Notes on Equilibrium Bidding in Complementary Auctions

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This note provides technical details of the presentation *Equilibrium Bidding in Complementary Auctions*. Section 1 presents the single object case with additional explanations and section 2 reveals the difficulty of applying single object case argument to two objects case. Section 3.1 shows a system of partial differential equations assuming implicit function theorem is applicable. However, conditions of the theorem cannot be easily checked globally due to complexity of the problem. Section 3.2 provides a proof regarding local solution existence at the origin where a neighborhood includes a point that a bidder values zero for both goods.\(^1\)

1 Single object case

1.1 Environment

- Object \(k \in \{1\}\)
- Agent \(i \in \{1, 2\}\)
- Value of Object, \(x_i\)
  - e.g. \(x_1\) ≡ agent 1’s valuation for the object.
  - \(x_i \sim F(\cdot)\) and \(x_i \in [0, 1]\); \(x_i\) is a random variable with a CDF \(F\), with a continuous and positive density \(f\).
- Bidding value, \(b_i \in [0, 1]\)
  - Each \(i\) makes a bid \(b_i\) on the object.
- Utility function of agent \(i\) (ex-post)
  In case of a tie, the object is not sold. An exception is that, if both agents bid 1, then each receives a perfect copy of the object.\(^3\)

\[
U_i(b_i, b_j, x_i, x_j) = \begin{cases} 
  x_i - b_i & \text{if } b_i > b_j \text{ or } b_1 = b_2 = 1, \text{ where } i \neq j \\
  0 & \text{otherwise}
\end{cases}
\]

1.2 Maximization problem and Equilibrium Bidding Strategy

Each agent maximizes their interim expected utility. At the interim, agents know their valuation of the object \((x_i)\) and the bid amount that they are making \((b_i)\). However, they don’t know about the other agent’s valuation and the bid. They only know the opposite person’s distribution of valuation \(F(x_j)\). Therefore, expected utility function is a function of \((b_i, x_i)\). In this section, we characterize agent 1’s expected utility and the maximization problem.

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\(^1\)The presentation was given by Edward Green, Department of Economics, The Pennsylvania State University. It was a part of Applied Math Seminar on September 23, 2013 in the Department of Mathematics. For further information or discussions on this note, please email to Wiroy Shin, wus130@psu.edu. All errors are mine.

\(^2\)For more details on the lattice theory approach with discrete bid space, please refer to the main paper, Shin(2013), “Auction + Auction = Auction”.

\(^3\)The usual specification of a first-price auction is that the object is allocated randomly in case of a tie. My specification slightly shortens some very long expressions (especially when taking derivatives), without changing the gist of the analysis.
1.2.1 Expected Utility of agent 1

Let $G$ denote the distribution function of agent 2’s bid $b_2$ and $g$ is the density function. Then, the expected utility of agent 1 is,

$$EU_1(b_1, x_1) = \Pr(\text{win}) \cdot (x_1 - b_1)$$

$$= \Pr(b_2 < b_1) \cdot (x_1 - b_1)$$

$$= G(b_1) \cdot (x_1 - b_1)$$

$$= \int_0^{b_1} g(y) \cdot (x_1 - b_1) \, dy$$

1.2.2 Symmetric Optimal Bidding

Agent 1 wants to maximize his expected utility. To maximize $EU_1(b_1, x_1)$, he needs to find an optimal bidding function $\beta_1(x)$ which obtains a maximum of the expected utility for any given $x_1$.

$$\max_{b_1} EU_1(b_1, x_1) = EU_1(\beta_1(x_1), x_1)$$

Also, we want to impose assumptions on $\beta(x)$ s.t.

