Chudnovsky’s conjecture for (very) general points in $\mathbb{P}^N_C$

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AMS Fall Eastern Sectional Meeting
Rutgers University, New Brunswick, NJ

November 15, 2015
Question

Given a finite set of $n$ points $X = \{p_1, \ldots, p_n\}$ in $\mathbb{P}^N_{\mathbb{C}}$, what is the minimum degree $\alpha_m(X)$ of a hypersurface $f \neq 0$ passing through each $p_i$ with multiplicity (at least) $m$?

Theorem (Waldschmidt, Skoda, 1977)

Let $X$ be a set of $n$ points in $\mathbb{P}^N_{\mathbb{C}}$. Then

\[
\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X)}{N},
\]

where $\alpha(X) = \alpha_1(X)$ is the minimum degree of a hypersurface $f \neq 0$ passing through every point of $X$. 

Background
Theorem (Waldschmidt, Skoda, 1977)

Let $X$ be a set of $n$ points in $\mathbb{P}^N_C$. Then

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X)}{N},$$

where $\alpha(X) = \alpha_1(X)$ is the minimum degree of a hypersurface $f \neq 0$ passing through every point of $X$.

Theorem (Chudnovsky, 1981)

Let $X$ be a set of $n$ points in $\mathbb{P}^2_C$. Then

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + 1}{2}.$$
Theorem (Chudnovsky, 1981)

Let $X$ be a set of $n$ points in $\mathbb{P}_C^2$. Then

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + 1}{2}.$$ 

Conjecture (Chudnovsky, 1981)

Let $X$ be a set of $n$ points in $\mathbb{P}_C^N$. Then

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + N - 1}{N}.$$
Chudnovsky’s conjecture was known in the following cases:

- Any set of finite points in general position in $\mathbb{P}^3$ (Dumnicki, 2012).
- Any set of binomial number of points in $\mathbb{P}^N$ that form a star configuration (Bocci and Harbourne, 2010).
- Any set of $N + 1$ simple points in general position in $\mathbb{P}^N$ (Dumnicki, 2012).
Let $R = \mathbb{C}[x_0, \ldots, x_N]$ be the homogeneous coordinate ring of $\mathbb{P}^N_{\mathbb{C}}$ and $I$ a homogeneous $R$-ideal. Recall the $m$-th symbolic power of $I$ is defined as

$$I^{(m)} = \cap_{p \in \text{Ass}(R/I)} I^m R_p \cap R$$

Denote by $\alpha(J)$ the smallest degree of a non-zero element in a homogeneous ideal $J$.

Let $X = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}^N_{\mathbb{C}}$ and $I_X = \cap_{i=1}^n I(p_i)$, where $I(p_i)$ is the ideal generated by all forms that vanish at $p_i$.

Nagata and Zariski showed that

$$\alpha_m(X) = \alpha(I_X^{(m)}).$$
Thus we can rewrite Chudnovsky’s conjecture as

**Conjecture (Chudnovsky, 1981)**

Let \( I = \bigcap_{i=1}^{n} l(p_i) \) be an ideal of \( n \) points in \( \mathbb{P}_{\mathbb{C}}^N \). Then

\[
\frac{\alpha(l^{(m)})}{m} \geq \frac{\alpha(l) + N - 1}{N}.
\]
Recall Ein, Lazarsfeld and Smith in 2001 proved that for any homogeneous ideal $I$ in $R = \mathbb{C}[x_0, \ldots, x_N]$, one has

$$I^{(Nm)} \subseteq I^m.$$

Harbourne and Huneke in 2011 observed that

$$I^{(Nm)} \subseteq I^m \implies \frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N}$$

actually holds for every homogeneous ideal $I$ in $R$. 
\[ I^{(Nm)} \subseteq I^m \implies \frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N} \]

**Conjecture (Harbourne-Huneke, 2011)**

Let \( I = \cap_{i=1}^{n} I(p_i) \) be an ideal of \( n \) points in \( \mathbb{P}^N_C \). Then

\[ I^{(Nm)} \subseteq M^{m(N-1)} I^m, \]

where \( M = (x_0, \ldots, x_N) \) the homogeneous maximal ideal of \( R \).

