Formulas for the multiplicity of graded algebras

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AMS Southeastern Section Meeting
University of Kentucky, Lexington, Kentucky

March 27, 2010
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Introduction

Setup

\[ A = k[A_1] \subseteq B = k[B_1] \] is a homogeneous inclusion of standard graded Noetherian domains over a field \( k \).

Goal

Find a formula for the multiplicity of \( A \) in terms of the multiplicities of \( B \) and of local multiplicities along \( \text{Proj}(B) \).
**Main application:** Compute the multiplicity of special fiber rings.

\[ R = k[R_1] \] is a standard graded Noetherian \( k \)-algebra.

\( I \) is an ideal of \( R \) generated by forms of the same degree \( \delta \).

The Rees algebra \( \mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^i \).

**The special fiber ring** \( F(I) = k \otimes_R \mathcal{R}(I) \cong k[I_\delta] \).

\( F(I) \) describes the homogeneous coordinate ring of the image of the rational map induced by \( I \). In particular, it yields homogeneous coordinate rings of Gauss images and of secant varieties.
To compute the multiplicity of $F(I)$, we reduce to the setup:

$$A = F(I) \cong k[I_\delta] \subseteq B = k[R_\delta] = R^{(\delta)},$$

where

- $k[I_\delta]$ is the $k$-algebra generated by the forms in $I$ of degree $\delta$,
- $R^{(\delta)}$ is the $\delta$-th Veronese subring of $R$. 
Back to the setup

\[ A = k[A_1] \subseteq B = k[B_1] \] is a homogeneous inclusion of standard graded Noetherian domains over a field \( k \).

Let

\[ K = \text{Quot}(A), \ L = \text{Quot}(B). \]

Observe

\[ \dim A \leq \dim B. \]

\[ \dim A = \dim B \iff \left[ L : K \right] = r < \infty. \]

\( B \) is integral over \( A \) \( \iff \dim B/A_1B = 0 \iff re(A) = e(B). \)

**Question**: What is the formula when \( \dim B/A_1B \geq 1 \)?
Theorem (Simis-Ulrich-Vasconcelos, 2001)

Let $A = k[A_1] \subseteq B = k[B_1]$ be a homogeneous inclusion of standard graded Noetherian domains over a field $k$ with the same dimension. Let $K = \text{Quot}(A)$, $L = \text{Quot}(B)$. Let $r = [L : K]$. If $\dim B/A_1B = 1$, then

$$e(B) = re(A) + \sum_{q \in Q} e_{A_1B_q}(B_q) e(B/q), \quad (1)$$

where $Q = \{q \in V(A_1B) \cap \text{Proj}(B) \mid \dim B/q = \dim B/A_1B\}$.

Question: What is the formula when $\dim B/A_1B \geq 2$?
**j-multiplicity** (Achilles-Manaresi)

$(R, m)$ is a Noetherian local ring of dimension $d$.

$I$ is an $R$-ideal. $G = \text{gr}_I(R)$.

$\Gamma_m(G) = 0 :_G m^\infty = \oplus_{i=0}^{\infty} \Gamma_m(I^i/I^{i+1})$. $\dim \Gamma_m(G) \leq d$.

Define

$$j(I) = (d - 1)! \lim_{i \to \infty} \frac{\lambda(\Gamma_m(I^i/I^{i+1}))}{i^{d-1}}.$$

Observe

$$j(I) \neq 0 \iff \text{the analytic spread } \ell(I) = d.$$

If $I$ is primary to $m$, then $j(I) = e_I(R)$.
Proposition (Validashti, 2007)

Let $A = k[A_1] \subseteq B = k[B_1]$ be a homogeneous inclusion of standard graded Noetherian domains over a field $k$ with the same dimension. Let $K = \text{Quot}(A)$, $L = \text{Quot}(B)$. Let $r = [L : K]$. Then

$$e(B) \geq re(A) + \sum_{q \in \mathcal{Q}'} j(A_1 B_q) e(B/q),$$

(2)

where $\mathcal{Q}' = \{ q \in V(A_1 B) \cap \text{Proj}(B) \mid \ell(A_1 B_q) = \dim B_q \}$.

This proposition improved the inequality when $\dim B/A_1 B \geq 2$. 
Questions:

If \( \dim A = \dim B \), what is the formula when \( \dim B/A_1B \geq 2 \)?

What is the formula if \( \dim A < \dim B \)?
**Main results**

**\(j_s\)-multiplicity:**

\((R, m)\) is a Noetherian local ring.

\(I\) is an \(R\)-ideal with analytic spread \(\ell(I) = s\).

\(G = \text{gr}_I(R)\). \(\Gamma_m(G) = 0 :_G m^\infty = \bigoplus_{i=0}^{\infty} \Gamma_m(I^i/I^{i+1})\).

