ABSTRACT. Given a unipotent bundle of smooth manifolds we construct its secondary transfer map and show that this map determines the higher smooth torsion of the bundle. This approach to higher torsion provides a new perspective on some of its properties. In particular it yields in a natural way a formula for torsion of a composition of two bundles.

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1. Introduction

By a smooth bundle of manifolds we will understand here a smooth submersion \( p: E \to B \) where \( E \) and \( B \) are smooth compact manifolds. A smooth bundle with the fiber \( F \) is unipotent if \( B \) is path connected and the graded vector space \( H_\ast(F; \mathbb{Q}) \) admits a filtration such that \( \pi_1 B \) acts trivially on the filtration quotients. Igusa and Klein [10, 13] showed using fiberwise Morse theory that to any unipotent bundle one can associate the higher torsion invariant which depends not only on the topological structure of \( p \), but also on its smooth structure. Higher torsion proved to be a useful tool in the study of smooth bundles. In [9] Igusa showed, for example, that it can be used to detect exotic disc bundles constructed by Hatcher.

In [2] and [1] the present authors in collaboration with Klein and Williams extended ideas of Dwyer, Weiss, and Williams [7] to obtain an alternative construction of torsion of unipotent bundles based on the machinery of homotopy theory. This construction can be briefly described as follows. Let \( K(\mathbb{Q}) \) be the infinite loop space underlying the algebraic \( K \)-theory spectrum of the field of rational numbers. Given a smooth bundle \( p: E \to B \) we can construct a map \( c_p : B \to K(\mathbb{Q}) \) which, roughly speaking, assigns to each \( b \in B \) the point of \( K(\mathbb{Q}) \) represented by the singular chain complex \( C_\ast(p^{-1}(b); \mathbb{Q}) \). The smooth Riemann-Roch theorem of [7] implies that \( c_p \) admits a factorization

\[
\begin{array}{ccc}
B & \xrightarrow{c_p} & K(\mathbb{Q}) \\
p^! \downarrow & & \downarrow \lambda_E \\
Q(E_+) & \xrightarrow{\alpha_E} & \\
\end{array}
\]

where \( Q(E_+) = \Omega^\infty \Sigma^\infty (E_+) \), \( p^! \) is the Becker-Gottlieb transfer and \( \lambda_E \) is the linearization map (3.1).

If \( p \) is a unipotent bundle then the map \( c_p \) is homotopic via a preferred homotopy to a constant map. As a consequence we obtain a lift of \( p^! \) to the space \( \text{Wh}_\mathbb{Q}^\infty (E) \) which is the homotopy fiber of \( \lambda_E \):

\[
\begin{array}{ccc}
B & \xrightarrow{c_p} & K(\mathbb{Q}) \\
p^! \downarrow & & \downarrow \lambda_E \\
Q(E_+) & \xrightarrow{\alpha_E} & \\
\end{array}
\]

The lift \( \tau^s(p) \) is the smooth torsion of the bundle \( p \).
The homotopy class of $\tau^s(p)$ is an invariant of the smooth structure of $p$ in the following sense. If $p': E' \to B$ is another smooth bundle and $f: E' \to E$ is a smooth bundle map then $f$ induces a map

$$f_*: \text{Wh}^Q_s(E') \to \text{Wh}^Q_s(E)$$

The map $f_\ast \tau^s(p')$ need not be homotopic to $\tau^s(p)$ in general, but this property does hold provided that $f$ is a fiberwise diffeomorphism of bundles.

The map $\tau^s(p)$ gives rise to a certain cohomology class

$$t^s(p) \in \bigoplus_{k>0} H^{4k}(B; \mathbb{R})$$

[1, 4.10] which we will call the cohomological torsion of the bundle $p$.

In [11, Section 9] Igusa showed that the cohomological torsion of the composition $pq$ can be, in some cases, computed from the torsion of the bundles $p$ and $q$. Namely, if $q$ is an oriented linear sphere bundle then

(1-1) $$t^s(pq) = \chi(F_q)t^s(p) + \text{tr}^E_B(t^s(q))$$

where $\chi(F_q) \in \mathbb{Z}$ is the Euler characteristic of the fiber of $q$ and

$$\text{tr}^E_B: H^*(E; \mathbb{R}) \to H^*(B; \mathbb{R})$$

is the transfer map associated to $p$. In [11] Igusa calls the formula (1-1) the transfer axiom and shows that taken together with a few other properties it uniquely determines the cohomological torsion.

Igusa’s arguments can be used to show that the formula (1-1) holds under more general conditions on $p$ and $q$, e.g. if dimensions of fibers of these bundles have the same parity. In [1, Thm 7.1] we verified that the same is true in the case when $p$ is an arbitrary unipotent bundle and $q$ satisfies the assumptions of the Leray-Hirsch isomorphism theorem. One of our goals in this paper is to show that this formula holds in general:

1.1. Theorem. The formula (1-1) holds for any unipotent bundles $p: E \to B$ and $q: D \to E$.

In order to prove Theorem 1.1 we develop a new construction of smooth torsion based on the notion of the secondary transfer of unipotent bundles. The starting point for this construction is the following fact:
1.2. Theorem. Given a smooth bundle of compact manifolds \( p: E \to B \) with fiber \( F_p \) consider the diagram

\[
\begin{array}{ccc}
Q(B_+) & \xrightarrow{Q(p')} & Q(E_+) \\
\downarrow{\lambda_B} & & \downarrow{\lambda_E} \\
K(Q) & \xrightarrow{\chi(F_p)} & K(Q)
\end{array}
\]  

where the lower horizontal map is given by the multiplication by the Euler characteristic \( \chi(F_p) \in \mathbb{Z} \) of \( F_p \) and the upper horizontal map is the Becker-Gottlieb transfer of \( p \). If \( p \) is a unipotent bundle then this diagram commutes up to a preferred homotopy \( \eta_p: Q(B_+) \times [0,1] \to K(Q) \).

As an intermediate step in the proof of this result it will be convenient to work in a more general setting of unipotent fibrations, i.e. fibrations \( p: E \to B \) satisfying some finiteness assumptions and such that the action of \( \pi_1(B) \) on homology of the fiber satisfies the same unipotency condition as in the case of unipotent bundles (see Definition 3.4). We show (3.5) that for any unipotent fibration an analog Theorem 1.2 holds, with spaces \( Q(B_+) \) and \( Q(E_+) \) replaced with Waldhausen’s algebraic \( K \)-theory spaces \( A(B) \) and \( A(E) \), and with \( A \)-theory transfer taken in place of the Becker-Gottlieb transfer.

For a unipotent bundle \( p: E \to B \) the homotopy \( \eta_p \) defines a map of homotopy fibers

\[
\text{Wh}_h^Q(p') : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E)
\]

We call this map the smooth secondary transfer of the bundle \( p \). Likewise, for any unipotent fibration \( p \) we construct its homotopy secondary transfer

\[
\text{Wh}_h^Q(p') : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E)
\]

where \( \text{Wh}_h^Q(B) = \text{hofib}(A(B) \to K(Q)) \).

The smooth secondary transfer shares some of the basic properties of the Becker-Gottlieb transfer. It is additive (7.3) and it preserves composition of bundles:

1.3. Theorem. If \( p: E \to B \) and \( q: D \to E \) are unipotent bundles then

\[
\text{Wh}_h^Q((pq)') \simeq \text{Wh}_h^Q(q') \circ \text{Wh}_h^Q(p')
\]

Analogous additivity and composition properties hold for the homotopy secondary transfer (7.1, 9.1).

The relationship between the smooth secondary transfer and the smooth torsion is as follows. If \( B \) is a compact, smooth manifold then the identity map \( \text{id}_B: B \to B \) is a unipotent bundle. We have
1.4. Theorem. If \( p: E \to B \) is a unipotent bundle then
\[
\tau^s(p) \simeq \text{Wh}_s^Q(p!) \circ \tau^s(\text{id}_B)
\]
This shows that the smooth secondary transfer of a unipotent bundle determines the smooth torsion of the bundle. Since smooth torsion can distinguish bundles that are fiberwise homotopy equivalent, but not fiberwise diffeomorphic, Theorem 1.4 implies that the smooth secondary transfer carries information about the smooth structure of a bundle. In Proposition 6.1 we show that, in contrast, the homotopy secondary transfer is invariant with respect to fiberwise homotopy equivalences of unipotent fibrations.

Combining Theorems 1.4 and 1.3 we obtain

1.5. Corollary. If \( p: E \to B \) and \( q: D \to E \) are unipotent bundles then
\[
\tau^s(pq) \simeq \text{Wh}_s^Q(q!) \circ \tau^s(p)
\]

Theorem 1.1 can be obtained as a direct consequence of this fact. Notice that in this way we exhibit the simple principle underlying the formula (1-1): the torsion of a composition of bundles \( p \) and \( q \) is a composition of two maps, one depending on \( p \) and the other on \( q \).

1.6. Note. As it was pointed out to us by the referee the main results of this paper bear some resemblance to the work of Lott and Bunke on the secondary \( K \)-theory pushforward map. In [16] Lott constructed for a smooth manifold \( B \) the secondary \( K \)-theory group \( \tilde{K}^0(B) \) which is generated by flat complex vector bundles over \( B \) with trivial Borel classes. He also constructed for a smooth bundle \( p: E \to B \) a pushforward map \( p!: \tilde{K}^0(E) \to \tilde{K}^0(B) \). Just as the secondary smooth transfer considered in this paper contains information about the smooth torsion of a bundle, the construction of the pushforward map involves higher analytic torsion forms of Bismut and Lott [3]. Lott’s pushforward map was studied by Bunke [6] who showed that it preserves composition of bundles: if \( q: D \to E \) and \( p: E \to B \) are smooth bundles then \((pq)! = p!q!\). This parallels our Theorem 1.3. Beside the difference in setting between Lott’s and Bunke’s results and the ones described in this paper the direction of their work is opposite to ours. While Lott’s construction of the pushforward map uses explicitly the analytic torsion form, we construct the secondary transfer first and then show that the smooth torsion of a bundle can be recovered from it. Also, while we use the composition property of the secondary transfer to obtain the composition formula for cohomological torsion (1-1), Bunke derives his result using a theorem of Ma [17] which states that an analog of the formula (1-1) holds for higher analytic torsion.

