Minimal $j$-multiplicity

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Outline

- Introduction
- Main results
- An example
In this talk, we will fix the following setting:

- \((R, m, k)\) is a Noetherian local ring with maximal ideal \(m\) and residue field \(k\).
- \(d = \dim R\).
- \(I\) is an \(R\)-ideal.
- \(G = \text{gr}_I(R) = \bigoplus_{j=0}^{\infty} I^j / I^{j+1}\) is the associated graded ring of \(R\) with respect to \(I\).

Observe

\[
\dim G = \dim R = d.
\]
Hilbert-Samuel multiplicity

If $I$ is primary to the maximal ideal $m$, then the Hilbert-Samuel multiplicity of $R$ with respect to $I$ is defined as

$$e(I) = (d - 1)! \lim_{t \to \infty} \frac{\lambda(I^t / I^{t+1})}{t^{d-1}}.$$  

Define $e(R) = e(m)$ the multiplicity of the local ring $R$. 
If $R$ is Cohen-Macaulay, one has the following inequality:

$$e(I) \geq \lambda(I/I^2) - (d - 1)\lambda(R/I).$$

- if $I = m$, it gives **Abhyankar’s inequality (1967):**

  $$e(R) = e(m) \geq \mu(m) - (d - 1) = 1 + (\mu(m) - d),$$

  where $\mu(m)$ is the number of generators of $m$.

- if $\dim R = 1$, then

  $$e(I) \geq \lambda(I/I^2).$$
$R$ is said to have \textit{minimal multiplicity} with respect to $I$, if

$$e(I) = \lambda(I/I^2) - (d - 1)\lambda(R/I).$$

$R$ is said to have \textit{almost minimal multiplicity} with respect to $I$, if

$$e(I) = \lambda(I/I^2) - (d - 1)\lambda(R/I) + 1.$$
Theorem (Sally, 1977)

Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \(d\). If \(R\) has minimal multiplicity (i.e., \(e(R) = \mu(m) - (d - 1)\)), then \(G = \text{gr}_m(R)\) is Cohen-Macaulay.
Theorem (Rossi-Valla (1996), Wang (1997))

Let $(R, m)$ be a Cohen-Macaulay local ring of dimension $d$. If $R$ has almost minimal multiplicity (i.e., $e(R) = \mu(m) - (d - 1) + 1$), then

$$\text{depth}(\text{gr}_m(R)) \geq d - 1.$$
The above two theorems have been generalized to Cohen-Macaulay modules with $I$-filtrations, where $I$ is an ideal with $\lambda(M/IM) < \infty$ (Huckaba, Corso-Polini-Pinto, Rossi, Elias, Rossi-Valla).

Our goal is to generalize their results to Cohen-Macaulay modules with $I$-filtrations, where $I$ is arbitrary, i.e., we DO NOT assume $\lambda(M/IM) < \infty$. 
Main results

**j-multiplicity** (Achilles-Manaresi)

For an arbitrary ideal $I$, recall $\ell(I) = \dim(G \otimes_R R/m)$.

Let $\Gamma_m(G) = 0 :_G m^\infty = \bigoplus_{j=0}^{\infty} \Gamma_m(I^j/I^{j+1})$.

Since $\dim \Gamma_m(G) \leq d$, define

$$j(I) = (d - 1)! \lim_{t \to \infty} \frac{\lambda(\Gamma_m(I^t/I^{t+1}))}{t^{d-1}}.$$

Observe

- $j(I) \neq 0 \iff \dim \Gamma_m(G) = d \iff \ell(I) = d$.
- If $I$ is primary to $m$, then $j(I) = e(I)$. 
Let $x \in I$ and $x^*$ the initial form of $x$ in $G$.

Recall $x^*$ is said to be \textit{filter-regular} for $G$ with respect to $G_+$, if there exists a non-negative integer $c$ such that for $t \geq c$,

$$(I^{t+1} : x) \cap I^c = I^t.$$ 

A sequence $x_1^*, \ldots, x_s^*$ of elements of $G$ is said to be a \textit{filter-regular sequence} for $G$ with respect to $G_+$, if $x_i^*$ is \textit{filter-regular} for $G/(x_1^*, \ldots, x_{i-1}^*)G$ with respect to $G_+$ for $i = 1, \ldots, s$. 


Assume $\ell(I) = d = \dim R$.

A sequence of elements $x_1, \ldots, x_d$ of $I$ will be called a super-reduction for $I$ if

- the initial forms $x_1^*, \ldots, x_d^*$ in $G$ are of degree one and form a filter-regular sequence for $G$ with respect to $G_+$.

- for every relevant highest-dimensional prime ideal $q$ of $G$ with $d(q) = \dim G/(mG + q)$, the initial forms $x_1^*, \ldots, x_d^*(q)$ are a system of parameters for $G/(mG + q)$. 
Let \( I = (a_1, \ldots, a_n) \).