\(\dim \Gamma_m(G) \leq s\). Define

\[
j_s(I) = (s - 1)! \lim_{i \to \infty} \frac{\lambda(\Gamma_m(I^i/I^{i+1}))}{i^{s-1}}.
\]

Observe

\[
j_s(I) = e(\Gamma_m(G)) \iff \dim \Gamma_m(G) = s.
\]

If \(\ell(I) = \dim R\), \(j_s(I) = j(I)\).

If \(I\) is \(m\)-primary, \(j_s(I) = j(I) = e_I(R)\).
The case when \( \dim A = \dim B \)

**Theorem (-)**

Let \( A = k[A_1] \subseteq B = k[B_1] \) be a homogeneous inclusion of standard graded Noetherian domains over an infinite field \( k \). Let \( \dim A = \dim B = d \). Let \( K = \text{Quot}(A) \), \( L = \text{Quot}(B) \). Let \( r = [L : K] \). Let \( g = \text{ht} A_1 B \). Let \( A_1 A = (a_1, \ldots, a_n)A \), where \( a_1, \ldots, a_n \) are homogeneous elements of degree one. For general elements \( \Lambda = (\lambda_{ij}) \in k^{dn} \), let \( x_i = \sum_{j=1}^n \lambda_{ij} a_j \), \( J_{i-1} = (x_1, \ldots, x_{i-1}) : I_\infty \), \( Q_i = \{ q \in \text{Min}(J_{i-1} + I) \mid \dim R/q = d - i \} \), for \( 1 \leq i \leq d \). Then

\[
e(B) = re(A) + \sum_{i=g}^{d-1} \sum_{q \in Q_i} j_1(\frac{A_1 B_q}{(x_1, \ldots, x_{i-1}) B_q}) e(B/q).
\]
Main results, Continued

The case when $\dim A \neq \dim B$

In the setting

\[ A = k[A_1] \subseteq B = k[B_1] \]

is a homogeneous inclusion of standard graded Noetherian domains over a field $k$.

$\dim A = s$, $\dim B = d$.

If $s = d$, then $r = [L : K]$, $K = \text{Quot}(A)$, $L = \text{Quot}(B)$.

If $s < d$, let $t = d - s$. Let $B = k[b_1, \ldots, b_\tau]$, $A' = A[y_1, \ldots, y_t]$, $y_i = \sum_{j=1}^{\tau} \lambda_{ij} b_j$, $1 \leq i \leq t$, for general elements $(\lambda_{ij}) \in k^{t \tau}$. Let $K' = \text{Quot}(A')$, then $r = [L : K']$.

One can show that $r = e(K \otimes_A G)$, $G = \text{gr}_{A_1 B}(B)$. 
Main results, Continued

**Theorem (-)**

Let $A = k[A_1] \subseteq B = k[B_1]$ be a homogeneous inclusion of standard graded Noetherian domains over an infinite field $k$. Let $s = \dim A$, $d = \dim B$. Let $K = \text{Quot}(A)$, $G = \text{gr}_{A_1B}(B)$. Let $r = e(K \otimes_A G)$. Let $g = \text{ht} A_1B$. Let $A_1A = (a_1, \ldots, a_n)A$, where $a_1, \ldots, a_n$ are homogeneous elements of degree one. For general elements $\Lambda = (\lambda_{ij}) \in k^{sn}$, define $x_i, J_{i-1}, \mathcal{Q}_i$ as before for $1 \leq i \leq s$. Then

$$e(B) = re(A) + \sum_{i=g}^{s-1} \sum_{q \in \mathcal{Q}_i} j_1 \left( \frac{A_1B_q}{(x_1, \ldots, x_{i-1})B_q} \right) e(B/q).$$
Main results, Continued

**Corollaries**

Assume $B$ is Cohen-Macaulay.

Assume $A_1 B$ satisfies condition $G_s$. Then

$$e(B) = re(A) + e(B/A_1 B) + \sum_{i=g+1}^{s-1} e(B/(J_{i-1} + A_1 B)).$$

Assume $A_1 B$ is a complete intersection on $\text{Proj}(B)$. Let $F_{i-1} = \text{Fitt}_0(A_1 B/(x_1, \ldots, x_{i-1}) B)$. Then

$$e(B) = re(A) + e(B/A_1 B) + \sum_{i=g+1}^{s-1} e(B/(F_{i-1} + A_1 B)).$$
Assume $A_1B$ is a perfect ideal of height 2 which satisfies condition $G_s$. Let $n = \mu(A_1B)$ and $X_{n \times (n-1)}$ a matrix such that $A_1B = I_{n-1}(X)$. Then

$$e(B) = re(A) + e(B/A_1B) + \sum_{i=3}^{s-1} e(B/(I_n(X \mid \Lambda_{i-1}^T) + A_1B)).$$