- $\beta_1(x) = \beta_2(x) = \beta(x) \iff \text{symmetric equilibrium}$
- $\beta(x)$ is (strictly) increasing in $x$
- $\beta(0) = 0$

1.2.3 Maximization problem

Suppose that agent 2 follows a (strictly) increasing bidding strategy $b_2(x_2)$. Then, the expected utility of agent 1 is,

$$EU_1(b_1 | x_1) = \Pr(\text{win}) \cdot (x_1 - b_1)$$

$$= \Pr(b_2 < b_1) \cdot (x_1 - b_1)$$

$$= G(b_1) \cdot (x_1 - b_1)$$

We want to maximize

$$EU_1(b_1, x_1) = F(b_2^{-1}(b_1)) \cdot (x_1 - b_1)$$

Taking derivative w.r.t. $b_1$, the first order condition is

$$\left[ f(b_2^{-1}(b_1)) \cdot \frac{1}{b_2'(b_2^{-1}(b_1))} \cdot (x_1 - b_1) - F(b_2^{-1}(b_1)) \right]_{b_2(x) = \beta(x)} = 0$$

Substitute $b_2(x) = b_1(x) = \beta(x)$ (symmetric equilibrium). Then the F.O.C is,

$$f(x_1) \cdot \frac{1}{\beta'(x_1)} \cdot (x_1 - \beta(x_1)) - F(x_1) = 0$$

Finally,

$$\beta'(x_1) = \frac{f(x_1)}{F(x_1)} \cdot (x_1 - \beta(x_1))$$

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4I conjecture that the opponent’s bid will be (strictly) increasing in his valuation, so that $G$ will be absolutely continuous. When I have solved for the opponent’s optimal bid as a function of his valuation, then I will be able to verify this conjecture.

5The event such that $[b_1 = b_2 = 1]$ is omitted since it is clear that the event almost never happens in the equilibrium, when the distribution of bids is absolutely continuous. At the cost of omitting complicated expressions, this exceptional case could have been handled explicitly.
2 2 objects case

2.1 Environment

- Object \( k \in \{1, 2\} \)
- Agent \( i \in \{1, 2\} \)
- Values of Objects, \( \vec{x}_i = (x_{i1}, x_{i2}) \)
  - e.g. \( \vec{x}_i = (x_{i1}, x_{i2}) \equiv \) agent 1’s valuation for the two objects.
  - \( x_{ik} \sim F(\cdot) \) and \( x_{ik} \in [0, 1] \).
  - \( x_{ik} \) is a random variable with a CDF \( F(\cdot) \), with a continuous and positive density \( f \).
  - Each object is distributed i.i.d.

\[ \phi(x_{i1}, x_{i2}) = f(x_{i1}) \cdot f(x_{i2}) \]

Let \( x_{i3} \) denote i’s value of consuming both objects, and let

\[ \lambda(x_{i1}, x_{i2}) = x_{i3} - (x_{i1} + x_{i2}) \]

Objects 1 and 2 are complementary if \( \lambda \) is (strictly) positive in the interior of \([0, 1] \times [0, 1] \).

We further assume symmetry, differentiability, and increasing differences (\( \lambda_{12} > 0 \)), and that \( \lambda(x_{ik}, 0) = 0 \).

- Bidding value, \( \vec{b}_i = (b_{i1}, b_{i2}) \in [0, 1 + \lambda(1, 1)] \times [0, 1 + \lambda(1, 1)] \)
  - Each \( i \) makes bids \( \vec{b}_i \) on both objects.
  - Let \( \bar{b} \) be the maximum bidding value for one object, \( \bar{b} = 1 + \lambda(1, 1) \).

- Utility function of agent i (ex-post)
  - In case of a tie for an object, the item is not sold—except that, if the tie occurs at the maximum possible value, \( \bar{b} = 1 + \lambda(1, 1) \), then each agent receives a copy of the object

\[
U_i(\vec{b}_i, \vec{b}_j, \vec{x}_i, \vec{x}_j) = \begin{cases} 
\lambda(x_i) + x_{i1} + x_{i2} - (b_{i1} + b_{i2}) & \text{if } [b_{i1} > b_{j1} \text{ or } b_{i1} = \bar{b}] \text{ and } [b_{i2} > b_{j2} \text{ or } b_{i2} = \bar{b}], \\
x_{i1} - b_{i1} & \text{if } [b_{i1} > b_{j1} \text{ or } b_{i1} = \bar{b}] \text{ and } b_{i2} < b_{j2} \\
x_{i2} - b_{i2} & \text{if } b_{i1} < b_{j1} \text{ and } [b_{i2} < b_{j2} \text{ or } b_{i2} = \bar{b}] \\
0 & \text{otherwise}
\end{cases}
\]