If the residue field \( k = R/m \) is infinite, for general elements \( (\lambda_{ij}) \in k^{dn} \), let \( x_i = \sum_{j=1}^{n} \lambda_{ij}a_j \) for \( 1 \leq i \leq d \), then \( x_1, \ldots, x_d \) form a super-reduction of \( I \).
The $j$-multiplicity can be computed using super-reductions of $I$. Indeed, for any super-reduction $x_1, \ldots, x_d$ of $I$,

$$ j(I) = \lambda(R/((x_1, \ldots, x_{d-1}) : I^\infty + x_d)) $$
$$ = e(I, R/((x_1, \ldots, x_{d-1}) : I^\infty)) = e(\overline{I}) $$

where $\overline{R} = R/((x_1, \ldots, x_{d-1}) : I^\infty)$ and $\overline{I} = I\overline{R}$.

Notice that $\overline{R}$ is an 1-dimensional Cohen-Macaulay local ring and $\overline{I}$ is an ideal which is $\overline{m}$-primary. Thus

$$ j(I) = e(\overline{I}) \geq \lambda(\overline{I}/\overline{I}^2). $$
Lemma (Polini, -)

Let $R$ be a Noetherian local ring with infinite residue field $k$. Let $I = (a_1, \ldots, a_n)$. For general elements $(\lambda_{ij}) \in k^{dn}$, let $x_i = \sum_{j=1}^{n} \lambda_{ij} a_j$ for $1 \leq i \leq d$, $\overline{R} = R/((x_1, \ldots, x_{d-1}) : I^\infty)$ and $\overline{I} = I \overline{R}$, then $\lambda(\overline{I}/\overline{I}^2)$ does not depend on $x_1, \ldots, x_d$. 
Definition (Polini, -)

Let $R$ be a Noetherian local ring with infinite residue field $k$. Let $I = (a_1, \ldots, a_n)$ be an $R$-ideal with $\ell(I) = d = \dim R$.

We say that $I$ has **minimal $j$-multiplicity** if for general elements $(\lambda_{ij}) \in k^{dn}$, let $x_i = \sum_{j=1}^{n} \lambda_{ij}a_j$ for $1 \leq i \leq d$, $\overline{R} = R/((x_1, \ldots, x_{d-1}) : I^\infty)$ and $\overline{I} = I\overline{R}$,

$$j(I) = \lambda(\overline{I}/\overline{I}^2).$$

We say $I$ has **almost minimal $j$-multiplicity** if

$$j(I) = \lambda(\overline{I}/\overline{I}^2) + 1.$$
The ideal $I$ is said to satisfy the condition $G_d$, if for every $p \in V(I)$ with $ht p < d$, $\mu(I_p) \leq ht p$.

Let $H = (x_1, \ldots, x_t) : I$, where $(x_1, \ldots, x_t) \subsetneq I$. $H$ is said to be a geometric $t$-residual intersection of $I$, if

- $ht (x_1, \ldots, x_t) : I \geq t \geq ht I$.
- $ht (((x_1, \ldots, x_t) : I) + I) \geq t + 1$.

The ideal $I$ has the Artin-Nagata property $\text{AN}_t$ if for every $i \leq t$ and every geometric $i$-residual intersection $H$ of $I$, $R/H$ is Cohen-Macaulay.
Main results, Continued

Theorem 1 (Polini, -)

Let $R$ be a Cohen-Macaulay local ring of dimension $d$ of infinite residue field. Let $I$ be an $R$-ideal with $\ell(I) = d$. Assume $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$, $I$ satisfies $G_d$ and $AN_{d-2}^{-}$. If $R$ has minimal $j$-multiplicity with respect to $I$. Then $G = \operatorname{gr}_I(R)$ is Cohen-Macaulay.

Note: The conditions are always satisfied if $I$ is primary to the maximal ideal. Thus this theorem generalizes the case of finite length.
Theorem 2 (Polini, -)

Let $R$ be a Cohen-Macaulay local ring of dimension $d$ of infinite residue field. Let $I$ be an $R$-ideal with $\ell(I) = d$. Assume $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$, $I$ satisfies $G_d$ and $\text{AN}_{d-2}$. If $R$ has almost minimal $j$-multiplicity with respect to $I$. Then

$$\text{depth}(\text{gr}_I(R)) \geq d - 1.$$
Example (Nishida, Ulrich)

Let $S = k[x, y, z]_{(x, y, z)}$ be a 3-dimensional regular local ring. Set

$$R = S/(x^4 - y^2 z^2)S, \quad I = (x^2, y^2)R,$$

then $R$ is a Cohen-Macaulay local ring of dimension 2, $I$ is a Cohen-Macaulay ideal of height 1 which is generically a complete intersection and $\ell(I) = 2$.

Let $\overline{R} = R/(x^2 R : I) = R/(x^2, z^2)R = S/(x^2, z^2)S$.

$$j(I) = e(I\overline{R}) = \lambda(\overline{R}/y^2 \overline{R}) = \lambda(S/(x^2, y^2, z^2)S) = 8.$$

$$= \lambda(I\overline{R}/I^2 \overline{R}) = \lambda((x^2, y^2, z^2)/(x^2, y^4, z^2)) = 8.$$

Thus $R$ has minimal $j$-multiplicity with respect to $I$.

By our theorem, $\text{gr}_I(R)$ is Cohen-Macaulay.

Indeed, $\text{gr}_I(R) \cong \text{gr}_{(x^2, y^2)} S(S)/(x^4 - y^2 z^2)^*).$