1.7. Organization of the paper. Section 2 contains a brief review of Waldhausen categories which provide the technical setting for the majority of construction of this paper. In Section 3 we take a closer look at the statements of Theorem 1.2 and its analog for unipotent fibrations, Theorem 3.5. The proof of both of these facts
uses a theorem of Brown\cite{brown}, which states that the singular chain complex of the total space of a fibration is quasi-isomorphic to a twisted tensor product of the chain complexes of the base and the fiber. In Section 4 we give an overview of this result and describe some properties of Brown’s quasi-isomorphism. In Section 5 we complete proofs of Theorems 1.2 and 3.5, which lets us complete the construction of the secondary transfers $\text{Wh}_h^q(p')$ and $\text{Wh}_h^q(p')$. In §6 we show that the homotopy secondary transfer if a fiberwise homotopy invariant. In Section 7 we obtain additivity formulas for both smooth and homotopy secondary transfers, and in Sections 8 and 9 we study secondary transfers of compositions of unipotent bundles and fibrations, which leads us to the proof of Theorem 1.3. Finally, in Section 10 we prove the relationship between the smooth secondary transfer and the smooth torsion of a bundle $p$ described by Theorem 1.4. We also show that it implies the statement on Theorem 1.1. Several arguments of the paper involve constructions of maps between homotopy fibers and constructions of homotopies of such maps. The appendix (§11) gives the basic outline of such constructions.

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## 2. Technical setup

A great majority of constructions described in this paper is set within the realm of Waldhausen categories \cite{waldhausen}, i.e. categories with distinguished classes of weak equivalences and cofibrations that satisfy certain axioms. Our basic setup in this respect will be largely the same as that of \cite[Section 3]{brown}, so we summarize it here only briefly. Given a Waldhausen category $\mathcal{C}$ we will denote by $K(\mathcal{C})$ the $K$-theory of $\mathcal{C}$. The standard construction of $K(\mathcal{C})$ proceeds using Waldhausen’s $S$-construction. For our purposes it will be more convenient though to use its variant, the $S'_+$-construction described by Blumberg and Mandell in \cite[§2]{blumberg_mandell}.

### 2.1. We will work mainly with two specific instances of Waldhausen categories. For a topological space $X$ the category $\mathcal{R}_{fd}^h(X)$ has as its objects homotopy finitely dominated retractive spaces over $X$, while its morphisms are maps of retractive spaces. It is a Waldhausen category with cofibrations given by closed embeddings having the homotopy extension property and weak equivalences defined as homotopy equivalences. The $K$-theory of $\mathcal{R}_{fd}^h(X)$ is the Waldhausen algebraic $K$-theory of $X$ and it is denoted by $A(X)$\footnote{See also \cite{igusa_klein} for a nice description of the relationship of Brown’s work to the higher torsion of Igusa-Klein.}.

\footnote{If $X$ is a path connected space then by abuse of notation by $\mathcal{R}_{fd}^h(X)$ we will understand the Waldhausen category of path connected retractive spaces over $X$. From the perspective of $K$-theory this change is of little consequence: the functor that embeds this category into the category of all retractive spaces over $X$ induces a homotopy equivalence of the associated $K$-theory spaces.}
Next, by $\mathcal{C}h^{fd}(\mathbb{Q})$ we will denote the category of homotopy finitely dominated chain complexes of $\mathbb{Q}$-vector spaces. This is a Waldhausen category with degreewise monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. We will denote by $K(\mathbb{Q})$ the $K$-theory of $\mathcal{C}h^{fd}(\mathbb{Q})$. This space describes the algebraic $K$-theory of the field of rational numbers: $K(\mathbb{Q}) \simeq \Omega BGL(\mathbb{Q})^+$.

In order to obtain a convenient combinatorial model of the space $Q(X_\ast)$ we will use one more instance of a construction of $K$-theory which, while it does not come from a Waldhausen category, is very closely related to the $S_\ast$-construction. We outline it briefly here and refer to Waldhausen’s papers [19, 21] where it was originally developed and [2] for details. Let $X$ be a smooth compact manifold. A partition of $X \times I$ is a codimension 0 submanifold $P \subseteq X \times I$ such that $X \times [0, \frac{1}{2}] \subseteq P$ and that satisfies some further conditions. By $\mathcal{P}_0(X \times I)$ we will denote the poset of partitions ordered by inclusion. Similarly, for $k \geq 0$ we define $\mathcal{P}_k(X \times I)$ to be the poset of locally trivial bundles of partitions over the standard simplex $\Delta^k$. These categories taken together form a simplicial category $\mathcal{P}_\ast(X \times I)$. In each category $\mathcal{P}_k(X \times I)$ we can introduce an analog of the Waldhausen category structure where, roughly, every morphism is a cofibration and weak equivalences are the identity morphisms. While in general pushouts do not exists in $\mathcal{P}_k(X \times I)$ and so the $S_\ast$-construction cannot be performed in this setting, it is possible to use its variant, the $\mathcal{T}_\ast$-construction to obtain a space $[\mathcal{T}_\ast \mathcal{P}_k(X \times I)]$. Denote by $[\mathcal{T}_\ast \mathcal{P}_\ast(X \times I)]$ the space obtained by applying the $\mathcal{T}_\ast$-construction to each category $\mathcal{P}_k(X \times I)$ and then taking geometrical realization of the resulting simplicial space. The final step in the construction of $Q(X_\ast)$ is stabilization. It is obtained by means of maps $[\mathcal{T}_\ast \mathcal{P}_\ast(X \times I^m)] \to [\mathcal{T}_\ast \mathcal{P}_\ast(X \times I^{m+1})]$ that are induced by functors $\mathcal{P}_k(X \times I^m) \to \mathcal{P}_k(X \times I^{m+1})$ that, roughly, send a partition $P \subseteq X \times I^m$ to $P \times I \subseteq X \times I^{m+1}$. Waldhausen showed that there exists a weak equivalence

$$Q(X_\ast) \simeq \Omega \lim_m [\mathcal{T}_\ast \mathcal{P}_\ast(X \times I^m)]$$

2.2. A functor $F : \mathcal{C} \to \mathcal{D}$ of Waldhausen categories is exact if it preserves weak equivalences, cofibrations and pushouts of diagrams

$$(2-1)\quad c' \leftarrow c \to c''$$

where one of the morphisms is a cofibration. Any such functor induces a map of infinite loop spaces $K(F) : K(\mathcal{C}) \to K(\mathcal{D})$. The advantage of working with the $S'_\ast$-construction is that we can obtain the map $K(F)$ under more relaxed assumptions on the functor $F$. Namely, following the terminology of [1, 3.4] we will say that a functor $F$ is almost exact if it preserves weak equivalences and cofibrations, and if it preserves pushouts of diagrams (2-1) up to a weak equivalence. An almost exact functor induces a functor of simplicial categories $F : S'_\ast \mathcal{C} \to S'_\ast \mathcal{D}$, and so it yields a map $K(F) : K(\mathcal{C}) \to K(\mathcal{D})$.

2.3. As we have already mentioned exact and almost exact functors between Waldhausen categories define maps between their associated $K$-theories. We will frequently need to construct homotopies of such maps. There are two main sources of such constructions. First, if $F, G : \mathcal{C} \to \mathcal{D}$ are (almost) exact functors then a natural
weak equivalence \( \varphi : F \to G \) defines a homotopy \( K(\varphi) \) between the induced maps \( K(F) \) and \( K(G) \). Second, if \( F_i : \mathcal{C} \to \mathcal{D} \) are (almost) exact functors for \( i = 0, 1, 2 \) and \( \varphi : F_0 \Rightarrow F_1, \psi : F_1 \Rightarrow F_2 \) are natural transformations such that

\[
F_0(c) \xrightarrow{\varphi} F_1(c) \xrightarrow{\psi} F_2(c)
\]

is a cofibration sequence for each \( c \in \mathcal{C} \), then the additivity theorem of Waldhausen [20, Theorem 1.4.2] provides a homotopy \( \mathcal{U} : K(\mathcal{C}) \times I \to K(\mathcal{D}) \) between the map \( K(F_1) \) and \( K(F_0 \lor F_2) \). A convenient combinatorial construction of this homotopy was given by Grayson in [8].

2.4. Our work will require us to go a step further beyond homotopies and consider homotopies of homotopies. If \( f_0, f_1 : X \to Y \) are two maps between topological spaces and \( h_0, h_1 : X \times I \to Y \) are homotopies between \( f_0 \) and \( f_1 \) then by a homotopy of homotopies we will understand a map \( H : X \times I \times I \to Y \) such that for each \( t \in I \) the map \( H_t = H(\cdot, \cdot, t) \) is a homotopy between \( f_0 \) and \( f_1 \) with \( H_0 = h_0, H_1 = h_1 \).