2.2 Maximization problem and Equilibrium Bidding Strategy for 2 objects case

Each \( i \) knows their valuation about the two objects \( (\vec{x}_i = (x_{i1}, x_{i2})) \) and decides bids \( (\vec{b}_i = (b_{i1}, b_{i2})) \). However, they don’t know about the other agent’s valuation and bids. They only know the fact that the opposite person’s valuation \( \vec{x}_j = (x_{j1}, x_{j2}) \) are distributed according to \( [F(x_{j1}) \cdot F(x_{j2})] \). Therefore, expected utility function is a function of \( (\vec{b}_i, \vec{x}_i) \). In this section, we characterize agent 1’s expected utility and the maximization problem.

2.2.1 Expected Utility of agent 1

Let \( G \) denote the joint distribution function of agent 2’s bid \( \vec{b}_2 \) and \( g \) is the joint density function. Then, the expected utility of agent 1 is,
\[ EU_1(\vec{b}_1, \vec{x}_1) = \Pr(\text{win 1 and win 2}) \cdot \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} + \Pr(\text{win 1 and lose 2}) \cdot \{x_{11} - b_{11}\} + \Pr(\text{lose 1 and win 2}) \cdot \{x_{12} - b_{12}\} + \Pr(\text{lose 1 and lose 2}) \cdot 0 \]

\[ EU_1(\vec{b}_1, \vec{x}_1) = \Pr(b_{21} < b_{11}, b_{22} < b_{12}) \cdot \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} + \Pr(b_{21} < b_{11}, b_{22} > b_{12}) \cdot \{x_{11} - b_{11}\} + \Pr(b_{21} > b_{11}, b_{22} < b_{12}) \cdot \{x_{12} - b_{12}\} + \Pr(b_{21} > b_{11}, b_{22} > b_{12}) \cdot 0^6 \]

\[ EU_1(\vec{b}_1, \vec{x}_1) = \int_{[0,b_{12}]} \int_{[0,b_{11}]} g(y_1, y_2) \cdot \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} dy_1 dy_2 + \int_{[b_{12}, \vec{b}]} \int_{[0,b_{11}]} g(y_1, y_2) \cdot \{x_{11} - b_{11}\} dy_1 dy_2 + \int_{[0,b_{12}]} \int_{[b_{11}, \vec{b}]} g(y_1, y_2) \cdot \{x_{12} - b_{12}\} dy_1 dy_2 \]

2.2.2 Symmetric Optimal Bidding

To maximize \( EU_1(\vec{b}_1, \vec{x}_1) \), agent 1 needs to find an optimal bidding function \( \vec{\beta}_1(\vec{x}_1) = (\beta_1^1(\vec{x}_1), \beta_2^1(\vec{x}_1)) \) which obtains a maximum of the expected utility for any given \( \vec{x}_1 \).

\[
\max_{\vec{b}_1} EU_1(\vec{b}_1, \vec{x}_1) = EU_1(\vec{\beta}_1(\vec{x}_1), \vec{x}_1)
\]

Also, we want to impose assumptions on \( \vec{\beta}(\vec{x}) \) s.t.

- \( \vec{\beta}_1(\vec{x}) = \vec{\beta}_2(\vec{x}) = \vec{\beta}(\vec{x}) \Leftrightarrow \text{symmetric equilibrium} \)
- \( \vec{\beta}(\vec{x}) \) is (strictly) increasing in \( \vec{x} \) in partial order sense.
- \( \beta^1(0, x) = 0, \beta^2(x, 0) = 0 \) and \( \vec{\beta}(0, 0) = (0, 0) \Leftrightarrow \text{optimal bidding for the object with value 0 is 0} \)

