Cohen-Macaulayness of conormal modules

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Outline

- Introduction
- Main results
- Negative examples
Recall: For a Noetherian local ring $R$ and an $R$-ideal $I$, $I/I^2$ is called the first conormal module of $I$.

Goal: We study the Cohen-Macaulayness of $I/I^2$ for some classes of ideals, namely ideals defining stretched algebras, short algebras and algebras with low multiplicity.
Motivation: $I/I^2$ plays an important role in the process of understanding the structure of $I$.

For example,

- Theorem (Vasconcelos, 1967): Assume $\text{Proj}_R I < \infty$. Then the freeness of $I/I^2$ (as $R/I$ module) is equivalent to $I$ being a complete intersection.

- Conjecture 1 (Vasconcelos, 1978): Assume $\text{Proj}_R I < \infty$. Then $\text{Proj}_{R/I} I/I^2 < \infty$ is equivalent to $I$ being a complete intersection.

(Conjecture 1 has been proved for some classes of ideals (0-dimensional ideals, almost complete intersection ideals, perfect ideals of height two or Gorenstein of height three, ideals of low projective dimension) in a Noetherian local ring (Vasconcelos, 1978) and for ideals defining finitely generated graded $k$-algebras. (Avramov-Herzog, 1994))
Conjecture 2 (Vasconcelos, 1987): Assume $R$ is a Gorenstein local ring and $I$ is a height 3 syzygetic perfect ideal. If $I/I^2$ is Cohen-Macaulay then $R/I$ is Gorenstein.

(Conjecture 2 is still wide-open. It was done for almost complete intersection ideals and homogeneous ideals of type 2 having a pure resolution (Vasconcelos, 1987))
Conjecture 2 was extended later to the following question:

**Question A**

Let $R$ be a regular local ring and $I$ a perfect ideal that is generically a complete intersection (i.e., $I_p$ is a complete intersection $R_p$-ideal for every $p \in \text{ass}_R(R/I)$). If $I/I^2$ is Cohen-Macaulay, then does $R/I$ have to be Gorenstein?

Question A has been proved to be true for licci ideals (Huneke-Ulrich, 1989). In particular, it is true for all perfect ideals of height 2. Recently it has been shown to be true for all squarefree-monomial ideals (Terai-Yoshida).
In our work, by studying the Cohen-Macaulayness of $I/I^2$, we give positive answer to Question A for some classes of ideals. More precisely, we prove Question A is true for the following classes of ideals:

- $I$ defines a stretched algebra and the residue field of $R$ has characteristic zero.
- $I$ defines a short algebra with socle degree at least 3.
- $I$ defines a short algebra with socle degree 2 and its multiplicity satisfies some numerical conditions.
- $I$ defines an algebra with multiplicity less than or equal to height $I + 4$ and the residue field of $R$ has characteristic zero.
Furthermore, using CoCoa, J. C. Migliore found that the ideal $I$ defining a set of 10 (general) points in $\mathbb{P}^5$ gives a negative answer to Question A. Since this example has multiplicity $10 = \text{height } I + 5$, this shows the sharpness of our results.

Using tools from linkage theory, we deform this homogeneous level ideal $I$ to a perfect prime ideal $p$ in a regular local ring with $p/p^2$ Cohen-Macaulay but $p$ is not Gorenstein, proving that the general answer to Question A is negative even for prime ideals. This method shows in particular that Question A can be reduced to the case of prime ideals.
Main results

From now on, for simplicity, we will always assume

- $(R, m)$ is a regular local ring with maximal ideal $m$ and $\text{char } R/m = 0$.
- $I \subseteq m^2$ is an $R$-ideal of height $c$.

Remark:
1. The condition $\text{char } R/m = 0$ is not need for the case of short algebras.
2. We can always reduce to the case that $I \subseteq m^2$.
3. Our method also works in the homogeneous settings.
Notation: Let \((A, \mathfrak{n})\) be an Artinian local ring.

The *socle degree* \(\text{socdeg}(A)\) is the positive integer \(s\) with \(\mathfrak{n}^{s+1} = 0\) and \(\mathfrak{n}^s \neq 0\). The *socle* \(\text{soc}(A) = 0 :_A \mathfrak{n}\).