In our constructions of homotopies of homotopies we will almost always have \( Y = K(\mathbb{Q}) \). The homotopies of homotopies we will consider will be obtained as an application of one of the following lemmas:

2.5. Lemma. Let \( \mathcal{C} \) be a Waldhausen category, let \( F_1, F_2 : \mathcal{C} \to \mathcal{C}h^{fd}(\mathbb{Q}) \) be almost exact functors, and let \( \varphi_1, \varphi_2 : F_1 \Rightarrow F_2 \) be natural weak equivalences. If \( \Phi \) is a natural chain homotopy between \( \varphi_1 \) and \( \varphi_2 \) then \( \Phi \) defines a homotopy of homotopies \( K(\Phi) \) between \( K(\varphi_1) \) and \( K(\varphi_2) \).

Proof: We start with the following observation. Assume that we are given three almost exact functors \( G_i : \mathcal{C} \to \mathcal{C}h^{fd}(\mathbb{Q}), i = 0, 1, 2 \) and two natural weak equivalences \( \psi_0 : G_0 \Rightarrow G_1 \) and \( \psi_1 : G_1 \Rightarrow G_2 \). In such situation we obtain a map

\[
K(\mathcal{C}) \times \Delta^2 \to K(\mathbb{Q})
\]

If we consider it as a family of maps \( K(\mathcal{C}) \to K(\mathbb{Q}) \) parametrized by \( \Delta^2 \) then vertices of \( \Delta^2 \) correspond to the functions \( K(G_i) \) and edges of \( \Delta^2 \) correspond to the homotopies \( K(\psi_0), K(\psi_1) \), and \( K(\psi_1 \psi_0) \).

For a chain complex \( C \) let \( \text{Cyl}(C) \) denote the mapping cylinder of the identity function \( \text{id} : C \to C \) [22, 1.5.5]. We have \( \text{Cyl}(C)_n = C_n \oplus C_{n-1} \oplus C_n \). Let \( j_0, j_1 : C \to \text{Cyl}(C) \) be the chain maps given by \( j_0(c) = (c, 0, 0) \) and \( j_1(c) = (0, 0, c) \). Recall that two chain maps \( f_0, f_1 : C \to D \) are chain homotopic if and only if there exists a chain map \( h : \text{Cyl}(C) \to D \) such that \( h j_i = f_i \) for \( i = 0, 1 \).

Going back to the setting of the lemma the natural chain homotopy \( \Phi \) gives a natural weak equivalence of functors \( \overline{\Phi} : \text{Cyl}(F_1) \Rightarrow F_2 \). Let \( \overline{\Psi} : \text{Cyl}(F_1) \Rightarrow F_1 \) denote the natural weak equivalence corresponding to the natural chain homotopy from the identity natural transformation \( \text{id} : F_1 \Rightarrow F_1 \) to itself. We obtain the
following commutative diagram:

Each vertex of this diagram corresponds to a functor $\mathcal{C} \to \mathcal{Ch}^{fd}(Q)$ and edges are natural weak equivalences of such functors. By the observation above each commutative triangle in this diagram induces a map $K(\mathcal{C}) \times \Delta^2 \to K(Q)$. Taken together these maps define a map $H: K(\mathcal{C}) \times I^2 \to K(Q)$. Considering $H$ as a family of functions $K(\mathcal{C}) \to K(Q)$ parametrized by $I^2$ we obtain that two adjacent edges of the square $I^2$ parametrize the homotopies $K(\varphi_1)$ and $K(\varphi_2)$ and each of remaining two edges parametrizes the homotopy defined by the identity natural transformation $\text{id}: F_1 \Rightarrow F_1$. Using the identity $\varphi_i \text{id} = \varphi_i$ we can further modify $H$ to a homotopy of homotopies between $K(\varphi_1)$ and $K(\varphi_2)$.

\[\square\]

2.6. Lemma. Assume that $F_i, G_i: \mathcal{C} \to \mathcal{Ch}^{fd}(Q)$ ($i = 0, 1, 2$) are almost exact functors, and that we have a commutative diagram of natural transformations

\[
\begin{array}{ccc}
F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 \\
\eta_0 & & \eta_1 & & \eta_2 \\
G_0 & \longrightarrow & G_1 & \longrightarrow & G_2
\end{array}
\]

where both rows are cofibration sequences and vertical arrows are natural weak equivalences. This yields a diagram

\[
\begin{array}{ccc}
K(F_1) & \longrightarrow & K(F_0 \vee F_2) \\
K(\eta_1) & & K(\eta_0 \vee \eta_2) \\
K(G_1) & \longrightarrow & K(G_0 \vee G_2)
\end{array}
\]

(2-2)

In this diagram every vertex represents a map $K(\mathcal{C}) \to K(Q)$ and each edge is a homotopy of such maps. In this setting there exists a homotopy of homotopies that fills this diagram, i.e. a homotopy of homotopies between the concatenation of $\mathcal{U}$ with $K(\eta_0 \vee \eta_2)$ and the concatenation of $K(\eta_1)$ with $\mathcal{U}$.

Proof. Denote by $\mathcal{E}\mathcal{Ch}^{fd}(Q)$ the Waldhausen category of short exact sequences in $\mathcal{Ch}^{fd}(Q)$. The functors $F_i$ and $G_i$ define functors $F, G: \mathcal{C} \to \mathcal{E}\mathcal{Ch}^{fd}(Q)$, and the
natural weak equivalences $\eta_i$ define a natural weak equivalence $\eta: F \Rightarrow G$. On the level of $K$-theory this yields a homotopy

$$K(\eta): K(C) \times I \to K(\mathcal{E}Ch^{fd}(Q))$$

For $i = 0, 1, 2$ let $ev_i: \mathcal{E}Ch^{fd}(Q) \to Ch^{fd}(Q)$ denote the functor given by

$$ev_i(C_0 \to C_1 \to C_2) = C_i$$

The Waldhausen additivity theorem can be equivalently stated by saying that there exists a homotopy $\mathcal{E}U: K(\mathcal{E}Ch^{fd}(Q)) \times I \to K(Q)$ between the map $K(ev_1)$ and $K(ev_0 \vee ev_2)$. The homotopy of homotopies in the diagram (2-2) is obtained by composing the map

$$K(\eta) \times \text{id}_I: K(C) \times I \times I \to K(\mathcal{E}Ch^{fd}(Q)) \times I$$

with $\mathcal{E}U$. □

2.7. Lemma. Consider the following a commutative diagram of almost exact functors $\mathcal{C} \to Ch^{fd}(Q)$ and their natural transformations:

$$\begin{array}{ccc}
A_0 & \longrightarrow & B_0 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & B_1 \\
\downarrow & & \downarrow \\
A_2 & \longrightarrow & B_2 \\
\end{array}$$

Assume that each row and each column of this diagram is a short exact sequence of functors and that the canonical natural transformation $A_1 \cup_{A_0} B_0 \Rightarrow B_1$ is a cofibration. In this situation we obtain two homotopies between the map $K(B_1)$ and the map $K(A_0) + K(C_0) + K(A_2) + K(C_2)$:

1) the homotopy $U_{rc}$ is obtained by applying the additivity theorem to the middle row which produces a homotopy $K(B_1) \simeq K(A_1) + K(C_1)$, and then applying additivity theorem to the left and right columns which gives a homotopy $K(A_1) + K(C_1) \simeq K(A_0) + K(A_2) + K(C_0) + K(C_2)$.

2) the homotopy $U_{cr}$ which is obtained in the same manner, but applying additivity to the middle column first, and then to the top and bottom rows.

There exists a homotopy of homotopies between $U_{rc}$ and $U_{cr}$

Proof. Let $\mathcal{E}Ch^{fd}(Q)$ denote the Waldhausen category of short exact sequences in $Ch^{fd}(Q)$ and let $\mathcal{E}_2Ch^{fd}(Q)$ be the Waldhausen category of short exact sequences in $\mathcal{E}Ch^{fd}(Q)$. The diagram (2-3) can be interpreted as a functor $F: \mathcal{C} \to \mathcal{E}_2Ch^{fd}(Q)$
The assignment \( X \mapsto \text{fib}(X) \). Assume that we have a fibration \( E \to B \). Let \( p \) denote the pullback diagram of \( E \to B \). Assume that we have a fibration \( E \to B \). Let \( p \) denote the pullback diagram of \( E \to B \).

This functor is almost exact, so it induces a map \( \lambda : C(E) \to C(B) \). We refer to [2, §3] for details.

Next, recall that the structure of a retractive space \( X \) is given by the composition of \( C(E) \) and \( C(B) \). The homotopy of homotopies between \( U_{rc} \) and \( U_{cr} \) is then given by the composition

\[
K(E) \times I^2 \xrightarrow{\lambda} K(\text{fib}(E)) \times I^2 \xrightarrow{\lambda} K(E) \times I \xrightarrow{U} K(Q)
\]

\[\square\]

3. The linearization and transfer maps

3.1. In preparation for the proof of Theorem 1.2 we start this section by reviewing briefly the construction of the linearization map \( \lambda_B : Q(B) \to K(Q) \). For an arbitrary space \( B \) we have the assembly map \( a_B : Q(B) \to A(B) \). The combinatorial construction of this map can be outlined as follows. Recall (2.1) that we are working with a model of \( Q(B) \) built using the category of partitions, and that the space \( A(B) \) is constructed using the category \( \text{fib}(B) \). By definition a partition \( P \subseteq B \times I \) contains \( B \times \{0\} \) as a subspace which gives \( P \) the structure of a retractive space over \( B \). In this way we can regard \( P \) as an object of \( \text{fib}(B) \). It is possible to extend this assignment to all parametrized and stabilized partitions (using parametrized and stabilized retractive spaces over \( B \) are an intermediate step). The map \( a_B \) is induced by this assignment. We refer to [2, §3] for details.