2.2.3 Maximization problem

\[
\max_{\vec{b}_1} EU_1(\vec{b}_1, \vec{x}_1) = \max_{\vec{b}_1} \int_{[0,b_{12}]} \int_{[0,b_{11}]} g(y_1, y_2) \cdot \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} dy_1 dy_2 + \int_{[b_{12}, \vec{b}]} \int_{[0,b_{11}]} g(y_1, y_2) \cdot \{x_{11} - b_{11}\} dy_1 dy_2 + \int_{[0,b_{12}]} \int_{[b_{11}, \vec{b}]} g(y_1, y_2) \cdot \{x_{12} - b_{12}\} dy_1 dy_2
\]

\(^6\)The event such that \( b_{ij} = b_{jk} = b \) is omitted in this expression, since it is clear that the event almost never happens in the equilibrium, when the distribution of bids is absolutely continuous. At the cost of omitting complicated expressions, this exceptional case could have been handled explicitly.
2.3 Bids Density transformation

2.3.1 Transformations of absolutely continuous random vectors

Bierens(2004), Introduction to the Mathematical and Statistical Foundations of Econometrics;

**Theorem 4.4** Let $X$ be $k$-variate, absolutely continuously distributed with joint density $f(x)$, and let $Y = G(X)$, where $G(x)$ is a one-to-one mapping with inverse mapping $x = G^{-1}(y)$ whose components are differentiable in the components of $y$. Let $J(y) = \frac{\partial x}{\partial y} = \frac{\partial G^{-1}(y)}{\partial y}$, which is called the Jacobian. Then $Y$ is $k$-variate, absolutely continuously distributed with joint density

$$h(y) = f(G^{-1}(y)) \cdot |\text{det}(J(y))|$$

for $y$ in the set $G(\mathbb{R}^k) = \{y \in \mathbb{R}^k : y = G(x), f(x) > 0, x \in \mathbb{R}^k\}$ and $h(y) = 0$ elsewhere.

2.3.2 Transformation from bids density to values density

Consider a differentiable bid function, $\beta$, that maps unit square 1–1 to $[0, \bar{b}] \times [0, \bar{b}]$. Let $B$ be the image of unit square under $\beta$. Now define $\gamma : B \rightarrow [0, 1] \times [0, 1]$ to be the inverse function of $\beta$.

By symmetry and Theorem 4.4 in Bierens(2004), we define the joint density of opponent’s bids induced by $\gamma$ and $\phi$ as follows. (Note that $\gamma^i$ denotes $i$th coordinate.)

$$g(y_1, y_2) = \phi(\gamma^1(y_1, y_2), \gamma^2(y_1, y_2)) \cdot \frac{\partial \gamma}{\partial y}$$

$$[x_1 \perp x_2] \rightarrow = f(\gamma^1(y_1, y_2)) \cdot f(\gamma^2(y_1, y_2)) \cdot \frac{\partial \gamma}{\partial y}$$

2.4 Representations of the expected utility function

Suppose that agent2 follows a bidding strategy $\beta$, which is 1–1 and has the inverse function $\gamma$ as defined in the previous section. Then, agent 1’s interim expected utility function can be represented as follows.

$$EU_1(\vec{b}_1, \vec{x}_1) = \text{Pr}(b_{21} < b_{11}, b_{22} < b_{12}) \cdot \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\}$$

$$+ \text{Pr}(b_{21} < b_{11}, b_{22} > b_{12}) \cdot \{x_{11} - b_{11}\}$$

$$+ \text{Pr}(b_{21} > b_{11}, b_{22} < b_{12}) \cdot \{x_{12} - b_{12}\}$$

$$= \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} \cdot \int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2$$

$$+ \{x_{11} - b_{11}\} \cdot \int_{b_{12}}^b \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2$$

$$+ \{x_{12} - b_{12}\} \cdot \int_0^{b_{12}} \int_{b_{11}}^b g(y_1, y_2) \, dy_1 \, dy_2$$

$$\text{eqn}(3) \rightarrow = \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} \cdot \int_0^{b_{12}} \int_0^{b_{11}} f(\gamma^1(y_1, y_2)) \cdot f(\gamma^2(y_1, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_1 \, dy_2$$