The *type* \(\tau(A) = \dim_{A/\mathfrak{n}} \text{soc}(A)\).

The *socle degree* \(\text{socdeg}(R)\), the *socle* \(\text{soc}(R)\) and the *type* \(\tau(R)\) are then defined respectively to be \(\text{socdeg}(R/J)\), \(\text{soc}(R/J)\) and \(\tau(R/J)\), where \(J\) is any minimal reduction of \(m\).

The ring \(R\) is *Gorenstein* if in addition \(\tau(R) = 1\).

We use \(e(R)\) to denote the Hilbert-Samuel multiplicity of \(R\) with respect to its maximal ideal \(m\).
Case 1. Stretched algebras.

Definition

An Artinian local ring \((A, \mathfrak{n})\) is said to be stretched if \(\mathfrak{n}^2\) is a principal ideal. Let \(I\) be a perfect ideal in \(R\), \(I\) is stretched (or \(R/I\) is stretched) if \(R/(I + (\underline{y}))\) is a stretched Artinian local ring for any \(\underline{y} = y_1, \ldots, y_t\) a minimal reduction of \(m_{R/I}\).

As a consequence, the Hilbert Function of a stretched Artinian local ring \((A, \mathfrak{n})\) is:

\[
\begin{array}{ccccccccc}
1 & c & 1 & 1 & \ldots & 1 & 0 \\
\end{array}
\]

Thus if \(I\) is stretched of \(\text{socdeg}(R/I) = s\), then \(e(R/I) = c + s\).
Theorem (Mantero-Xie)

Assume $I$ is a stretched perfect ideal that is generically a complete intersection. Recall $c = \text{height } I$.

(a) If $c \leq 3$ and $R/I$ is not Gorenstein then $I/I^2$ is not Cohen-Macaulay.

(b) If $c \geq 4$ then $I/I^2$ is not Cohen-Macaulay.

Corollary (Mantero-Xie)

Question A is true for ideals defining stretched algebras.
**Sketch of the proof:** Since $I$ is generically a complete intersection of height $c$, by the Associativity formula, 
\[ e(R/I^2) = (c + 1)e(R/I) = (c + 1)(c + s). \]

Assume by contradiction that $I/I^2$ is Cohen-Macaulay. Since $R/I$ is also Cohen-Macaulay, the short exact sequence 
\[ 0 \rightarrow I/I^2 \rightarrow R/I^2 \rightarrow R/I \rightarrow 0 \]
shows that $R/I^2$ is Cohen-Macaulay as well, hence there exists a regular sequence $y$ on $R$ that generates a minimal reduction of $m_{R/I}$ and $m_{R/I^2}$.

After modulo $y$, we may assume that $R$ is a $c$-dimensional regular local ring, $I$ is a stretched $m$-primary ideal of socle degree $s$ and 
\[ \lambda(R/I^2) = (c + 1)(c + s). \]
However, since $R/I$ is a stretched Artinian local ring of socle degree $s$, our next step is to use information on the Hilbert Function of $R/I$ (sometimes together with the non-Gorenstein assumption on $R/I$) to estimate the length of $R/I^2$.

Finally these estimates shows that $\lambda(R/I^2) > (c + 1)(c + s)$, contradicting the above equality and showing that if $R/I$ is not Gorenstein then $I/I^2$ cannot be Cohen-Macaulay.
To estimate the length of $R/I^2$, we need the following structure theorem for ideals defining 0-dimensional stretched local rings.

**Theorem (Sally, Elias-Valla)**

Assume $\dim R = c$. Let $I \subseteq m^2$ be an $m$-primary ideal with $R/I$ stretched of socle degree $s$. Write $\tau(R/I) = r + 1$ for some $0 \leq r \leq c - 1$. Then there exist minimal generators $x_1, \ldots, x_c$ for the maximal ideal $m$ and elements $u_{r+1}, \ldots, u_{c-1} \notin m$ with

$$I = (x_1m, \ldots, x_rm) + J$$

where

$$J = (x_{r+i}x_{r+j} \mid 1 \leq i < j \leq c-r) + (x_c^s - u_{r+i}x_{r+j}^2 \mid 1 \leq i \leq c-r-1)$$

if $r < c - 1$ and $J = (x_c^{s+1})$ if $r = c - 1$. 
We now employ the knowledge of this class of ideals to obtain precise information about their square.