Assume $A_1B$ is a perfect Gorenstein ideal of height 3 which satisfies condition $G_s$. Let $n = \mu(A_1B)$ and $X_{n \times n} = (x_{ij})$ an alternating matrix such that $A_1B = \text{Pf}_{n-1}(X)$. Let

$$T_{i-1} = \begin{pmatrix} X & \Lambda_{i-1}^T \\ -\Lambda_{i-1} & 0 \end{pmatrix}$$

and $J(T_{i-1})$ the $B$-ideal generated by the Pfaffians of all principal sub-matrices of $T_{i-1}$ which contain $X$ for $4 \leq i \leq s-1$. Then

$$e(B) = re(A) + e(B/A_1B) + \sum_{i=4}^{s-1} e(B/(J(T_{i-1}) + A_1B)).$$
The formula for special fiber rings

**Theorem (-)**

Let $R = k[R_1]$ be a standard graded Noetherian domain of dimension $d$ over an infinite field $k$. Let $I$ be an $R$-ideal generated by forms in $R$ of the same degree $\delta$. Let $g = \text{ht } I$. The special fiber ring $F(I) \cong k[I_\delta]$. Let $s = \ell(I)$. Let $K = \text{Quot}(k[I_\delta])$, $R^{(\delta)}$ the $\delta$-th Veronese subring of $R$, $G = \text{gr}_{(I_\delta)}R^{(\delta)}(R^{(\delta)})$. Let $r = e(K \otimes k[I_\delta] G)$. For general elements $\Lambda = (\lambda_{ij}) \in k^{sn}$, define $x_i$, $J_{i-1}$, $\Omega_i$, $1 \leq i \leq s$, as before. Then

$$e(F(I)) = e(R) \frac{\delta^{d-1}}{r} - \sum_{i=g}^{s-1} \sum_{q \in \Omega_i} j_1 \left( \frac{l_q}{(x_1, \ldots, x_{i-1})_q} \right) e(R/q) \frac{\delta^{d-i-1}}{r}.$$
Applications to dual varieties

Dual varieties and the class of a variety

\( \mathbb{P}^n \) is the projective \( n \)-space over an algebraically closed field of characteristic 0.

\( X \) is a closed subvariety of \( \mathbb{P}^n \) with degree \( \delta \), dimension \( d - 1 \), codimension \( \rho \). So \( d + \rho = n + 1 \).

\( \text{Sing}(X) \) is the set of singular points of \( X \).

\( \text{Sm}(X) \) is the set of smooth points of \( X \).

\( \mathring{\mathbb{P}}^n \) is the dual projective \( n \)-space.

\( CX \) is the conormal variety in \( X \times \mathring{\mathbb{P}}^n \).
Let $p : CX \to X$, $p' : CX \to \mathbb{P}^n$ be the projections. Then $X' = p'CX$ is the dual variety of $X$.

The class $\delta'$ of $X$ (see for example Kleiman’s paper) is defined as an intersection number:

$$\delta' = \int \ell'^{n-1}[CX],$$

where $\ell' = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

By this definition, one can see

If $X$ has non-deficient dual (i.e., $X'$ is a hypersurface), then $\delta' = \text{degree}(X')$. In this case, $X'$ is Cohen-Macaulay of dimension $n - 1$.

If $\dim X' < n - 1$, then $\delta' = 0$. 
Plücker formulas

(Plücker, 1834) $X$ is a complex plane curve of degree $\delta$ in $\mathbb{P}^2$ with at most nodes and cusps as singularities:

$$\delta' = \delta(\delta - 1) - 3(\text{number of cusps}) - 2(\text{number of nodes}).$$

(Teissier, 1975) $X$ is a hypersurface of dimension $d - 1$ and degree $\delta$ with at most isolated singularities:

$$\delta' = \delta(\delta - 1)^{d-1} - \sum_{x \in \text{Sing}(X)} e(x)$$

where $e(x)$ is the Hilbert-Samuel multiplicity of the Jacobian ideal at $x$. 
(Kleiman, 1994) $X$ is any projective variety with at most isolated singularities.

(Thorup, 1997) $X$ is any projective variety.

**Remark**

Both of the above formulas apply only when $X$ has nondeficient dual, i.e., $X'$ is a hypersurface.

Using our formulas, we can generalize their formulas to any varieties with arbitrary dual.
To generalize their formulas, let $s - 1 = \dim X'$, where $s \leq n$. Define

$$
\delta' = \int \ell^{n-s} \ell'^{s-1}[CX],
$$

where $\ell = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\ell' = c_1(\mathcal{O}_{\tilde{\mathbb{P}}^n}(1))$. We can show $\delta' = \text{degree} \ (X')$ (at least when $X$ is a hypersurface).
Example 1 (Fulton, Thorup)

\(X = V(y_1^2y_3 - y_2^2y_0)\) is a hypersurface in \(\mathbb{P}^3\).

