$j$-multiplicity and depth of associated graded rings

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Outline

- Introduction
- Main results
- An example
In this talk, we will fix the following setting:

- $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and residue field $k$.
- $d = \dim R$.
- $I$ is an $R$-ideal.
- $G = \text{gr}_I(R) = \bigoplus_{j=0}^{\infty} I^j / I^{j+1}$ is the associated graded ring of $R$ with respect to $I$. 

It is well known that $G$ reflects various algebraic and geometric properties of the ideal $I$. For instance, $\text{proj}(G)$ is the exceptional fibre of the blow-up of $\text{spec}(R)$ along the subvariety $V(I)$.

One is particularly interested in $\text{depth}(G)$.

- $\text{depth}(G)$ gives information on the vanishing of cohomology groups of the exceptional fiber of the blow-up and of the blow-up itself.

- The Cohen-Macaulayness or Gorensteinness of $G$ provide a wealth information about how $I$ sits in $R$ and one could compute various numerical invariants of the exceptional fiber and of the blow-up (Castelnuovo-Mumford regularity, number of defining equations, ...).
Method 1. \( \sqrt{I} = \mathfrak{m} \).

Use assumptions on the Hilbert-Samuel multiplicity.

(Corso, Elias, Vaz Pinto, Polini, Rossi, Valla, Sally, Wang, Puthenpurakal, ...)

Remark: In the first method, the Hilbert function is also completely determined.

Method 2: \( I \) is any ideal.

Use (i) assumptions on residual intersections and depth of the powers of \( I \) (this is automatic if \( \sqrt{I} = \mathfrak{m} \)) and (ii) conditions on reduction number of \( I \) (replace the assumptions on the multiplicity).

(Aberbach, Goto, Herzog, Huneke, Huckaba, Johnson, Nakamura, Nishida, Simis, Trung, Ulrich, Vasconcelos, ...)
Introduction

In our approach: $I$ is any ideal.

Use assumptions on $j$-multiplicity of $I$ and residual intersections.

Remark: In our approach, we also defined the generalized Hilbert function and determine its shape (in progress).
Case 1. \( l = m \).

(Abhyankar, 1967)

\[
e(R) = e(m) \geq (\mu(m) - d) + 1 = \text{ecodim}(R) + 1,
\]

where \( e(m) \) is the Hilbert-Samuel multiplicity of the maximal ideal \( m \), \( \mu(m) \) is the number of a minimal generating set of \( m \) and \( \text{ecodim}(R) \) is the embedding codimension of \( R \).

\( R \) has **minimal multiplicity** if \( e(R) = \text{ecodim}(R) + 1 \).
Theorem

1. (Sally, 1980’s) If $e(R) = \text{ecodim}(R) + 1$,
or $e(R) = \text{ecodim}(R) + 2$ and $\text{type}(R) < \text{ecodim}(R)$,
or $e(R) = \text{ecodim}(R) + 3$ and $R$ is Gorenstein,
then $G$ is Cohen-Macaulay.

2. (Rossi-Valla, 1996, Wang, 1997) If $e(R) = \text{ecodim}(R) + 2$,
then $G$ is almost Cohen-Macaulay, i.e., $\text{depth } G \geq d - 1$.

3. (Rossi-Valla, 2000) If $e(R) = \text{ecodim}(R) + 3$ and $\text{type}(R) < \text{ecodim}(R)$,
then $G$ is almost Cohen-Macaulay.
Case 2. $I$ is $m$-primary.

The generalized Abhyankar’s inequality:

$$e(I) \geq \left[ \lambda(I/I^2) - (d - 1)\lambda(R/I) \right] + \lambda(I^2/JI)$$

$$\geq \lambda(I/I^2) - (d - 1)\lambda(R/I)$$

where $e(I)$ is the Hilbert-Samuel multiplicity of the ideal $I$, $\lambda(I/I^2)$ is the length of $I/I^2$.

$I$ has **minimal multiplicity** if

$$e(I) = \lambda(I/I^2) - (d - 1)\lambda(R/I).$$
The case of $m$-primary ideals was done by Corso-Polini-Vaz Pointo, Rossi, Elias.

Rossi and Valla generalized the result to $R = M$ with $\lambda(M/IM) < \infty$.

The tools used are superficial sequences, Sally machine, Valabrega-Valla Criterion and Ratliff-Rush filtrations.
Main results

Case 3. \( I \) is any ideal.

**j-multiplicity** (Achilles-Manaresi, 1993)

Consider \( H^0_m(G) = 0 :_G m^\infty \).

Observe \( H^0_m(G) \) is a finite graded module over \( G/m^sG \) for some \( s > 0 \) of dimension \( \leq d \). Thus for \( t >> 0 \),

\[
\lambda([H^0_m(G)]_t) = \frac{j(I)t^{d-1}}{(d-1)!} + \text{lower terms}.
\]

The normalized leading coefficient \( j(I) \) is called the **j-multiplicity** of \( I \).