**Proposition (Mantero-Xie)**

*Let R and I be the same as in the above theorem. Then*\n
\[ I^2 \subseteq L = (x_1, \ldots, x_{c-1})H + (x_1, \ldots, x_{c-1})x_c^{s+1} + (x_c^{2s}), \]

*where H is the monomial ideal generated by all monomials of degree 3 in x_1, \ldots, x_c except for x_3^c. The above inclusion is strict if and only if \( \tau(R/I) \geq c - 1 \), and in this case \( \lambda(R/I^2) \geq \lambda(R/L) + 2 \).*

The result is used to obtain tight estimates for the Hilbert Function of \( R/I^2 \), indeed it shows that one can replace \( I^2 \) by the ‘monomial’ ideal \( L \) without changing too much the Hilbert Function.
The Hilbert Function of $R/L$ has the shape:

1 $c$ $h_2$ $h_3$ ... 

where $h_{j+1} \geq c$, $3 \leq j \leq s$ and $h_j \geq 1$, $s + 2 \leq j \leq 2s - 1$.

Therefore

\[
\lambda(R/I^2) \geq \lambda(R/L) \geq 1 + c + \binom{c+1}{2} + \binom{c+2}{3} + c(s - 2) + s - 2
\]

which is strictly greater than $(c + 1)(c + s)$ if $c \geq 4$.

Hence, if $c \geq 4$, $\lambda(R/I^2) > (c + 1)(c + s)$, a contradiction.

If $c = 3$, this amount actually equals $(c + 1)(c + s)$. However, since $R/I$ is not Gorenstein, $\tau(R/I) \geq 2 = c - 1$. Again by the above proposition, $\lambda(R/I^2) > \lambda(R/L) = (c + 1)(c + s)$. 
Case 2. Short algebras.

Set \( n_j = \binom{c+j-1}{j-1} \) and \( N_j = \binom{c+j}{j} \).

Definition

An Artinian local ring \((A, n)\) is short if there exist integers \( c \) and \( s \) with \( HF_A(j) = n_j \) for every \( j < s \), \( HF_A(s) = n_s - q \), \( 0 \leq q < n_s \), and \( HF_A(s+1) = 0 \).

Let \( I \) be a perfect ideal in \( R \), \( I \) is short (or \( R/I \) is short) if \( R/(I + (y)) \) is a short Artinian local ring for any \((y)\) a minimal reduction of \( m_{R/I} \).
The Hilbert function of a short Artinian local ring \((A, \mathfrak{n})\):

\[
1 \quad c \quad n_2 \quad n_3 \quad \ldots \quad n_{s-1} \quad n_s - q \quad 0 \rightarrow
\]

Thus if \(I\) is short with \(\text{socdeg}(R/I) = s\), then

\[
e(R/I) = \sum_{i=0}^{s} n_i - q = N_s - q.
\]
Theorem (Mantero-Xie)

Assume $I$ is a short perfect ideal that is generically a complete intersection.

(a) If $s = \text{socdeg}(R/I) \geq 3$ and $c \geq 2$, then $I/I^2$ is not Cohen-Macaulay.

(b) If $s = \text{socdeg}(R/I) = 2$ and its multiplicity satisfies some numerical conditions, then $I/I^2$ is not Cohen-Macaulay.

Corollary (Mantero-Xie)

Question A is true for any short ideal $I$ with socle degree at least 3 or socle degree 2 with multiplicity satisfying some numerical conditions.
Sketch of the proof: Part(a). As before, assume $I/I^2$ is Cohen-Macaulay. After moding out a minimal reduction, we may assume $R/I$ is a short Artinian local ring of socle degree $s$ with length $\lambda(R/I^2) = (c + 1)e(R/I) = (c + 1)(N_s - q)$.