Similarly,

\[ I^{(Nm)} \subseteq M^{m(N-1)} I^m \implies \frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + N - 1}{N}. \]

The converse holds if \( I \) is generated by a single degree.
Theorem (Fouli-Mantero-Xie)

Let $I = \bigcap_{i=1}^{n} I(p_i)$ be an ideal of $n$ (very) general points in $\mathbb{P}^N_C$. Then

$$\frac{\alpha(I^m)}{m} \geq \frac{\alpha(I) + N - 1}{N}.$$

Corollary (Fouli-Mantero-Xie)

Let $I$ be an ideal of $n$ (very) general points in $\mathbb{P}^N_C$, where $n = \binom{N+\beta-1}{N}$ for some $\beta \geq 1$. Then

$$I^{Nm} \subseteq M^{m(N-1)} I^m,$$

where $M = (x_0, \ldots, x_N)$ the homogeneous maximal ideal of $R$. 
Let 
\[ R = \mathbb{C}[x_0, \ldots, x_N] \subseteq S = \mathbb{C}(z)[x_0, \ldots, x_N], \]
where \( \mathbb{C} \subseteq \mathbb{C}(z) \) is an extension of fields by joining \( nN(N + 1) \)
variables \( z = (z_{ijr}), \ 1 \leq i \leq n, \ 1 \leq j \leq N, \ 0 \leq r \leq N. \)
For \( 1 \leq i \leq n \) and \( 1 \leq j \leq N \), define
\[ L_{ij} = z_{ij0}x_0 + z_{ij1}x_1 + \ldots + z_{ijn}x_N. \]
For every \( 1 \leq i \leq n \), let \( P_i \) be the point in \( \mathbb{P}^N_{\mathbb{C}(z)} \) defined by linear
forms \( L_{i1}, \ldots, L_{iN} \) and \( I(P_i) = (L_{i1}, \ldots, L_{iN})S. \)
Set 
\[ H = \cap_{i=1}^n I(P_i) = (L_{11}, \ldots, L_{1N})S \cap \ldots \cap (L_{n1}, \ldots, L_{nN})S. \]
For any vector $\lambda = (\lambda_{ijr}) \in \mathbb{C}^{nN(N+1)}$, we define for $1 \leq i \leq n$ and $1 \leq j \leq N$,

$$l_{ij} = \lambda_{ij0}x_0 + \lambda_{ij1}x_1 + \ldots + \lambda_{ijn}x_N.$$ 

For $1 \leq i \leq n$, let $p_i$ be a point that vanishes on the linear forms $l_{i1}, \ldots, l_{iN}$, we set $I(p_i) = (l_{i1}, \ldots, l_{iN})$ and define

$$H(\lambda) = \cap_{i=1}^n I(p_i) = (l_{11}, \ldots, l_{1N}) \cap \ldots \cap (l_{n1}, \ldots, l_{nN}).$$

We say that $P_1, \ldots, P_n$ are **generic** points in $\mathbb{P}^N_{\mathbb{C}(z)}$, and that $p_1, \ldots, p_n$ are **general** points in $\mathbb{P}^N_{\mathbb{C}}$ if the coefficient vector $\lambda$ is in a Zariski dense open subset of $\mathbb{C}^{nN(N+1)}$, i.e., $\lambda \in \mathbb{P}^N_{\mathbb{C}} \setminus C$, where $C$ is a proper Zariski closed subset of $\mathbb{C}^{nN(N+1)}$. 
We say that $P_1, \ldots, P_n$ are **generic** points in $\mathbb{P}_\mathbb{C}(z)$, and that $p_1, \ldots, p_n$ are **general** points in $\mathbb{P}_\mathbb{C}^N$ if the coefficient vector $\lambda$ is in a Zariski dense open subset of $\mathbb{C}^{nN(N+1)}$, i.e., $\lambda \in \mathbb{P}_\mathbb{C}^N \setminus C$, where $C$ is a proper Zariski closed subset of $\mathbb{C}^{nN(N+1)}$.

The points $p_1, \ldots, p_n$ are **(very) general points** in $\mathbb{P}_\mathbb{C}^N$, if $\lambda \in \mathcal{U} = \mathbb{C}^{nN(N+1)} \setminus \bigcup_{i=1}^{\infty} C_i$, where $C_i$ are proper Zariski-closed subsets of $\mathbb{C}^{nN(N+1)}$.

Observe the set $\mathcal{U}$ is a dense (but not necessarily open) subset of $\mathbb{C}^{nN(N+1)}$. 
Recall \( H = \bigcap_{i=1}^{n} I(P_i) \) is the ideal of \( n \) generic points in \( \mathbb{P}^{N}_\mathbb{C}(z) \).

**Theorem (Fouli-Mantero-Xie)**

Fix \( m \geq 1 \). Then for every \( \lambda \in \mathbb{C}^{nN(N+1)} \),

\[
\alpha(H^{(m)}) \geq \alpha(H(\lambda)^{(m)}).
\]

Moreover, there exists a Zariski dense open subset of \( \mathbb{C}^{nN(N+1)} \) for which the equality holds.
Proof.