Next, recall that the \( K(Q) \) was constructed from the Waldhausen category of chain complexes \( \text{fib}(Q) \). Consider the functor

\[
\Lambda_B : \text{fib}(B) \to \text{fib}(Q)
\]

that assigns to a retractive space \( X \) the relative singular chain complex \( C_*(X,B) \). This functor is almost exact, so it induces a map \( \lambda_B^h : A(B) \to K(Q) \). We will call \( \lambda_B^h \) the A-theory linearization. The linearization map \( \lambda_B : Q(B) \to K(Q) \) is given by the composition

\[
\lambda_B = \lambda_B^h a_B
\]

3.2. Assume that we have a fibration \( p : E \to B \). For a retractive space \( X \in \text{fib}(B) \) let \( p^*X \) denote the pullback

\[
p^*X := \lim(X \to B \leftarrow E)
\]

The assignment \( X \mapsto p^*X \) defines an exact functor \( \text{fib}(B) \to \text{fib}(E) \). We will call the induced map \( A(p^*) : A(B) \to A(E) \) the A-theory transfer of \( p \).
If $p$ is a smooth bundle of manifolds then in a similar way we can define a map $Q(p^!): Q(B_+) \to Q(E_+)$. This map is induced by functors of categories of partitions $\mathcal{P}_k(B \times I^m) \to \mathcal{P}_k(E \times I^m)$ that, roughly, associate to a partition $P \subseteq B \times I^m$ the partition $(p \times \text{id})^{-1}(P) \subseteq E \times I^m$. The map $Q(p^!)$ obtained in this way coincides with the Becker Gottlieb transfer [2, §4].

3.3. Let $p: E \to B$ be a fibration with a homotopy finitely dominated fiber $F_p$. The maps described above can be assembled into a diagram

$$
\begin{array}{ccc}
Q(B_+) & \xrightarrow{Q(p^!)} & Q(E_+) \\
\downarrow a_B & & \downarrow a_E \\
A(B) & \xrightarrow{A(p^!)} & A(E) \\
\downarrow A^h_B & & \downarrow A^h_E \\
K(Q) & \xrightarrow{\chi(F_p)} & K(Q)
\end{array}
$$

(3-1)

The outer rectangle in this diagram coincides with the diagram (1-2). If $p: E \to B$ is a smooth bundle then directly from the constructions described above it follows that the upper square commutes up to a homotopy induced by the natural transformation that for a partition $P \subseteq B \times I$ is given by the isomorphism

$$(p \times \text{id})^{-1}(P) \to p^*P$$

of retractive spaces over $E$. We will denote this homotopy by $\mu_p^3$. In order to obtain Theorem 1.2 it is then enough to show that the lower square of (3-1) is homotopy commutative. We will show that this fact holds for any unipotent fibration $p$:

3.4. **Definition.** A fibration $p: E \to B$ is unipotent if $B$ is a path connected space, both $B$ and the fiber $F_p$ of $p$ have the homotopy type of a finite CW-complex, and $H_*(F_p)$ admits a filtration by $\pi_1 B$-modules such that the action of $\pi_1 B$ on the filtration quotients is trivial.

3.5. **Theorem.** Let $p: E \to B$ be a unipotent fibration with fiber $F_p$. The diagram

$$
\begin{array}{ccc}
A(B) & \xrightarrow{A(p^!)} & A(E) \\
\downarrow A^h_B & & \downarrow A^h_E \\
K(Q) & \xrightarrow{\chi(F_p)} & K(Q)
\end{array}
$$

(3-2)

While all maps in the upper square of (3-1), i.e. the Becker-Gottlieb transfer, the $A$-theory transfer and assembly maps are defined for any fibration, smoothness of $p$ is essential for commutativity. This diagram does not commute, in general, when $p$ is a fibration. See e.g. the proof of Theorem F in [14].
commutes up to a preferred homotopy \( \eta^p_B : A(B) \times I \to K(\mathbb{Q}) \).

In the next section we describe some technical tools that we will use in the proof of this fact. The proof itself is given in Section 5.

4. Twisted tensor products

Consider the diagram (3-2). The lower horizontal map in this diagram can be described combinatorially as follows. Let \( H_* (F_p) \) be the chain complex of rational homology groups of \( F_p \) with trivial differentials. The functor

\[
- \otimes H_* (F_p) : \text{Ch}^{fd}(\mathbb{Q}) \to \text{Ch}^{fd}(\mathbb{Q})
\]

is exact and \( \chi(F_p) : K(\mathbb{Q}) \to K(\mathbb{Q}) \) is the map induced by this functor. As a consequence the map \( \chi(F_p)_B \) comes from the functor \( \mathcal{R}^{fd}(B) \to \mathcal{C}^{fd}(\mathbb{Q}) \) that associates to a retractive space \( X \) the chain complex \( C_*(X, B) \otimes H_* (F_p) \). On the other hand the composition \( A^h_E A(p^f) \) is induced by the functor \( \mathcal{R}^{fd}(B) \to \mathcal{C}^{fd}(\mathbb{Q}) \) that assigns to a space \( X \) the chain complex \( C_*(p^f X, E) \). The main ingredient of the proof of Theorem 3.5 is the fact that under the assumption that \( p : E \to B \) is a unipotent fibration for any \( X \in \mathcal{R}^{fd}(B) \) we can construct a path in \( K(\mathbb{Q}) \) joining the points represented by these two chain complexes. This path is natural to the extent that it gives rise to a homotopy filling the diagram (3-2). Our main tool in the construction of this path will be the theorem of Brown [5] which shows that the chain complex of the total space of a fibration is quasi-isomorphic to a twisted tensor product of the chain complex of the base and the homology of the fiber. We begin this section by reviewing the relevant notions in homological algebra. Subsequently we describe Brown’s result and develop some of its properties that we will need later on.

4.1. Twisting cochains and twisted tensor products. Let \( A \) be a differential graded \( \mathbb{Q} \)-algebra with multiplication \( \mu : A \otimes A \to A \), and let \( K \) be a d.g. \( \mathbb{Q} \)-coalgebra with comultiplication \( \nabla : K \to K \otimes K \). Given homomorphisms of graded vector spaces \( \varphi, \psi : K \to A \) the cup product \( \varphi \cup \psi : K \to A \) is given by the formula

\[
\varphi \cup \psi := \mu(\varphi \otimes \psi) \nabla
\]

If \( M \) is a left \( A \)-module with multiplication \( \nu : A \otimes M \to M \) then for \( \varphi \) as above and \( c \in K \otimes M \) the cap product \( \varphi \cap c : K \otimes M \to K \otimes M \) is given by

\[
\varphi \cap c := (\text{id}_K \otimes \nu)(\text{id}_K \otimes \varphi)(\text{id}_M)(\nabla \otimes \text{id}_M)(c)
\]

For a fixed \( \varphi \) the map

\[
\varphi \cap - : K \otimes M \to K \otimes M
\]

is a homomorphism of graded vector spaces.
4.2. Definition. Let $A$, $K$, $M$ be respectively a d.g. $\mathbb{Q}$-algebra, coalgebra, and a left $A$-module as above.

(i) A twisting cochain is a homomorphism of graded vector spaces $\varphi: K \to A$ that lowers grading by 1 and satisfies the identity
\[ \partial \varphi - \varphi \partial + \varphi \cup \varphi = 0 \]

(ii) If $\varphi: K \to A$ is a twisting cochain then the twisted tensor product $K \otimes_{\varphi} M$ is a chain complex such that $K \otimes_{\varphi} M = K \otimes M$ as a graded vector space, and the differential in $K \otimes_{\varphi} M$ is given by
\[ \partial_{\varphi} := \partial \otimes \text{id} + \text{id} \otimes \partial + \varphi \cap - \]

4.3. Twisted chain complex of a fibration. Let $X$ be a topological space with a basepoint $x_0$. By $C'_*(X)$ we will denote the subcomplex of the singular chain complex $C_*(X)$ with coefficients in $\mathbb{Q}$ generated by all singular simplices $\sigma: \Delta^n \to X$ that send all vertices of $\Delta^n$ into $x_0$. If $X$ is a path connected space then the inclusion $C'_*(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. The chain complex $C'_*(X)$ can be equipped with the usual d.g. coalgebra structure with comultiplication $\nabla: C'_*(X) \to C'_*(X) \otimes C'_*(X)$ defined by $\nabla(\sigma) = \sum_{i=0}^n f_i(\sigma) \otimes b_{n-i}(\sigma)$ where $\sigma \in C'_*(X)$ is a singular $n$-simplex and $f_i(\sigma)$, $b_{n-i}(\sigma)$ denote, respectively, the front $i$-th face and the back $(n-i)$-th face of $\sigma$.

For a space $X$ we can also consider its associated d.g. homology algebra $\text{End}(H_*(X))$ defined as follows. Let $\text{End}_n(H_*(X))$ denote the vector space of all maps of graded vector spaces $H_*(X) \to H_*(X)$ that increase the grading by $n$, and let
\[ \text{End}(H_*(X)) = \bigoplus_{n \geq 0} \text{End}_n(H_*(X)) \]

We view $\text{End}(H_*(X))$ as a chain complex with trivial differentials. The d.g. algebra structure on $\text{End}(H_*(X))$ comes from composition of maps. Naturally $H_*(X)$ is a module over this d.g. algebra.

The main result of [5] says that given a fibration $p: E \to B$ with a path connected base space and a fiber $F_p$, we can find a twisting cochain $\varphi_p: C'_*(B) \to \text{End}(H_*(F_p))$ such that the twisted tensor product $C'_*(B) \otimes_{\varphi_p} H_*(F_p)$ is naturally quasi-isomorphic to $C_*(E)$. For our purposes it will be convenient to state this fact in the following form. Let $\mathcal{S}_*$ denote the category of path connected, pointed spaces, and $\mathcal{S}_* \downarrow B$ be the over category of $\mathcal{S}_*$ over a space $B$. Given a fibration $p: E \to B$ and an object $X \in \mathcal{S}_* \downarrow B$ denote by $p_X: p^*X \to X$ the fibration induced from $p$.