On the other hand, since $I^2 \subseteq \mathfrak{m}^{2s}$, 
$\lambda(R/I^2) \geq \lambda(R/\mathfrak{m}^{2s}) = \sum_{i=0}^{2s-1} n_i = N_{2s-1}$.

Hence, to finish the proof, it is enough to show that $N_{2s-1} > (c + 1)(N_s - q)$ for all $c \geq 2$ and $s \geq 3$.

Let $Q(c, s) = N_{2s-1} - (c + 1)N_s$. We can prove that if $Q(\bar{c}, \bar{s}) > 0$ for some positive integers $\bar{c} \geq 2$ and $\bar{s} \geq 3$, then $Q(c, s) > 0$ for every $c \geq \bar{c}$ and $s \geq \bar{s}$. 
Since $Q(5, 4) = 6 > 0$, $Q(4, 5) = 17 > 0$, $Q(3, 6) = 7 > 0$ and $Q(2, 8) = \frac{1}{3} > 0$, it follows that $Q(c, s) > 0$ either if $c \geq 5$, $s \geq 4$, or if $c \geq 4$, $s \geq 5$, or if $c \geq 3$, $s \geq 6$ or if $c \geq 2$, $s \geq 8$.

The few cases left out can be done by direct computations.

Part (b) is proved by using similar methods and estimates.
Case 3. Algebras having low multiplicity.

Let $I \subseteq m^2$ be a perfect ideal in $R$. It is well-known (by Abhyankar’s inequality) that $e(R/I) \geq c + 1$, where $c = \text{height } I$. We will refer to ideals $I$ with $e(R/I) \leq c + 4$ as ‘ideals with low multiplicity’.

Theorem (Mantero-Xie)

Let $I$ be a perfect ideal that is generically a complete intersection. Assume $e(R/I) \leq c + 4$. If $R/I$ is not Gorenstein, then $I/I^2$ is not Cohen-Macaulay.
One application of the previous result is to show that for some ideals defining a monomial curve, the Cohen-Macaulayness of the first conormal module forces the ideal (hence the monomial curve) to be Gorenstein.

**Corollary (Mantero-Xie)**

Let $S = k[t^{a_1}, \ldots, t^{a_n}]$ for some $a_1 \leq a_2 \leq \cdots \leq a_n$ and assume that none of the $a_i$’s is redundant. Let $R = k[X_1, \ldots, X_n]$ and $I$ the homogeneous $R$-ideal defining $S$. If $a_1 \leq n + 3$ then Question A holds true for $I$. 


Negative examples

We showed that for ideals with multiplicity up to $c + 4$, Question A has a positive answer. However, in multiplicity $c + 5$, there is an example for which Question A has a negative answer. It has been found by J. C. Migliore using CoCoA.

Example

(a) (Migliore) Assume $S = \mathbb{Q}[a, b, c, d, e, f]$ is a polynomial ring. Then for a homogeneous level ideal $I$ in $S$ defining 10 general points, Question A has a negative answer. In particular, one has $\dim(S/I) = 1$, $c = 5$, $e(S/I) = c + 5$, $S/I$ is a short algebra with $\text{socdeg}(S/I) = 2$ and $\tau(S/I) = 4$.

(b) There exist a regular local ring $R$ and a prime ideal $\mathfrak{p}$ with $e(R/\mathfrak{p}) = c + 5$ for which Question A has a negative answer.
Using CoCoA, it has been found that also for the ideal defining 12 (general) points in $\mathbb{P}^6$, Question A has a negative answer. Similarly for 15 (general) points in $\mathbb{P}^7$. We suggest that for every $c \geq 5$, the smallest set of (general) points of $\mathbb{P}^c$ defining an ideal that is not covered by our theorem gives an ideal for which Question A has a negative answer. If true, this would provide a class of examples in any codimension at least 5 for which Question A has a negative answer.

Let $\lceil x \rceil$ denote the smallest integer bigger than or equal to $x$.

**Question B**

Is it true that, for any $c \geq 5$, the ideal defined by a set of $1 + c + \lceil \frac{c(c-1)}{6} \rceil$ general points in $\mathbb{P}^c$ gives a negative answer to Question A?

Computations made by J. C. Migliore confirmed that Question B holds true for any $c \leq 9$. 
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