$$+ \{x_{11} - b_{11}\} \cdot \int_{b_{12}}^b \int_0^{b_{11}} f(\gamma^1(y_1, y_2)) \cdot f(\gamma^2(y_1, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_1 \, dy_2$$

$$+ \{x_{12} - b_{12}\} \cdot \int_0^{b_{12}} \int_{b_{11}}^b f(\gamma^1(y_1, y_2)) \cdot f(\gamma^2(y_1, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_1 \, dy_2$$
A naive approach for solving the maximization problem of this expected utility function is change of variables. In the domain of the double integral, if we can replace each \( b_{1k} \) by \( \gamma_k \), we can simplify and separate these integrals. Then, differentiation results would have clearer forms or they might be able to provide further steps in deriving partial differential equations. However, it does not work in that way. Though the inverse bidding function \( \gamma \) is one to one, projection to the each coordinate of \( \gamma \) is not unique. That is, there is a chance that \( \gamma_1(b_{11}, b_{12}) = \gamma_1(b'_{11}, b'_{12}) \) where \( (b_{11}, b_{12}) \neq (b'_{11}, b'_{12}) \), so the double integral is non-separable in general.

### 2.5 First Order Conditions

#### 2.5.1 from eqn(5), w.r.t. \( \bar{b}_1 \)

Let’s differentiate eqn(5) w.r.t. \( b_{11} \) and \( b_{12} \).

\[
\frac{\partial E U_1(b_{11}, b_{12}, x_{11}, x_{12})}{\partial b_{11}} = \frac{\partial}{\partial b_{11}} \left[ \{(\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12})\} \cdot \int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 \right. \\
+ \left\{ x_{11} - b_{11} \right\} \cdot \int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 \\
+ \left\{ x_{12} - b_{12} \right\} \cdot \int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 \\
\left. \right] (7)
\]

\[
= -\int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 + \int_0^{b_{12}} g(b_{11}, y_2) \cdot \{ (\lambda(x_1) + x_{11} + x_{12}) - (b_{11} + b_{12}) \} \, dy_2 \\
- \int_0^{b_{12}} g(y_1, y_2) \, dy_1 \, dy_2 + \int_0^{b_{12}} g(b_{11}, y_2) \cdot \{ x_{11} - b_{11} \} \, dy_2 \\
- \int_0^{b_{12}} g(b_{11}, y_2) \cdot \{ x_{12} - b_{12} \} \, dy_2 (8)
\]

\[
= -\int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 \\
+ (\lambda(x_1) + x_{11} + x_{12}) \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 - b_{11} \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 \\
+ x_{11} \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 - x_{12} \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 (9)
\]

\[
= -\int_0^{b_{12}} \int_0^{b_{11}} g(y_1, y_2) \, dy_1 \, dy_2 \\
+ \{ x_{11} - b_{11} \} \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 \\
+ \lambda(x_1) \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2 (10)
\]

Respectively,
\[
\frac{\partial E_{U1}(b_{11}, b_{12}, x_{11}, x_{12})}{\partial b_{12}} = -\int_0^b \int_0^{b_{12}} g(y_1, y_2) \, dy_2 dy_1 \\
+ \{x_{12} - b_{12}\} \cdot \int_0^b g(y_1, b_{12}) \, dy_1 \\
+ \lambda(x_1) \cdot \int_0^{b_{11}} g(y_1, b_{12}) \, dy_1
\] (11)

2.5.2 Transformation of first order conditions

We might want to transform eqn(10) so that the equation is represented by values and \( f(\gamma^1, \gamma^2) \) is reduced to \( f(z_1, z_2) \). However, this cannot be arranged due to the same argument described in section 2.4. Eqn(14) is the furthest step I can proceed.