4.4. Theorem. Let $p: E \to B$ be a fibration with a path connected, pointed base space $B$ and a fiber $F_p$. 

1) For every $X \in S_* \downarrow B$ there exists a twisting cochain
\[ \varphi_{px}: C'_X(X) \to \text{End}(H_*(F_p)) \]
and a quasi-isomorphism
\[ \beta_{px}: C_*(p^*X) \xrightarrow{\approx} C'_X(X) \otimes_{\varphi_{px}} H_*(F_p) \]

2) On $C'_X(X)$ the twisting cochain $\varphi_{px}: C'_X(X) \to \text{End}_0(H_*(F_p))$ is given as follows. If $\sigma$ is a singular simplex in $C'_X(X)$ then $\sigma$ is a loop in $X$, and so it represents an element $[\sigma] \in \pi_1 X$. For $z \in H_*(F_p)$ we have
\[ \varphi_{px}(\sigma)(z) = [\sigma]z - z \]
where the product $[\sigma]z$ is defined by the action of $\pi_1 X$ on $H_*(F_p)$.

3) The assignment $X \mapsto C'_X(X) \otimes_{\varphi_{px}} H_*(F_p)$ defines a functor
\[ F: S_* \downarrow B \to \mathcal{C}(\mathbb{Q}) \]
where $\mathcal{C}(\mathbb{Q})$ is the category of chain complexes over $\mathbb{Q}$. If $f: X \to Y$ is a morphism in $S_* \downarrow B$ then $F(f) = f_* \otimes \text{id}_{H_*(F_p)}$.

4) The quasi-isomorphisms $\beta_{px}$ define a natural transformation of functors.

4.5. Notation. For simplicity from now on we will write $C'_X(X) \otimes_{\varphi_{px}} H_*(F_p)$ to denote the complex $C'_X(X) \otimes_{\varphi_{px}} H_*(F_p)$.

4.6. While we refer to Brown’s paper [5] for the proof of Theorem 4.4, a few comments will be useful later on. Brown constructs the quasi-isomorphisms $\beta_{px}$ in two stages. First, he shows that given a path connected space $X$ one can construct a twisting cochain $\psi_X: C'_X(X) \to C_*(\Omega X)$ [5, Theorem 4.1], where the d.g. algebra structure on $C_*(\Omega X)$ is obtained by composing the Eilenberg-Zilber map and the map $C_*(\Omega X \times \Omega X) \to C_*(\Omega X)$ induced by the loop multiplication. The twisting cochain $\psi_X$ depends on the space $X$ only, not on a fibration over $X$. Moreover, the construction of $\psi_X$ is natural on the category of path connected spaces. The action of $\Omega X$ on the fiber $F_p$ of $p_X$ defines a $C_*(\Omega X)$-module structure on $C_*(F_p)$. Brown shows [5, Theorem 4.2] that the twisted tensor product $C'_X(X) \otimes_{\psi_X} C_*(F_p)$ is chain homotopy equivalent to $C_*(p^*X)$ via a chain homotopy equivalence natural in $X$. A minor technical point here is that in order to get a suitable action of $\Omega X$ on $F_p$ one needs to specify a weakly transitive lifting function for the fibration $p$. This can be taken care of by first replacing the fibration $p: E \to B$ by the homotopy equivalent fibration $\tilde{p}: PB \times_B E \to B$ where $PB$ is the space of Moore paths in $B$. The fibration $\tilde{p}$ admits a canonical lifting function [5, p.225] which can be used to get a chain homotopy equivalence
\[ C_*(p^*X) \xrightarrow{\approx} C_*(\tilde{p}^*X) \xrightarrow{\approx} C'_X(X) \otimes_{\psi_X} C_*(\tilde{p}) \]
where $F_{\bar{p}}$ is the fiber of $\bar{p}$. Since $F_{\bar{p}} \simeq F_{\rho}$ we have $C_*(F_{\bar{p}}) \simeq C_*(F_{\rho})$.\footnote{In [5] Brown gives a quasi-isomorphism going in the opposite direction, $C_*(X) \otimes_{\phi_X} C_*(F_{\rho}) \to C_*(p^*X)$, However, since his argument relies on the method of acyclic models it also produces a natural homotopy inverse of that map, and we work here with this inverse for convenience.}

To complete the construction of $\beta_{pX}$ it suffices to show that we have quasi-isomorphisms

$$C'_*(X) \otimes_{\phi_X} C_*(F_{\rho}) \xrightarrow{\cong} C'_*(X) \otimes_{px} H_*(F_{\rho})$$

One can proceed as follows. Using the fact that we deal here with chain complexes over a field we can find chain maps

$$j_{F_{\rho}}: H_*(F_{\rho}) \xrightarrow{\cong} C_*(F_{\rho})$$

such that for $z \in H_*(F_{\rho}) \cong H_*(F_{\rho})$ the element $j_{F_{\rho}}(z) \in C_*(F_{\rho})$ is a chain representing $z$, $r_{F_{\rho}}j_{F_{\rho}} = \text{id}_{H_*(F_{\rho})}$, and $j_{F_{\rho}}r_{F_{\rho}} = \text{id}_{C_*(F_{\rho})}$. The maps $j_{F_{\rho}}$ and $r_{F_{\rho}}$ define a strong deformation retraction of (untwisted) tensor products

$$\text{id} \otimes j_{F_{\rho}}: C'_*(X) \otimes_{px} H_*(F_{\rho}) \xrightarrow{\cong} C'_*(X) \otimes C_*(F_{\rho})$$

The Basic Perturbation Lemma (see e.g. [15, 2.6]) shows that in such situation there is a twisting cochain $\phi_{px}: C_*(X) \to \text{End}(H_*(F_{\rho}))$ and a strong deformation retraction of twisted tensor products

$$(\text{id} \otimes j_{F_{\rho}})^\circ: C'_*(X) \otimes_{px} H_*(F_{\rho}) \xrightarrow{\cong} C'_*(X) \otimes_{\phi_X} C_*(F_{\rho})$$

Following our convention (4.5) by $C'_*(X) \otimes_{px} H_*(F_{\rho})$ we denote here the complex $C'_*(X) \otimes_{\phi_X} H_*(F_{\rho})$. We define $\beta_{pX}$ as the composition

$$\beta_{pX}: C_*(p^*X) \to C'_*(X) \otimes_{\phi_X} C_*(F_{\rho}) \xrightarrow{(\text{id} \otimes r_{F_{\rho}})^\circ} C'_*(X) \otimes_{px} H_*(F_{\rho})$$

### 4.7. Homological filtration.

Let $\varphi: K \to A$ be a twisting cochain and let $M$ be an $A$-module. Directly from the definition of a twisted tensor product it follows that the chain complex $K \otimes_{\varphi} M$ admits an increasing filtration

$$U_0 \subset U_1 \subset \cdots \subset K \otimes_{\varphi} M$$

where $U_n := (\bigoplus_{q \leq n} K_q) \otimes_{\varphi} M$. In the case where $M$ has trivial differentials we have also a decreasing filtration

$$K \otimes_{\varphi} M = L_0 \supset L_1 \supset \cdots$$

given by $L_n := K \otimes_{\varphi} (\bigoplus_{q \geq n} M_q)$. Since we will consider this filtration in the situation where $M$ is the homology of some chain complex we will call it the homological filtration of $K \otimes_{\varphi} M$.

Let $p: E \to B$ be a fibration with fiber $F_{\rho}$, let $X \in S_{\rho} \downarrow B$, and let $\{L_n(pX)\}$ denote the homological filtration of the chain complex $C'_*(X) \otimes_{px} H_*(F_{\rho})$. We will need an explicit description of the quotients $L_n(pX)/L_{n+1}(pX)$. On the level of graded vector spaces we have isomorphisms

$$L_n(pX)/L_{n+1}(pX) \cong C'_*(X) \otimes H_*(F_{\rho})$$
In order to describe the differential in the filtration quotients notice that the differential in \( C'_s(X) \otimes_{p_X} H_*(F_p) \) is given by

\[
\partial(\sigma \otimes z) = \partial \sigma \otimes z + \sum_{i=0}^n f_i(\sigma) \otimes \varphi_{p_X}(b_{n-i}(\sigma))(z)
\]

where \( \sigma \) is a singular simplex in \( C'_s(X) \), \( z \in H_*(F_p) \) and \( f_i(\sigma) \), \( b_{n-i}(\sigma) \) are, respectively, the \( i \)-th front face and the \( (n-i) \)-th back face of \( \sigma \). Using part 2) of Theorem 4.4 we get from here

\[
\partial(\sigma \otimes z) = \partial \sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z) \quad \text{mod } L_{n+1}(p_X)
\]

As a consequence we obtain

**4.8. Proposition.** Let \( p : E \to B \) be a fibration with a fiber \( F_p \). For \( X \in S_x \downarrow B \) let \( C'_s(X) \otimes_{\pi_1X} H_*(F_p) \) denote the chain complex such that

\[
(C'_s(X) \otimes_{\pi_1X} H_*(F_p))_k = C'_{k-n}(X) \otimes H_*(F_p)
\]

with differential given by

\[
\partial(\sigma \otimes z) = \partial \sigma \otimes z + f_{n-1}(\sigma) \otimes ([b_1(\sigma)]z - z)
\]

for a singular simplex \( \sigma \in C'_s(X) \) and \( z \in H_*(F_p) \). We have a canonical isomorphism

\[
C'_s(X) \otimes_{\pi_1X} H_*(F_p) \cong L_n(p_X)/L_{n+1}(p_X)
\]

