Example 1 *By definition,1*

$$P_x = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} |x(t)|^2 dt \tag{1}$$

First consider the case $T_o = NT$. Since T is a finite quantity, $T_0 \to \infty \Rightarrow N \to \infty$. The integral in (1) can be split as (2).

$$\begin{aligned} \textstyle\int_{-T_o}^{T_o} |x(t)|^2 dt = \int_{-NT}^{-(N-1)T} |x(t)|^2 dt + \int_{-(N-1)T}^{-(N-2)T} |x(t)|^2 dt + ... \\ + \textstyle\int_{-T}^{0} |x(t)|^2 dt + \int_{0}^{T} |x(t)|^2 dt + ... + \int_{(N-1)T}^{NT} |x(t)|^2 dt \end{aligned} \tag{2}$$

Consider one of the terms $I_k = \int_{kT}^{(k+1)T} |x(t)|^2 dt$. Substitue $t - kT = u$. Then, $dt = du$,$t = u + kT$. we have (3).

$$\begin{aligned} I_k &= \textstyle\int_0^T |x(u+kT)|^2 du \\ &= \textstyle\int_0^T |x(u)|^2 du = I_0 \end{aligned} \tag{3}$$

The last equality follows since $x(u + kT) = x(u)$ due to the periodicity of $x(t)$. Using this in (2), we get (4).

$$\therefore \int_{-T_o}^{T_o} |x(t)|^2 dt = I_o + I_o + ... + I_o \quad (2N\ times) \tag{4a}$$

$$= 2NI_o \tag{4b}$$

$$= 2N \int_0^T |x(t)|^2 dt \tag{4c}$$

$$\therefore P_x = \lim_{N \to \infty} \frac{1}{2NT} 2N \int_0^T |x(t)|^2 dt \tag{4d}$$

$$= \frac{1}{T} \int_0^T |x(t)|^2 dt \tag{4e}$$

Next suppose $T_o = NT + t_o$. In this case, we have (5).

$$\begin{aligned} \textstyle\int_{-T_o}^{T_o} |x(t)|^2 dt = & \textstyle\int_{-(NT+t_o)}^{-NT} |x(t)|^2 dt + \int_{-NT}^{-(N-1)T} |x(t)|^2 dt + ... + \int_{-(N-1)T}^{NT} |x(t)|^2 dt \\ & + \textstyle\int_{NT}^{NT+t_o} |x(t)|^2 dt \end{aligned} \tag{5}$$

Following the same procedure as above, we get (6).

$$\int_{-T_o}^{T_o} |x(t)|^2 dt = 2NI_o + \int_{-(NT+t_o)}^{-NT} |x(t)|^2 dt + \int_{NT}^{NT+t_o} |x(t)|^2 dt \tag{6}$$

Substituting (6) into (1).

$$\begin{aligned} P_x &= \lim_{N \to \infty} \tfrac{1}{2NT} \left(2NI_o + \textstyle\int_{-(NT+t_o)}^{-NT} |x(t)|^2 dt + \int_{NT}^{NT+t_o} |x(t)|^2 dt \right) \\ &= I_o + \lim_{N \to \infty} \left(\textstyle\int_{-(NT+t_o)}^{-NT} |x(t)|^2 dt + \int_{NT}^{NT+t_o} |x(t)|^2 dt \right) \end{aligned} \tag{7}$$

By assumption, the terms within the parantheses are finite. Hence, as $N \to \infty$, those terms $\to 0$. This gives $P = I_o = \frac{1}{T} \int_0^T |x(t)|^2 dt$.

Example 2 *Let* $Z_1 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$*,*$Z_2 = 1 + j\sqrt{3}$*. Their Polar respresentations are given below.*

$$\begin{aligned} r_1 &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \\ \phi_1 &= \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{\pi}{6} \\ r_2 &= \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2 \\ \phi_2 &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} \end{aligned} \tag{8}$$

$$\therefore Z_1 = 1 \cdot e^{j\pi/6}, Z_2 = 2 \cdot e^{j\pi/3} \tag{9}$$

Consider their product.

$$\begin{aligned} Z_1 Z_2 &= \left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)\left(1 + j\sqrt{3}\right) \\ &= \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + j\left(\frac{1}{2} + \frac{3}{2}\right) \\ &= 2j \end{aligned} \tag{10}$$

In polar form

$$\begin{aligned} Z_1 Z_2 &= (1 \times 2)\, e^{j(\pi/6 + \pi/3)} \\ &= 2e^{j\pi/2} \\ &= 2\left(\cos\pi/2 + j\sin\pi/2\right) = 2j \end{aligned} \tag{11}$$

Next,

$$\begin{aligned} \frac{Z_2}{Z_1} &= \frac{1 + j\sqrt{3}}{\frac{\sqrt{3}}{2} + j\frac{1}{2}} \\ &= \frac{\left(1 + j\sqrt{3}\right)\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{3} + j \end{aligned} \tag{12}$$