\[
\frac{\partial E_{U1}(b_{11}, b_{12}, x_{11}, x_{12})}{\partial b_{11}} = -\int_0^b \int_0^{b_{11}} g(y_1, y_2) \, dy_1 dy_2 \\
+ \{x_{11} - b_{11}\} \cdot \int_0^b g(b_{11}, y_2) \, dy_2 \\
+ \lambda(x_1) \cdot \int_0^{b_{12}} g(b_{11}, y_2) \, dy_2
\] (12)

\[
[g(y_1, y_2) \Rightarrow f(\gamma^1, \gamma^2)] \Rightarrow -\int_0^b \int_0^{b_{11}} f(\gamma^1(y_1, y_2), \gamma^2(y_1, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_1 dy_2 \\
+ \{x_{11} - b_{11}\} \cdot \int_0^b f(\gamma^1(b_{11}, y_2), \gamma^2(b_{11}, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_2 \\
+ \lambda(x_1) \cdot \int_0^{b_{12}} f(\gamma^1(b_{11}, y_2), \gamma^2(b_{11}, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_2
\] (13)

\[
[(b_{11}, b_{12}) = \tilde{b}(x_{11}, x_{12})] \Rightarrow -\int_0^b \int_0^{\tilde{b}(x_{11}, x_{12})} f(\gamma^1(y_1, y_2), \gamma^2(y_1, y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_1 dy_2 \\
+ \{x_{11} - \tilde{b}(x_{11}, x_{12})\} \cdot \int_0^b f(\gamma^1(\tilde{b}(x_{11}, x_{12}), y_2), \gamma^2(\tilde{b}(x_{11}, x_{12}), y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_2 \\
+ \left[ \lambda(x_1) \cdot \int_0^{\tilde{b}(x_{11}, x_{12})} f(\gamma^1(\tilde{b}(x_{11}, x_{12}), y_2), \gamma^2(\tilde{b}(x_{11}, x_{12}), y_2)) \cdot \left| \frac{\partial \gamma}{\partial y} \right| \, dy_2 \right]
\] (14)
3 Applications of Implicit function theorem

As an alternative approach, I tried to apply implicit function theorem to show existence of partial differential equations for an equilibrium bidding function. However, due to complexity of Jacobian matrix and other components, it was hard to verify conditions for implicit function theorem. Lemma1 shows the theorem is applicable in the neighborhood of the point in which agent values (0,0) for both objects. A complete form of matrix(eqn(18)) assuming IFT is applicable is presented in the accompanying file(iftAppendix) which is produced by Mathematica.\(^7\)

Original Implicit function theorem requires an open set \(A \subset \mathbb{R}^2 \times \mathbb{R}^2\) as a domain of \(M\) (it is defined below), neither a positive orthant nor an arbitrary closed set. Therefore, we should think about validity of the theorem on the boundary pts. Lemma1 solves a part of this issue at \(x_1 = (0,0)\).

Consider a function \(M : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2\) s.t.

\[
M_1 = \frac{\partial E_U(x_1, x_2, b_1, b_2)}{\partial b_1}; \quad M_2 = \frac{\partial E_U(x_1, x_2, b_1, b_2)}{\partial b_2}.
\]

Suppose \(M(x_1, x_2, b_1, b_2) = 0\), and let \(J\) be a Jacobian matrix of \(M\) w.r.t. \(b\). That is,

\[
J = \begin{bmatrix}
\frac{\partial M_1}{\partial b_1} & \frac{\partial M_1}{\partial b_2} \\
\frac{\partial M_2}{\partial b_1} & \frac{\partial M_2}{\partial b_2}
\end{bmatrix}.
\]

Assume that \(\det(J) \neq 0\). Then, by Implicit function theorem, there exist an open neighborhood of \(x, U \subset \mathbb{R}^2\), an open neighborhood of \(b, V \subset \mathbb{R}^2\), and a unique function \(\beta : U \to V\) s.t.