**4.9. Maps of fibrations.** Assume that that we have a map of fibrations over \( B \):

\[
\begin{array}{ccc}
E & \xrightarrow{g} & D \\
p & \downarrow & q \\
B & \xrightarrow{p} & B
\end{array}
\]

For \( X \in S_x \downarrow B \) let \( g_X : p^*X \to q^*X \) be the map of the induced fibrations over \( X \). Consider the diagram

\[
\begin{array}{ccc}
C_*(p^*X) & \xrightarrow{g_X^*} & C_*(q^*X) \\
\beta_{p_X} & & \beta_{q_X} \\
C_*(X) \otimes_{p_X} H_*(F_p) & \longrightarrow & C_*(X) \otimes_{q_X} H_*(F_q)
\end{array}
\]

where \( F_p, F_q \) denote, respectively, the fibers of \( p \) and \( q \). We would like to construct a natural lower horizontal map such that the resulting diagram commutes up to a homotopy. The obvious candidate for such map is \( \text{id} \otimes (g|_{F_p})_* \), where the homomorphism \( (g|_{F_p})_* : H_*(F_p) \to H_*(F_q) \) is induced by restriction of \( g \) to the fibers,
but this map is not a chain map in general. We can, however, proceed as follows. By the construction of quasi-isomorphisms \( \beta_{pX} \) and \( \beta_{qX} \) (4.6) we have a diagram

\[
\begin{array}{ccc}
C_*(p^*X) & \overset{g_{X^*}}{\longrightarrow} & C_*(q^*X) \\
\cong & & \cong \\
C'_*(X) \otimes_{\phi_X} C_*(F_p) & \overset{\text{id} \otimes (g|_{F_p})_*}{\longrightarrow} & C'_*(X) \otimes_{\phi_X} C_*(F_q)
\end{array}
\]

(4-2)

\[
\begin{array}{ccc}
C'_*(X) \otimes_{pX} H_*(F_p) & \longrightarrow & C'_*(X) \otimes_{qX} H_*(F_q) \\
\text{(id} \otimes r_{F_q})^\infty & & (\text{id} \otimes r_{F_q})^\infty \\
\text{id} \otimes (g|_{F_p})_*
\end{array}
\]

The compositions of the vertical maps give \( \beta_{pX} \) and \( \beta_{qX} \). The upper square commutes by the naturality properties of Brown’s theorem [5, Theorem 4.2]. Recall that the map \( \text{(id} \otimes r_{F_p})^\infty \) is a part of the strong deformation retraction data

\[
\text{id} \otimes (g|_{F_p})_* : C'_*(X) \otimes_{pX} H_*(F_p) \longrightarrow C'_*(X) \otimes_{qX} H_*(F_q)
\]

Define a map \( g_X^\infty : C'_*(X) \otimes_{pX} H_*(F_p) \rightarrow C'_*(X) \otimes_{qX} H_*(F_q) \) by

\[
g_X^\infty := (\text{id} \otimes r_{F_q})^\infty \circ (\text{id} \otimes (g|_{F_p})_*) \circ (\text{id} \otimes j_{F_p})^\infty
\]

The lower square in the diagram (4-2) commutes then up to a chain homotopy. The Basic Perturbation Lemma gives explicit formulas for this chain homotopy and for the maps \( \text{(id} \otimes r_{F_q})^\infty \) and \( \text{(id} \otimes j_{F_p})^\infty \). Direct computations involving these formulas yield the following fact:

**4.10. Proposition.** Let \( B \) be a pointed, path connected space. Let \( p : E \rightarrow B \) and \( q : D \rightarrow B \) be fibrations, and let \( g : E \rightarrow D \) be a map of fibrations.

1) The maps \( g_X^\infty \) define a natural transformation of functors \( S_* \downarrow B \rightarrow \mathcal{C}h(Q) \).

2) The diagram

\[
\begin{array}{ccc}
C_*(p^*X) & \overset{g_{X^*}}{\longrightarrow} & C_*(q^*X) \\
\beta_{pX} & = & \beta_{qX} \\
C'_*(X) \otimes_{pX} H_*(F_p) & \overset{g_X^\infty}{\longrightarrow} & C'_*(X) \otimes_{qX} H_*(F_q)
\end{array}
\]

commutes up to a chain homotopy that is natural in \( X \).

3) For \( X \in S_* \downarrow B \) consider the homological filtrations \( \{L_n(pX)\} \) and \( \{L_n(qX)\} \) of the complexes \( C'_*(X) \otimes_{pX} H_*(F_p) \) and, respectively, \( C'_*(X) \otimes_{qX} H_*(F_q) \) (4.7). The map \( g_X^\infty \) preserves these filtrations. Moreover, for every \( n \) the following diagram
commutes:

\[
\begin{array}{ccc}
L_n(p_X)/L_{n-1}(p_X) & \xrightarrow{g_X^\infty} & L_n(q_X)/L_{n-1}(q_X) \\
\cong & & \cong \\
C'_s(X) \otimes_{\pi_1 X} H_n(F_p) & \xrightarrow{id \otimes (g|_{F_p})_*} & C'_s(X) \otimes_{\pi_1 X} H_n(F_q)
\end{array}
\]

The vertical isomorphisms in this diagram come from Proposition 4.8.

5. The secondary transfer

We are now ready to give

Proof of Theorem 3.5. Consider the diagram (3-2). We want to construct a homotopy \( \eta \) between the maps \( \lambda E_A(p!) \) and \( \chi(F_p) \lambda B \). Recall that the map \( \lambda E_A(p!) \) is induced by the functor

\[ \Phi : \mathcal{R}^{ld}(B) \to \mathcal{C}^{rd}(Q) \]

that assigns to a retractive space \( X \) the relative chain complex \( C_*(p^*X, E) \). While the map \( \chi(F_p) \lambda B \) is induced by the functor

\[ \Psi : \mathcal{R}^{ld}(B) \to \mathcal{C}^{rd}(Q) \]

given by \( \Psi(X) = C_*(X, B) \otimes H_*(F_p) \). We will build a sequence of intermediate functors between \( \Phi \) and \( \Psi \) and show that maps induced by these functors can be connected by homotopies.

First, for \( X \in \mathcal{R}^{ld}(B) \) the inclusion map \( i_X : B \hookrightarrow X \) induces an inclusion \( \bar{i}_X : E \hookrightarrow p^*X \). The naturality of the quasi-isomorphisms \( \beta_{px} \) described in Theorem 4.4 implies that we have a commutative diagram

\[
\begin{array}{ccc}
C_*(E) & \xrightarrow{i_X^*} & C_*(p^*X) \\
\downarrow \beta_p & & \downarrow \beta_{px} \\
C'_s(B) \otimes_p H_*(F_p) & \xrightarrow{i_X^* \otimes \text{id}} & C'_s(X) \otimes_{px} H_*(F_p)
\end{array}
\]

Define

\[ C'_s(X, B) \otimes_{px} H_*(F_p) := \text{coker}(i_{X*} \otimes \text{id}) \]

The assignment \( X \mapsto C'_s(X, B) \otimes_{px} H_*(F_p) \) defines an almost exact functor

\[ \Phi_1 : \mathcal{R}^{ld}(B) \to \mathcal{C}^{rd}(Q) \]

and the natural quasi-isomorphisms

\[ \beta_{px} : C_*(p^*X, E) \xrightarrow{\cong} C'_s(X, B) \otimes_{px} H_*(F_p) \]
define a natural weak equivalence $\beta: \Phi \Rightarrow \Phi_1$. Denote by $K(\Phi_1): A(B) \to K(\mathbb{Q})$ the map induced by $\Phi_1$. The natural weak equivalence $\beta$ defines a homotopy

\begin{equation}
\lambda^h_{\mathcal{E}} A(p^1) \simeq K(\Phi_1)
\end{equation}

Next, since the map $i_{\mathcal{E}, \mathcal{F}} \otimes \text{id}$ preserves the homological filtration of twisted tensor products we can define

\[ L_n(p_X, p) := \text{coker}(L_n(p) \xrightarrow{i_{\mathcal{E}, \mathcal{F}} \otimes \text{id}} L_n(p_X)) \]

The chain complexes $L_n(p_X, p)$ form a decreasing filtration of $C'_*(X, B) \otimes_{p_X} H_*(F_p)$. Proposition 4.8 shows that the filtration quotient $L_n(p_X, p)/L_{n+1}(p_X, p)$ can be identified with the chain complex

\[ C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) := \text{coker}(C'_*(B) \otimes_{\pi_1 B} H_n(F_p) \to C'_*(X) \otimes_{\pi_1 X} H_n(F_p)) \]

Since $F_p$ is a homotopy finite space we have $H_q(F_p) = 0$ for $q$ large enough, and so $\{L_n(p_X, p)\}$ is in fact a finite filtration. The assignments $X \mapsto L_n(p_X, p)$, and $X \mapsto C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p)$ define almost exact functors $\mathcal{R}^{\mathcal{E}}(B) \to \mathcal{C}^{h\mathcal{F}}(\mathbb{Q})$. These functors are connected by natural short exact sequences

\begin{equation}
0 \to L_{n+1}(p_X, p) \to L_n(p_X, p) \to C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) \to 0
\end{equation}