In polar form,

$$\begin{aligned} \frac{Z_2}{Z_1} &= \frac{2e^{j\pi/3}}{e^{j\pi/6}} = 2e^{j\pi/6} \\ &= 2\left(\cos\pi/6 + j\sin\pi/6\right) \\ &= 2\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) \\ &= \sqrt{3} + j \end{aligned} \tag{13}$$

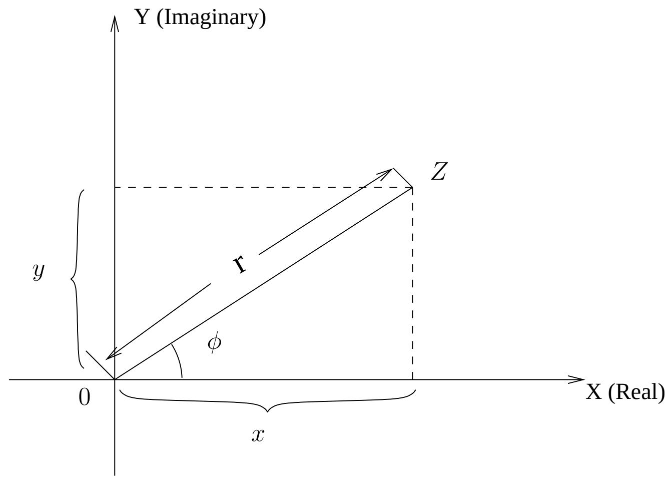

Figure 1: Figure 1

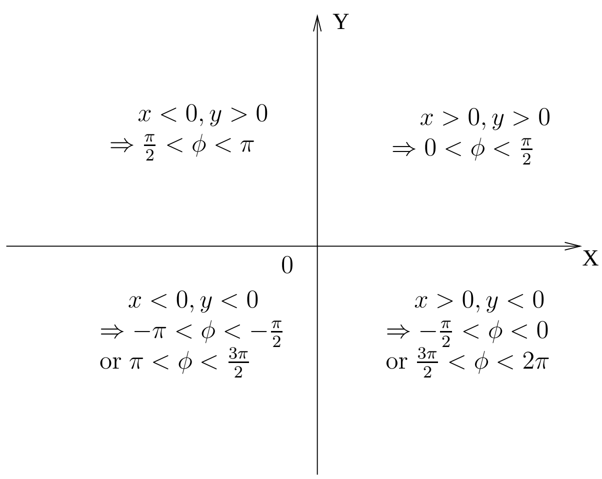

Figure 2: Figure 2

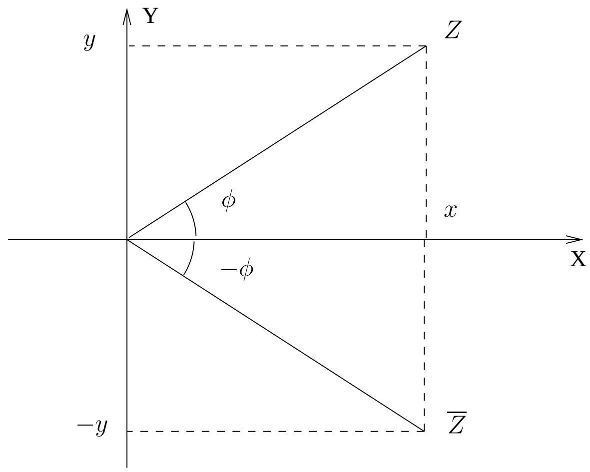

Figure 3: Figure 3

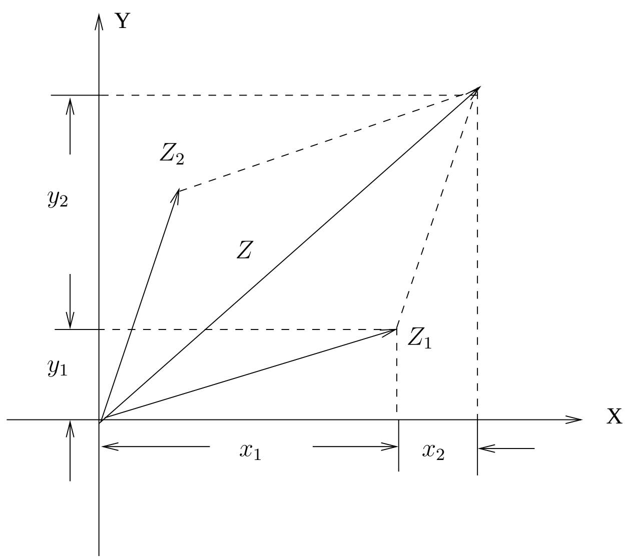

Figure 4: Figure 4