)$ to define a distribution, with the following kernel.

$$q(-x^2) = \frac{x^4 e^{-x^2}}{e^{-x^2} + e^{x^2}}$$

We will find its normalizing constant.

$$\int_{-\infty}^{\infty} \frac{x^4 e^{-x^2}}{e^{-x^2} + e^{x^2}} dx = 2 \int_0^{\infty} \frac{x^4 e^{-2x^2}}{1 + e^{-2x^2}} dx$$

Now note that on $(0, \infty)$, $e^{-2x^2} < 1$. Therefore the below holds.

$$\frac{1}{1 + e^{-2x^2}} = \sum_{n=0}^{\infty} (-1)^n e^{-2nx^2}$$

Using this, we obtain the expression below.

$$2 \int_0^{\infty} \frac{x^4 e^{-2x^2}}{1 + e^{-2x^2}} dx = 2 \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} x^4 e^{-2(n+1)x^2} dx$$

We apply a change of variable, $u = x^2$, $du = 2xdx = 2u^{\frac{1}{2}} dx$.

$$\begin{aligned}
2 \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} x^4 e^{-2(n+1)x^2} dx &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} u^{\frac{3}{2}} e^{-2(n+1)u} du \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{5}{2}\right)}{2^{\frac{5}{2}} (n+1)^{\frac{5}{2}}} \\
&= \frac{\Gamma\left(\frac{5}{2}\right)}{2^{\frac{5}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{\frac{5}{2}}} \\
&= \frac{\Gamma\left(\frac{5}{2}\right)}{\sqrt{2}^5} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\frac{5}{2}}} \\
&= \sqrt{2}^{-5} \Gamma\left(\frac{5}{2}\right) \eta\left(\frac{5}{2}\right)
\end{aligned}$$

Where Γ is the gamma function and η is the Dirichlet eta function. Therefore the following is a density of a random variable.

$$\frac{\sqrt{2}^5 x^4 e^{-x^2}}{\Gamma\left(\frac{5}{2}\right) \eta\left(\frac{5}{2}\right) \left(e^{-x^2} + e^{x^2}\right)}$$

We calculate the following which appears in the calculation of moments. Note

p is an integer, and when it is odd the integral is 0. So just write it as $p = 2n$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^{4+p}e^{-2x^2}}{e^{-2x^2}+1}dx &= 2\int_0^{\infty} \frac{x^{4+2n}e^{-2x^2}}{e^{-2x^2}+1}dx \\
&= 2\sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} x^{4+2n}e^{-2(1+k)x^2}dx \\
&= \sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} u^{2+n-\frac{1}{2}}e^{-2(1+k)u}du \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma\left(n+\frac{5}{2}\right)}{2^{n+\frac{5}{2}}(1+k)^{n+\frac{5}{2}}} \\
&= \frac{\Gamma\left(n+\frac{5}{2}\right)}{2^{n+\frac{5}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+k)^{n+\frac{5}{2}}} \\
&= \frac{\Gamma\left(n+\frac{5}{2}\right)}{2^{n+\frac{5}{2}}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{n+\frac{5}{2}}} \\
&= \frac{\Gamma\left(n+\frac{5}{2}\right)}{2^{n+\frac{5}{2}}} \eta\left(n+\frac{5}{2}\right)
\end{aligned}$$

Therefore we have the following result.

$$\begin{aligned}
E[X^{2n}] &= \frac{\Gamma\left(n+\frac{5}{2}\right)\eta\left(n+\frac{5}{2}\right)}{2^n\Gamma\left(\frac{5}{2}\right)\eta\left(\frac{5}{2}\right)} \\
&= \frac{\Gamma\left(n+\frac{5}{2}\right)\eta\left(n+\frac{5}{2}\right)}{2^n\frac{3}{4}\sqrt{\pi}\eta\left(\frac{5}{2}\right)}
\end{aligned}$$

We calculate a moment of interest below.

$$\begin{aligned}
E[X^2] &= \frac{\Gamma\left(\frac{7}{2}\right)\eta\left(\frac{7}{2}\right)}{2^n\Gamma\left(\frac{5}{2}\right)\eta\left(\frac{5}{2}\right)} \\
&= \frac{5\eta\left(\frac{7}{2}\right)}{2^{n+1}\eta\left(\frac{5}{2}\right)}
\end{aligned}$$

Therefore the characteristic function is the following.

$$E[e^{itX}] = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma\left(n+\frac{5}{2}\right)\eta\left(n+\frac{5}{2}\right)t^{2n}}{2^n\Gamma\left(\frac{5}{2}\right)\eta\left(\frac{5}{2}\right)(2n)!}$$

We now look at a slight modification.

$$\frac{(x^2+c)^2e^{-x^2}}{e^{-x^2}+e^{x^2}} = \frac{(x^4+2cx^2+c^2)e^{-x^2}}{e^{-x^2}+e^{x^2}}$$

We find the normalizing constant of this density.

$$\int_{-\infty}^{\infty} \frac{x^4 e^{-x^2}}{e^{-x^2}+e^{x^2}} dx + 2c \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2}}{e^{-x^2}+e^{x^2}} dx + c^2 \int_{-\infty}^{\infty} \frac{e^{-x^2}}{e^{-x^2}+e^{x^2}} dx$$
$$= \frac{\Gamma\left(\frac{5}{2}\right)\eta\left(\frac{5}{2}\right)}{2^{\frac{5}{2}}} + \frac{2c\Gamma\left(\frac{3}{2}\right)\eta\left(\frac{3}{2}\right)}{2^{\frac{3}{2}}} + \frac{c^2\Gamma\left(\frac{1}{2}\right)\eta\left(\frac{1}{2}\right)}{2^{\frac{1}{2}}}$$
$$= \frac{3\sqrt{\pi}}{16\sqrt{2}}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2\sqrt{2}}\eta\left(\frac{3}{2}\right) + \frac{c^2\sqrt{\pi}}{\sqrt{2}}\eta\left(\frac{1}{2}\right)$$