\(R = k[y_0, y_1, y_2, y_3]/(y_1^2y_3 - y_2^2y_0) = k[\overline{y}_0, \overline{y}_1, \overline{y}_2, \overline{y}_3]\) is the homogeneous coordinate ring of \(X\).

The Jacobian ideal \(I = V(\overline{y}_2^2, \overline{y}_1\overline{y}_3, \overline{y}_2\overline{y}_0, \overline{y}_1^2)\).

\(\text{Sing}(X) = V(I)\) is a line which consists of double points and 2 pinch points: \((0, 0, 0, 1), (1, 0, 0, 0)\).

The special fiber ring \(F(I)\) is the homogeneous coordinate ring of the dual variety \(X'\). Then we can use our formula to find the class \(\delta'\) of the variety \(X\).
Observe, \( \dim R = 3 \), \( I \) is generated by forms of the same degree 2, \( \text{ht} \ I = 1 \), \( \ell(I) = 3 \), \( r = 1 \) which is the degree of the Gauss map.

Let \( x_1 = y_1 y_3 - y_2 y_0 \), \( x_2 = y_1^2 - y_2^2 + y_1 y_3 \), \( \mathcal{Q}_1 = \{ (y_1, y_2) \} \), \( \mathcal{Q}_2 = \{ (y_0, y_1, y_2), (y_1, y_2, y_3), (y_0 - y_3, y_1, y_2) \} \).

Then

\[
\delta' = \text{degree}(X') = e(F(I)) \\
= 3 \cdot 2^2 - 2 \cdot \sum_{q \in \mathcal{Q}_1} j_1(I_q) - \sum_{q \in \mathcal{Q}_2} j_1(I_q/(y_1 y_3 - y_2 y_0)_q) \\
= 3 \cdot 2^2 - 2 \cdot 2 - (2 + 2 + 1) = 3.
\]

Indeed, \( X' \) is a hypersurface which is isomorphic to \( X \).
Example 2 (Kleiman)

$X = V(F_1, F_2)$ is a complete intersection in $\mathbb{P}^4$, where $F_1, F_2$ are two homogeneous polynomials of degree $d_1$ and $d_2$.

$R = k[y_0, y_1, y_2, y_3, y_4]/(F_1, F_2) = k[\overline{y}_0, \overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4]$ is the homogeneous coordinate ring of $X$.

The Jacobian matrix

$$M = \begin{bmatrix} F_0^1 & F_1^1 & F_1^1 & F_1^1 & F_1^1 \\ F_0^2 & F_1^2 & F_2^1 & F_3^1 & F_4^1 \\ F_0^2 & F_1^2 & F_2^2 & F_3^2 & F_4^2 \end{bmatrix}$$

where $F_j^i = \frac{\partial F^i}{\partial x_j}$.

The Jacobian ideal $I = l_{2 \times 2}(M)$. $\text{Sing}(X) = V(I)$. 
For general \((a_{ij}) \in k^{5 \times 3}\),

\[
N = \begin{bmatrix}
\sum_{i=0}^{4} F_{i}^{1} a_{i1} & \sum_{i=0}^{4} F_{i}^{1} a_{i2} & \sum_{i=0}^{4} F_{i}^{1} a_{i3} \\
\sum_{i=0}^{4} F_{i}^{2} a_{i1} & \sum_{i=0}^{4} F_{i}^{2} a_{i2} & \sum_{i=0}^{4} F_{i}^{2} a_{i3}
\end{bmatrix}
\]

and \(\Pi = V(l_{2 \times 2}(N)) = V((N_1, N_2, N_3))\).

Assume \(X\) has nondeficient dual. By Kleiman’s paper,

\[
\delta' = \sum_{q \in \text{Sm}(X), \dim R/q=1} \lambda(R_q/(N_1, N_2, N_3)_q).
\]
We can show that

\[
\delta' = \sum_{q \in \text{Sm}(X), \dim R/q=1} \lambda(R_q/(N_1, N_2, N_3)_q)
\]

\[
= \sum_{q \in \text{Sm}(X), \dim R/q=1} \lambda(R_q/(N_1, N_2)_q) = e(F(I)).
\]

Observe, \( \dim R = 3, e(R) = d_1d_2, I \) is generated by forms of the same degree \( (d_1 + d_2 - 2) \), \( \ell(I) = 3, r = 1. \) By applying our formula,

\[
\delta' = e(F(I)) = d_1d_2(d_1 + d_2 - 2)^2
\]

\[
- \sum_{i=1}^{2} \sum_{q \in \Omega_i} j_1(I_q/(x_1, \ldots, x_{i-1})_q) e(R/q)(d_1 + d_2 - 2)^{2-i}.
\]