Let \( t = \alpha(H^{(m)}) \). Set

\[
V_t = \{ \lambda = (\lambda_{ijr}) \in \mathbb{C}^{nN(N+1)} | \alpha(H(\lambda)^{(m)}) \leq t \}
\]

\[
= \{ \lambda = (\lambda_{ijr}) \in \mathbb{C}^{nN(N+1)} | \exists 0 \neq f \in H(\lambda)^{(m)} \text{ of degree } t \}.
\]

We prove \( V_t \) is a Zariski closed subset which also contains a dense open subset of \( \mathbb{C}^{nN(N+1)} \). Therefore \( V_t = \mathbb{C}^{nN(N+1)} \).
Indeed, let

$$A_{ij} = (-1)^j \det \begin{bmatrix} Z_{i10} & Z_{i11} & \cdots & \widehat{Z_{i1j}} & \cdots & Z_{i1N} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ Z_{iN0} & Z_{iN1} & \cdots & \widehat{Z_{iNj}} & \cdots & Z_{iNN} \end{bmatrix}.$$ 

Then \( P_i = (A_i) = (A_{i0}, \ldots, A_{iN}) \).

Let

$$f = \sum_{|\alpha| = t} C_\alpha x_\alpha,$$

where \(|\alpha| = \alpha_1 + \ldots + \alpha_N\) and \(x_\alpha = x_0^{\alpha_0} \cdot \ldots \cdot x_N^{\alpha_N}\).

The statement \( f \in H(\lambda)^{(m)} \) is equivalent to \( \partial_\beta f(P_i) = 0 \) for all \( \beta \) with \(|\beta| \leq m - 1\) and all points \( P_1, \ldots, P_n \).
Proposition (Fouli-Mantero-Xie)

Chudnovsky’s conjecture holds for $n$ generic points if it is proved for sets of $\binom{N+\beta-1}{N}$ generic points for all $\beta \geq 1$.

Proof.

$$t = \binom{N + \beta - 1}{N} \leq n < \binom{N + \beta}{N}.$$  

Then $H = \bigcap_{i=1}^{n} I(P_i) \subseteq J = \bigcap_{i=1}^{t} I(P_i)$ with $\alpha(J) = \alpha(H)$. 

Since $H^{(m)} \subseteq J^{(m)}$, $\alpha(H^{(m)}) \geq \alpha(J^{(m)})$. 

If Chudnovsky’s conjecture holds for $\binom{N+\beta-1}{N}$ generic points then

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(J^{(m)})}{m} \geq \frac{\alpha(J) + N - 1}{N} = \frac{\alpha(H) + N - 1}{N}.$$
Theorem (Fouli-Mantero-Xie)

Let $H = \cap_{i=1}^{n} I(P_i)$ be the ideal of $n$ generic points in $\mathbb{P}^N_{\mathbb{C}(z)}$. Then

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(H) + N - 1}{N}.$$ 

Proof. We may assume $n = \binom{N+\beta-1}{N}$ for some $\beta \geq 1$. Let $\lambda \in \mathbb{C}^{nN(N+1)}$ be such that $H(\lambda)$ is the defining ideal of $n$ points in $\mathbb{P}^N_{\mathbb{C}}$ which form a star configuration.

One has that $\alpha(H(\lambda)) = \alpha(H)$. Now

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(H(\lambda)^{(m)})}{m} \geq \frac{\alpha(H(\lambda)) + N - 1}{N} = \frac{\alpha(H) + N - 1}{N}.$$
Theorem (Fouli-Mantero-Xie)

Let $I = \bigcap_{i=1}^{n} I(p_i)$ be the ideal of $n$ (very) general points in $\mathbb{P}_{\mathbb{C}}^N$. Then

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + N - 1}{N}.$$ 

Proof. Let $H = \bigcap_{i=1}^{n} I(P_i)$. We have

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(H) + N - 1}{N}.$$ 

For each $m \geq 1$, there exists a proper Zariski closed subset $C_m \in \mathbb{C}^{nN(N+1)}$ such that if $\lambda \in \mathbb{C}^{nN(N+1)} \setminus C_m$, then

$$\alpha(H^{(m)}) = \alpha(H(\lambda)^{(m)}).$$

Now let $\lambda \in \mathcal{U} = \mathbb{C}^{nN(N+1)} \setminus \bigcup_{m=1}^{\infty} C_m$. Take $I = H(\lambda) = \bigcap_{i=1}^{n} I(p_i)$. 
Homogeneous ideals in $\mathbb{C}[x_0, \ldots, x_n]$

**Theorem (Fouli-Mantero-Xie)**

Let $I$ be a homogeneous ideal in $R$. There exists an integer $t_0 > 0$ such that if $t \geq t_0$, then the symbolic power $I^{(t)}$ satisfies Chudnovsky’s conjecture, i.e.,

$$\frac{\alpha((I^{(t)})^m)}{m} \geq \frac{\alpha(I^{(t)}) + N - 1}{N}.$$

**Corollary (Fouli-Mantero-Xie)**

Let $I = \bigcap_{i=1}^n I(p_i)$ be an ideal of $n$ points in $\mathbb{P}_\mathbb{C}^N$. Then for every $t \geq N - 1$, the symbolic power $I^{(t)}$ satisfies Chudnovsky’s conjecture.