Let $\Phi_2: \mathcal{R}^{\mathcal{E}}(B) \to \mathcal{C}^{h\mathcal{F}}(\mathbb{Q})$ denote the almost exact functor given by

\[ \Phi_2(X) := \bigoplus_n C'_*(X, B) \otimes_{\pi_1 X} H_n(F_p) \]

and let $K(\Phi_2): A(B) \to K(\mathbb{Q})$ be the map induced by $\Phi_2$. Applying repeatedly Waldhausen’s additivity theorem to the sequences (5-2) we obtain a homotopy

\begin{equation}
K(\Phi_1) = K(\Phi_2)
\end{equation}

Assume now for a moment that $p: E \to B$ is a fibration with the trivial action of $\pi_1B$ on $H_*(F)$. In this case the action of $\pi_1X$ on $H_*(F_p)$ is trivial as well, so we have isomorphisms

\[ \Phi_2(X) \cong C'_*(X, B) \otimes H_*(F_p) \]

Since by assumption $X$ and $B$ are path connected spaces we also have natural quasi-isomorphisms

\[ C'_*(X, B) \otimes H_*(F_p) \cong C_*(X, B) \otimes H_*(F_p) = \Psi(X) \]

As a consequence for every $X \in \mathcal{R}^{\mathcal{E}}(B)$ we obtain $\Phi_2(X) \cong \Psi(X)$ which induces a homotopy

\begin{equation}
K(\Phi_2) \simeq \chi(F_p)\lambda^h_{\mathcal{E}}
\end{equation}

Concatenating the homotopies (5-1), (5-3), and (5-4) we get the desired homotopy $\eta^h_{\mathcal{E}}$. If $p$ is an arbitrary unipotent fibration we need an additional step to pass between the maps $K(\Phi_2)$ and $\chi(F_p)\lambda^h_{\mathcal{E}}$. In this case the action of $\pi_1B$ need not
be trivial, but $H_*(F_p)$ admits a decreasing filtration $\{V^i\}$ such that $V^i$ is a $\pi_1B$-module and the action of $\pi_1B$ on the quotients $V^{i+1}/V^i$ is trivial. This defines a filtration $\{C'_*(X,B) \otimes_{\pi_1X} V^i\}$ of the complex $\Phi_2(X)$. The quotients of this filtration are the (untwisted) tensor products $C'_*(X,B) \otimes (V^i/V^{i+1})$. Define a functor $\Phi_3: R^fd(B) \to Ch^fd(Q)$ by

$$\Phi_3(X) := \bigoplus_i C'_*(X,B) \otimes (V^i/V^{i+1})$$

Naturally we also have a filtration $\{C'_*(X,B) \otimes V^i\}$ of the untwisted tensor product $C'_*(X,B) \otimes H_*(F_p)$ and $\Phi_3(X)$ is the direct sum of the quotients of this filtration. This means that using Waldhausen’s additivity theorem (and the quasi-isomorphisms $C'_*(X,B) \otimes H_*(F_p) \simeq C_*(X,B) \otimes H_*(F_p)$) we get homotopies

$$K(\Phi_2) \simeq K(\Phi_3) \simeq \chi(F_p) \lambda_B^h$$

The homotopy $\eta_p^h$ is then obtained as a concatenation of the homotopies (5-1), (5-3), and (5-5).

**Proof of Theorem 1.2.** The homotopy $\eta_p$ is obtained by concatenating the homotopy $\eta_p^h$ and the homotopy $\mu_p$ described in (3.3). □

Let $C \in Ch^fd(Q)$ be a chain complex. Notice that our construction of $K(Q)$ lets us identify $C$ with a point of $K(Q)$.

**5.1. Definition.** Let $B$ be a path connected space and let $C \in Ch^fd(Q)$. Denote by $Wh^Q_B(B)_C$ the homotopy fiber of the linearization map taken over the point $C \in K(Q)$.

$$Wh^Q_B(B)_C := \text{hofib}(\lambda_B: Q(B_+) \to K(Q))_C$$

For the zero chain complex $0 \in Ch^fd(Q)$ we will write $Wh^Q_B(B)$ to denote $Wh^Q_B(B)_0$.

Let $p: E \to B$ be a unipotent bundle with a fiber $F_p$. By Theorem 1.2 for any $C \in Ch^fd(Q)$ we have a map

$$Wh^Q_B(B)_C \to Wh^Q_B(E)_{C \otimes H_*(F_p)}$$

This gives rise to the following

**5.2. Definition.** The smooth secondary transfer of a unipotent bundle $p: E \to B$ is the map

$$Wh^Q(p^i): Wh^Q_B(B) \to Wh^Q_B(E)$$

determined by the Becker-Gottlieb transfer $Q(p^i)$ and the homotopy $\eta_p$ given by Theorem 1.2.

It will be convenient to consider a variant of this definition in the setting of unipotent fibrations:
5.3. **Definition.** For a path connected space $B$ let

$$\text{Wh}_h^Q(B) := \text{hofib}(\lambda^h: A(B) \to K(\mathbb{Q}))_0$$

The homotopy secondary transfer of a unipotent fibration $p: E \to B$ is the map

$$\text{Wh}_h^Q(p!) : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E)$$

determined by the transfer $A(p!)$ and the homotopy $\eta^h_p$ given by Theorem 3.5.

5.4. **Note.** Let $p: E \to B$ be a unipotent fibration with a fiber $F_p$. The construction of the homotopy $\eta^h_p$ described in the proof of Theorem 3.5 makes use of a choice of a strong deformation retraction $H_*(F_p) \cong C_*(F_p)$ and a choice of a unipotent filtration $\{V^i\}$ of $H_*(F_p)$. One can check though that the homotopy class of the map $\text{Wh}_h^Q(p!)$ is independent of these choices, and so it depends on the fibration $p$ only. Likewise, if $p$ is a unipotent bundle then the homotopy class of the smooth secondary transfer $\text{Wh}_h^Q(p!)$ depends only on the bundle $p$.

6. **Homotopy invariance of $\text{Wh}_h^Q(p!)$**

Let $f: E_1 \to E_2$ be a map of topological spaces. Such map defines an exact functor of Waldhausen categories

$$f_*: \mathcal{R}^{fd}(E_1) \to \mathcal{R}^{fd}(E_2)$$

given by $f_*(X) = X \cup_{E_1} E_2$ for $X \in \mathcal{R}^{fd}(E_1)$. This functor in turn induces a map $f_*: A(E_1) \to A(E_2)$. Consider the diagram

$$\begin{array}{ccc}
A(E_1) & \xrightarrow{f_*} & A(E_2) \\
\downarrow \lambda^h_{E_1} & & \downarrow \lambda^h_{E_2} \\
K(\mathbb{Q}) & & \\
\end{array}$$

Recall that the map $\lambda^h_{E_1}$ is induced by the functor $\mathcal{R}^{fd}(E_1) \to \mathcal{C}^{fd}(\mathbb{Q})$ given by $X \mapsto C_*(X, E_1)$. Similarly, the map $\lambda^h_{E_2} f_*$ comes from the functor defined by $X \mapsto C_*(X \cup_{E_1} E_2, E_2)$. The natural quasi-isomorphisms $C_*(X, E_1) \to C_*(X \cup_{E_1} E_2, E_2)$ define a homotopy $h_f$ between $\lambda^h_{E_1}$ and $\lambda^h_{E_2} f_*$. As a result we obtain a map

$$f_*: \text{Wh}_h^Q(E_1) \to \text{Wh}_h^Q(E_2)$$

Our goal in this section is to prove the following

6.1. **Proposition.** For $i = 1, 2$ let $p_i: E_i \to B$ be a unipotent fibration, and let $f: E_1 \to E_2$ be a fiberwise homotopy equivalence. There is a homotopy

$$f_*\text{Wh}_h^Q(p_1!) \simeq \text{Wh}_h^Q(p_2!)$$
Proof. Let \( F_{p_1} \) denote the fiber of \( p_i \). Using (11.2) we see that in order to obtain the desired homotopy it is enough to construct the following data:

1) a homotopy \( H^A : A(B) \times I \to A(E_2) \) between \( f_*A(p_1^1) \) and \( A(p_2^1) \);

2) a homotopy \( H^K : K(Q) \times I \to K(Q) \) between the maps \( \chi(F_{p_1}) \) and \( \chi(F_{p_2}) \);

3) a homotopy of homotopies that fills the following diagram:

\[
\begin{array}{ccc}
\lambda^h_{E_2} f_*A(p_1^1) & \xrightarrow{\lambda^h_{E_2} H^A_j} & \lambda^h_{E_2} A(p_2^1) \\
\downarrow h_f \circ (A(p_1^1) \times \text{id}_I) & & \downarrow \eta_{p_2} \\
\chi(F_{p_1}) \lambda^h_B & \xrightarrow{\chi(F_{p_2}) \lambda^h_B} & \chi(F_{p_2}) \lambda^h_B \\
\end{array}
\]

Each vertex of this diagram represents a map \( A(B) \to K(Q) \) and edges represent homotopies of such maps.

1) Construction of \( H^A_j \). The map \( f_*A(p_1^1) \) comes from the functor \( \mathcal{R}^{fd}(B) \to \mathcal{R}^{fd}(E_2) \) given by \( X \mapsto f_* p_1^1 X \) while \( A(p_2^1) \) is induced by the functor \( X \mapsto p_2^1 X \). Since \( f \) is a fiberwise homotopy equivalence the natural maps \( f_* p_1^1 X \to p_2^1 X \) induced by \( f \) are weak equivalences, and so they define the homotopy \( H^A_j \).