Therefore the below is a density.

$$\frac{1}{\frac{3\sqrt{\pi}}{16\sqrt{2}}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2\sqrt{2}}\eta\left(\frac{3}{2}\right) + \frac{c^2\sqrt{\pi}}{\sqrt{2}}\eta\left(\frac{1}{2}\right)} \frac{(x^2+c)^2 e^{-x^2}}{e^{-x^2}+e^{x^2}}$$

We look at the moments. Odd powers of X has an expectation of zero.

$$E[X^{2n}] = \frac{\frac{\Gamma\left(n+\frac{5}{2}\right)\eta\left(n+\frac{5}{2}\right)}{2^{n+2}} + \frac{2c\Gamma\left(n+\frac{3}{2}\right)\eta\left(n+\frac{3}{2}\right)}{2^{n+1}} + \frac{c^2\Gamma\left(n+\frac{1}{2}\right)\eta\left(n+\frac{1}{2}\right)}{2^n}}{\frac{3\sqrt{\pi}}{16}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2}\eta\left(\frac{3}{2}\right) + c^2\sqrt{\pi}\eta\left(\frac{1}{2}\right)}$$

Ofcourse you could write the characteristic function as an infinite degree polynomial. But I will not bother. Below is a particular moment.

$$E[X^2] = \frac{\frac{\Gamma\left(\frac{7}{2}\right)\eta\left(\frac{7}{2}\right)}{8} + \frac{2c\Gamma\left(\frac{5}{2}\right)\eta\left(\frac{5}{2}\right)}{4} + \frac{c^2\Gamma\left(\frac{3}{2}\right)\eta\left(\frac{3}{2}\right)}{2}}{\frac{3\sqrt{\pi}}{16}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2}\eta\left(\frac{3}{2}\right) + c^2\sqrt{\pi}\eta\left(\frac{1}{2}\right)}$$
$$= \frac{\frac{\frac{15\sqrt{\pi}}{8}\eta\left(\frac{7}{2}\right)}{8} + \frac{2c\frac{3\sqrt{\pi}}{4}\eta\left(\frac{5}{2}\right)}{4} + \frac{c^2\frac{\sqrt{\pi}}{2}\eta\left(\frac{3}{2}\right)}{2}}{\frac{3\sqrt{\pi}}{16}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2}\eta\left(\frac{3}{2}\right) + c^2\sqrt{\pi}\eta\left(\frac{1}{2}\right)}$$
$$= \frac{\frac{15\eta\left(\frac{7}{2}\right)}{64} + \frac{3c\eta\left(\frac{5}{2}\right)}{8} + \frac{c^2\eta\left(\frac{3}{2}\right)}{4}}{\frac{3}{16}\eta\left(\frac{5}{2}\right) + \frac{c}{2}\eta\left(\frac{3}{2}\right) + c^2\eta\left(\frac{1}{2}\right)}$$

We take the derivative of the density.

$$\frac{d}{dx}\left(\frac{(x^2+c)^2}{1+e^{2x^2}}\right) = \frac{4x(x^2+c)(1+e^{2x^2}) - 4x(x^2+c)^2 e^{2x^2}}{(1+e^{2x^2})^2}$$

Look at the numerator.

$$4x(x^2+c)(1+e^{2x^2} - (x^2+c)e^{2x^2}) = 4x(x^2+c)(1+e^{2x^2}(1-x^2-c))$$

This is 0 when x is 0 no matter what c is. If $c < 0$ it is 0 when $x = \pm\sqrt{2}$. When $2 \geq c \geq 0$ it is 0 when x is s.t. $e^{-2x^2} = x^2 + c - 1$. When $c \geq 2$, the distribution is unimodal. If $c = 0$ distribution is bimodal. And in all other cases it is quadrimodal. Now apply the change of variable $x = \frac{y-\mu}{\sigma}$, then $dx = \frac{dy}{\sigma}$.

$$\frac{1}{\sigma\left[\frac{3\sqrt{\pi}}{16\sqrt{2}}\eta\left(\frac{5}{2}\right) + \frac{c\sqrt{\pi}}{2\sqrt{2}}\eta\left(\frac{3}{2}\right) + \frac{c^2\sqrt{\pi}}{\sqrt{2}}\eta\left(\frac{1}{2}\right)\right]} \frac{\left[\frac{(y-\mu)^2}{\sigma^2}+c\right]^2 e^{-\frac{(y-\mu)^2}{\sigma^2}}}{e^{-\frac{(y-\mu)^2}{\sigma^2}} + e^{\frac{(y-\mu)^2}{\sigma^2}}}$$

This has mean μ and variance $\frac{\sigma^2\left[\frac{15\eta\left(\frac{7}{2}\right)}{64}+\frac{3c\eta\left(\frac{5}{2}\right)}{8}+\frac{c^2\eta\left(\frac{3}{2}\right)}{4}\right]}{\frac{3}{16}\eta\left(\frac{5}{2}\right)+\frac{c}{2}\eta\left(\frac{3}{2}\right)+c^2\eta\left(\frac{1}{2}\right)}$. I tried to show that the sum of two random variables of this kind with $c = 0$ is also of the same kind (did not even try for $c \neq 0$). But the calcultion becomes very ugly so I did not go on. Theory of stable distribution may be of help in this pursuit. One value is that for $c = 0$, the density is 0 at $x = 0$ which is a property you can not obtain for a mixture of Gaussian distributions. More properties about the Dirichlet eta function may help in revealing more about this distribution.