\[
M(x, \beta(x)) = 0,
\]

for all \(x \in U\). Also, the \(\frac{\partial \beta_i}{\partial x_k}\) are given by \(^8\)

\[
\begin{bmatrix}
\frac{\partial \beta_1}{\partial x_1} & \frac{\partial \beta_1}{\partial x_2} \\
\frac{\partial \beta_2}{\partial x_1} & \frac{\partial \beta_2}{\partial x_2}
\end{bmatrix} = -J^{-1} \begin{bmatrix}
\frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} \\
\frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2}
\end{bmatrix}
\]

\[
= -M_1 M_2 - M_1 b_1 M_2 b_2 \begin{bmatrix}
M_1 x_1 & M_1 x_2 \\
M_2 x_1 & M_2 x_2
\end{bmatrix}
\]

\[
= -\frac{1}{\det(J)} \begin{bmatrix}
M_2 b_2 M_1 x_1 - M_1 b_2 M_2 x_1 & M_2 b_2 M_1 x_2 - M_1 b_2 M_2 x_2 \\
-M_2 b_1 M_1 x_1 + M_1 b_1 M_2 x_1 & -M_2 b_1 M_1 x_2 + M_1 b_1 M_2 x_2
\end{bmatrix}.
\]

\(^7\)In the accompanied document, more general assumptions are used. Substitute \(u(x_k) = x_k, u(x_1 + x_2) = x_1 + x_2 + \lambda(x_1, x_2)\) and change \(\lambda\) to \(\beta\). Then it is equivalent to the assumptions in this note.

\(^8\) Denote \(M_x = \begin{bmatrix}
\frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} \\
\frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2}
\end{bmatrix}\)
3.1 Partial differentiation from FOC

Let’s recall $M(x_{i1}, x_{i2}, b_{i1}, b_{i2})$ from section 2.

\[ M_1(x_{i1}, x_{i2}, b_{i1}, b_{i2}) = \frac{\partial EU_i(x_{i1}, x_{i2}, b_{i1}, b_{i2})}{\partial b_{i1}} \]
\[ = - \int_0^b \int_0^{b_{i1}} g(y_1, y_2) \, dy_2 \, dy_1 + [x_{i1} - b_{i1}] \int_0^b g(b_{i1}, y_2) \, dy_2 + \lambda(x_i) \int_0^{b_{i1}} g(b_{i1}, y_2) \, dy_2 \]
\[ (19) \]

\[ M_2(x_{i1}, x_{i2}, b_{i1}, b_{i2}) = \frac{\partial EU_i(x_{i1}, x_{i2}, b_{i1}, b_{i2})}{\partial b_{i2}} \]
\[ = - \int_0^b \int_0^{b_{i2}} g(y_1, y_2) \, dy_1 \, dy_2 + [x_{i2} - b_{i2}] \int_0^b g(y_1, b_{i2}) \, dy_1 + \lambda(x_i) \int_0^{b_{i2}} g(y_1, b_{i2}) \, dy_1 \]
\[ (20) \]

Following formulas provide elements in (18).

\[ M_{1,b_{i1}} = \frac{\partial M_1}{\partial b_{i1}} = -2 \int_0^b g(b_{i1}, y_2) \, dy_2 + [x_{i1} - b_{i1}] \int_0^b g_1(b_{i1}, y_2) \, dy_2 + \lambda(x_i) \int_0^{b_{i1}} g_1(b_{i1}, y_2) \, dy_2 \]
\[ (21) \]

\[ M_{1,b_{i2}} = \frac{\partial M_1}{\partial b_{i2}} = \lambda(x_i) g(b_{i1}, b_{i2}) \]
\[ (22) \]

\[ M_{2,b_{i1}} = \frac{\partial M_2}{\partial b_{i1}} = \lambda(x_i) g(b_{i1}, b_{i2}) \]
\[ (23) \]

\[ M_{2,b_{i2}} = \frac{\partial M_2}{\partial b_{i2}} = -2 \int_0^b g(y_1, b_{i2}) \, dy_1 + [x_{i2} - b_{i2}] \int_0^b g_2(y_1, b_{i2}) \, dy_1 + \lambda(x_i) \int_0^{b_{i2}} g_2(y_1, b_{i2}) \, dy_1 \]
\[ (24) \]
\( M_{1,x_i} = \frac{\partial M_1}{\partial x_i} = 1 \cdot \int_0^b g(b_{11}, y_2) dy_2 
\)