Observe

- \( j(I) \neq 0 \iff \dim H^0_m(G) = d \iff \ell(I) := \dim G/mG = d. \)
- If \( I \) is \( m \)-primary then \( j(I) = e(I) \).
Main results, Continued

From now on, we assume $\ell(I) = d = \text{dim } R$. 
Theorem (Achilles-Manaresi, Nishida-Ulrich, Xie)

Assume $R$ has infinite residue field. For general elements $x_1, \ldots, x_d$ in $I$, set $a = (x_1, \ldots, x_{d-1})R$, then

$$j(I) = e(I, R/a : I^\infty) = \lambda(R/(a : I^\infty + x_d R)).$$

It follows from the following facts:

1. Achilles-Manaresi (1993): The formula holds for any super-reduction sequence $x_1, \ldots, x_d$ in $I$.
3. Xie (2009): General elements $x_1, \ldots, x_d$ in $I$ form super-reduction sequences.
Let $\overline{R} = R/\alpha : I^\infty$ and $\overline{I} = I \overline{R}$, then $\overline{R}$ is a 1-dimensional Cohen-Macaulay local ring and $\overline{I}$ is $\overline{m}$-primary. Furthermore,

$$j(I) = e(I) \geq \lambda(I/I^2) + \lambda(I^2/x_d I) \geq \lambda(I/I^2).$$

**Lemma (Polini-Xie)**

$\lambda(I/I^2)$ and $\lambda(I^2/x_d I)$ are independent of $x_1, \ldots, x_d$. 
Definition (Polini-Xie)

$I$ has **minimal** $j$-multiplicity if for general elements $x_1, \ldots, x_d$ in $I$, 

$$j(I) = \lambda(\bar{I}/\bar{I}^2).$$

$I$ has **almost minimal** $j$-multiplicity if for general elements $x_1, \ldots, x_d$ in $I$, 

$$j(I) = \lambda(\bar{I}/\bar{I}^2) + 1.$$

Observe

$$j(I) = \lambda(\bar{I}/\bar{I}^2) \iff \bar{I}^2 = x_d \bar{I} \iff I^2 = x_d I + (x_1, \ldots, x_{d-1} : I^\infty) \cap I^2.$$
Main results, Continued

The ideal $I$ has the **condition $G_d$**, if for every $p \in V(I) \setminus \{m\}$, $\mu(I_p) \leq \text{ht } p$, where $\mu(I_p)$ is the number of a minimal generating set of $I_p$ and $\text{ht } p$ is the height of $p$.

Let $H = (x_1, \ldots, x_t) : I$, where $(x_1, \ldots, x_t) \subsetneq I$. $H$ is said to be a **geometric $t$-residual intersection of $I$**, if

- $\text{ht } H \geq t \geq \text{ht } I$.
- $\text{ht } (H + I) \geq t + 1$.

The ideal $I$ has the **Artin-Nagata property $AN_{d-2}$** if for every $i \leq d - 2$ and every geometric $i$-residual intersection $H$ of $I$, $R/H$ is Cohen-Macaulay.
I has $G_d$ and $AN_{d-2}$ if

(1) $I$ is $m$-primary;

(2) $I$ is generically a complete intersection with $\dim R/I = 1$;

(3) $I$ has $G_d$ and sliding depth or strongly Cohen-Macaulay;

(4) $I$ has $G_d$ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \dim R/I - 1$.

**Theorem (Goto-Nakamura-Nishida, Johnson-Ulrich)**

Assume $I$ has $G_d$ and condition (4). If the reduction number $r(I) \leq \dim R/I + 1$, then $G$ is Cohen-Macaulay.
Main results, Continued

**Theorem (Polini-Xie)**

Assume \( \text{depth} \left( \frac{R}{I} \right) \geq \min \{ \dim \frac{R}{I}, 1 \} \) and \( I \) has \( G_d \) and \( AN_{d-2}^- \). One has the following statements:

1. If \( I \) has minimal \( j \)-multiplicity then \( G \) is Cohen-Macaulay.

   If in addition, \( R \) is Gorenstein and

   \[
   \text{depth} \left( \frac{R}{I^j} \right) \geq \dim \frac{R}{I} - j + 1 \text{ for } 1 \leq j \leq \dim \frac{R}{I} - 1,
   \]

   then \( G \) is Gorenstein.

2. If \( I \) has almost minimal \( j \)-multiplicity then \( G \) is almost Cohen-Macaulay.
Proof: 1. May assume that $I$ is not $m$-primary. Observe

$$j(I) = \lambda(\bar{I}/\bar{I}^2) \iff \bar{I}^2 = x_d \bar{I}.$$  Therefore

$$I^2 = x_d I + (x_1, \ldots, x_{d-1} : I^\infty) \cap I^2 = (x_1, \ldots, x_d)I$$

where the second equality follows from residual assumptions. This shows $r(I) = 1$. By Goto-Nakamura-Nishida, Johnson-Ulrich, $G$ is Cohen-Macaulay or Gorenstein.

2. Use a very deep combination of the methods used for $m$-primary ideals with tools in residual intersection theory.
Example (Nishida-Ulrich)
Let $S$ be a 3-dimensional Cohen-Macaulay local ring and $x, y, z$ are system of parameters for $S$. Set

$$R = S/(x^2 - yz)S, \quad I = (x, y)R.$$ 

Then $I$ has minimal $j$-multiplicity. Indeed, let $\xi$ be a general element in $I$, $\overline{R} = R/\xi R : I^\infty$ and $\overline{I} = I\overline{R}$, then

$$j(I) = \lambda(\overline{I}/\overline{I}^2) = \lambda_S(S/(x, y, z)S) \neq 0.$$ 

Observe $R$ is a Cohen-Macaulay local ring of dimension 2, $I$ is a Cohen-Macaulay ideal of $\dim R/I = 1$ which is generically a complete intersection. By our theorem, $G$ is Cohen-Macaulay and if in addition $S$ is Gorenstein, then $G$ is also Gorenstein.