\( + [\frac{\partial \lambda}{\partial x_i} - 1] \int_0^{b_{12}} g(b_{11}, y_2) dy_2 \) 

(25)

\( M_{1,x_i2} = \frac{\partial M_1}{\partial x_i2} = [\frac{\partial \lambda}{\partial x_i2} - 1] \int_0^{b_{12}} g(b_{11}, y_2) dy_2 \) 

(26)

\( M_{2,x_i1} = \frac{\partial M_2}{\partial x_i1} = [\frac{\partial \lambda}{\partial x_i1} - 1] \int_0^{b_{11}} g(y_1, b_{i2}) dy_1 \) 

(27)

\( M_{2,x_i2} = \frac{\partial M_2}{\partial x_i2} = 1 \cdot \int_0^b g(y_1, b_{i2}) dy_1 
\)

\( + [\frac{\partial \lambda}{\partial x_i2} - 1] \int_0^{b_{11}} g(y_1, b_{i2}) dy_1 \) 

(28)

3.2 When a bidder values both objects near (0,0)

**Definition 1.** (Dederick 1913, p171) A region\(^9\) will be called **uniformly connected** if there exists a constant \( G \) such that any two points of the region can be connected by a broken line lying wholly in the region and having a length not greater than \( G \) times the distance in a straight line between the points.

**Theorem 1.** (Dederick 1913, p178) If the given functions are of class \( C' \), the ordinary theorems on implicit functions can be extended to the case where the initial solution is a boundary point, with only such changes as are obvious, provided the region in the neighborhood of this boundary point is uniformly connected.

**Lemma 1.** Implicit function Theorem is applicable where \((x_{i1}, x_{i2}) = (0, 0) \in \mathbb{R}^2_+\).

**Proof.** Define a region for \( O = (0, 0) \) with an Euclidean norm \( d \) and an arbitrary small positive number \( \epsilon \) s.t.

\[ R_\epsilon(O) = \{ z \in \mathbb{R}^2_+ | d(O, z) \leq \epsilon \}, \]

which is a quarter circle with a radius \( \epsilon \) on \( \mathbb{R}^2_+ \).

Let \( A \in R_\epsilon(O) \) and \( d(O, A) = \delta(\leq \epsilon) \). We want to construct a broken line between \( O \) and \( A \) s.t. its length does not exceed \( \gamma \delta \), where \( \gamma \) is a constant. First, suppose \( A \in \text{int}(\mathbb{R}^2_+) \). Define \( A' \) as a projection point of \( A \) on the \( x \)-axis. Then, we have a broken line \( OA'A \) in \( R_\epsilon(O) \). Since \( A \in R_\delta(O) \) and \( A' \in R_\delta(O) \), we have \( d(O, A') \leq \delta \) and \( d(A', A) \leq \delta \). Therefore,

\[ \text{length}(OA'A) = d(O, A') + d(A', A) \leq \delta + \delta = 2\delta. \]  

Second, suppose \( A \in \partial \mathbb{R}^2_+ \). Without loss of generality, let \( A = (\delta, 0) \). Define \( A' = (0, \delta) \). Now we have a broken line \( OA'A \) in \( R_\epsilon(O) \), and the length is,

\[ \text{length}(OA'A) = d(O, A') + d(A', A) = \delta + \sqrt{2}\delta = (1 + \sqrt{2})\delta. \]  

\(^9\)By a region, unless the contrary is stated, will always be meant a complete region, i.e. a continuum together with all its boundary points.
Using (30) and (31), set $\gamma = \max\{2, 1 + \sqrt{2}\} = 1 + \sqrt{2}$. Finally, we have a broken line $OA_1A$ in $R_\epsilon(O)$ for $\forall A \in R_\epsilon(O)$, and the length is not greater than $\gamma d(O, A)$. Therefore, $R_\epsilon(O)$ is uniformly connected.

By the assumptions on this auction problem, (19) and (20) are $C'$. Thus, by Theorem 1, we can extend Implicit function theorem where $(x_{i1}, x_{i2}) = (0, 0)$. ♦

References