2) Construction of \( H^K_j \). Recall that for \( i = 1, 2 \) the map \( \chi(F_i) \) is induced by functor \( \mathcal{C}^{h^{fd}(Q)} \to \mathcal{C}^{h^{fd}(Q)} \) given by \( C \mapsto C \otimes H_*(F_i) \). Since the map \( f|_{F_1} : F_1 \to F_2 \) is a homotopy equivalence it induces an isomorphism of homology groups of the fibers

\[
(f|_{F_1})_* : H_*(F_1) \xrightarrow{\cong} H_*(F_2)
\]

This gives a natural isomorphism of functors

\[
- \otimes H_*(F_1) \Rightarrow - \otimes H_*(F_2)
\]

The homotopy \( H^K_j \) is defined by this natural isomorphism.
3) **Construction of the homotopy of homotopies.** Consider the following diagram:

Each vertex of this diagram represents a functor $\mathcal{R}^{dl}(B) \to \mathcal{C}H^{dl}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The maps $\beta_{p_{1X}}$ and $\beta_{p_{2X}}$ are the Brown quasi-isomorphisms (4.4) and the maps $f_{\infty}^X$ come from Proposition 4.10. On the level of $K$-theory each vertex of this diagram represents a map $A(B) \to K(\mathbb{Q})$ and the edges represent homotopies of such maps. The outer rectangle coincides with diagram (6-1).

In order to show that the diagram (6-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams (1)-(3) in the above diagram of functors can be filled by a homotopy of homotopies. In the case of subdiagram (1) such a homotopy of homotopies exists since this subdiagram commutes. By Proposition 4.10 subdiagram (2) commutes up to a natural chain homotopy, so it again can be filled by a homotopy of homotopies. Proposition 4.10 says also that the maps $f_{\infty}^X$ preserve the homological filtration of the twisted tensor products and that they induce the map $\text{id} \otimes (f|_{F_1})_* : H_*(F_1) \to H_*(F_2)$ is an isomorphism of $\pi_1 B$-modules, implies that we also have a homotopy of homotopies filling subdiagram (3).
7. Additivity of the secondary transfer

Our goal of this section is to prove that secondary transfer maps have additivity properties that are analogous to additivity of the Becker-Gottlieb transfer and the A-theory transfer. We start by considering additivity of the homotopy secondary transfer:

7.1. Theorem. For \( i = 0, 1, 2 \) let \( p_i: E_i \to B \) be a unipotent fibration. Assume that we have maps of fibrations

\[
\begin{array}{ccc}
E_1 & \xleftarrow{j} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
E_0 & \xrightarrow{p_0} & B
\end{array}
\]

where \( j \) is a cofibration over \( B \). Let \( E := E_1 \cup_{E_0} E_2 \) and let \( p: E \to B \) be the fibration given by \( p := p_1 \cup_{p_0} p_2 \). Then \( p \) is a unipotent fibration and we have

\[
[\operatorname{Wh}_h^Q(p)] = [k_1 \ast \operatorname{Wh}_h^Q(p_1)] + [k_2 \ast \operatorname{Wh}_h^Q(p_2)] - [k_0 \ast \operatorname{Wh}_h^Q(p_0)]
\]

Here \( k_i: \operatorname{Wh}_h^Q(E_i) \to \operatorname{Wh}_h^Q(E) \) is induced by the map \( k_i: E_i \to E \).

7.2. Lemma. Consider a diagram of chain complexes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g'} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

that commutes up to a chain homotopy \( h \). There exists a map \( g': \text{Cyl}(f) \to B' \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \text{Cyl}(f) \\
\downarrow{g} & & \downarrow{g'} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

commutes. Moreover \( g' \) is chain homotopic to the composition

\[
\text{Cyl}(f) \to B \xrightarrow{g'} B'
\]

Proof. Recall that \( \text{Cyl}(f)_n = A_n \oplus A_{n-1} \oplus B_n \). The map \( g'_n: \text{Cyl}(f)_n \to B'_n \) is given by

\[
g'_n(a_1, a_2, b) = f'g(a_1) + h(a_2) + g'(b)
\]

The second statement of the lemma is easy to verify. \( \square \)
Proof of Theorem 7.1. Let $F$ denote the fiber of the fibration $p$ and for $i = 0, 1, 2$ let $F_i$ be the fiber of $p_i$. The fact that the action of $\pi_1(B)$ on $F = F_1 \cup F_0 F_2$ is unipotent can be obtained using the Mayer-Vietoris sequence for homology of the fibers.

The strategy of the proof of additivity of the secondary transfer is as follows. We will construct maps $f_1^h : \text{Wh}^\mathcal{Q}_h(B) \rightarrow \text{Wh}^\mathcal{A}_h(E)$ and $f_2^h : \text{Wh}^\mathcal{Q}_h(B) \rightarrow \text{Wh}^\mathcal{Q}_h(E_2)$ such that $[f_1^h] = [k_2, f_2^h]$. We will also show that

$$[\text{Wh}^\mathcal{Q}_h(p_1')] = [k_1, \text{Wh}^\mathcal{Q}_h(p_1')] + [f_1^h] \quad \text{and} \quad [\text{Wh}^\mathcal{Q}_h(p_2')] = [j_*, \text{Wh}^\mathcal{Q}_h(p_0')] + [f_2^h].$$

Since $k_0 = k_2 j$ the second of these equations will give

$$[k_2, \text{Wh}^\mathcal{Q}_h(p_2')] = [k_0, \text{Wh}^\mathcal{Q}_h(p_0')] + [k_2, f_2^h] = [k_0, \text{Wh}^\mathcal{Q}_h(p_0')] + [f_2^h]$$

which, combined with the first equation, will yield the formula (7-1).

The construction of the map $f_1^h$ will proceed following the scheme outlined in (11.1). First, we will construct a map $f_1^A : A(B) \rightarrow A(E)$. Subsequently we will consider the diagram

$$
\begin{array}{ccc}
A(B) & \xrightarrow{f_1^A} & A(E) \\
\downarrow f_1^h & & \downarrow f_1^h \\
K(\mathbb{Q}) & \xrightarrow{\chi(\text{Cone}(k_1|_{F_i}))} & K(\mathbb{Q})
\end{array}
$$

(7-2)

Here $\text{Cone}(k_1|_{F_i})$ denotes the mapping cone of the map $k_1|_{F_i} : H_*(F_i) \rightarrow H_*(F)$, and the map

$$\chi(\text{Cone}(k_1|_{F_i})) : K(\mathbb{Q}) \rightarrow K(\mathbb{Q})$$

is induced by the functor $\mathcal{C}h^d(\mathbb{Q}) \rightarrow \mathcal{C}h^d(\mathbb{Q})$ given by tensoring by $\text{Cone}(k_1|_{F_i})$.

We will show that the diagram (7-2) commutes up to a preferred homotopy $h_1$. This homotopy together with the map $f_1^A$ will define the map $f_1^h$.

In order to obtain the map $f_1^A$ recall that the map $k_1 A(p_1')$ is induced by the functor that assigns to a space $X \in \mathcal{R}^d(B)$ the space $k_1 p_1^* X \in \mathcal{R}^d(E)$, and that the map $A(p_1')$ is induced by the functor $X \mapsto p^* X$. For $X \in \mathcal{R}^d(B)$ we have a cofibration $k_1 p_1^* X \rightarrow p^* X$. Let $M_X \in \mathcal{R}^d(E)$ denote the cofiber of this map. The assignment $X \mapsto M_X$ defines an exact functor $\mathcal{R}^d(B) \rightarrow \mathcal{R}^d(E)$. The map $f_1^A$ is induced by this functor. The above constructions give a short exact sequence of functors

$$k_1 p_1^* X \rightarrow p^* X \rightarrow M_X$$

(7-3)

Applying Waldhausen’s additivity theorem we obtain a homotopy

$$A(p_1') \simeq k_1 A(p_1') + f_1^A$$

(7-4)
Next, in order to describe a homotopy that fills the diagram (7-2) consider the following diagram of functors:

\[
\begin{align*}
C_*(k_1, p_1^*X, E) & \xrightarrow{=} C_*(M_X, E) \\
C_*(p_1^*X, E_1) & \xrightarrow{k_{X, *}} C_*(p^*X, E) \xrightarrow{=} \text{coker}(k_{X, *}) \\
C_*(p_1^*X, E_1) & \xrightarrow{=} C_*(p^*X, E) \\
C_*(X, B) \otimes p_1 X & \xrightarrow{\beta_{p_1 X}} C_*(X, B) \otimes p X \\
C_*(X, B) & \xrightarrow{g_{X}} \text{Cyl}(k_{X}) \xrightarrow{\text{additivity}} \text{Cone}(k_{X}) \\
C_*(X, B) \otimes H_*(F_1) & \xrightarrow{\text{additivity}} \text{Cyl}(k_{X}^\infty) \xrightarrow{\text{additivity}} \text{Cone}(k_{X}^\infty) \\
C_*(X, B) \otimes H_*(F_1) & \xrightarrow{\text{additivity}} C_*(X, B) \otimes \text{Cyl}(k_1|_{F_1, X}) \xrightarrow{\text{additivity}} C_*(X, B) \otimes \text{Cone}(k_1|_{F_1, X})
\end{align*}
\]

The map \(k_{X_*}\) in this diagram is induced by the map of fibrations \(k_X : p_1^*X \to p^*X\). The complexes \(\text{Cyl}(k_{X_*})\) and \(\text{Cone}(k_{X_*})\) are, respectively, the mapping cylinder and the mapping cone of \(k_{X_*}\). Similarly \(\text{Cyl}(k_{X}^\infty)\) and \(\text{Cone}(k_{X}^\infty)\) are the mapping cylinder and the mapping cone of the map \(k_{X}^\infty : C_*(X, B) \otimes H_*(F_1) \to C_*(X, B) \otimes H_*(F)\) given by Proposition 4.10. Finally, \(\text{Cyl}(k_1|_{F_1, X})\) and \(\text{Cone}(k_1|_{F_1, X})\) are the mapping cylinder and the mapping cone of the map \(k_1|_{F_1, X} : H_*(F_1) \to H_*(F)\). The horizontal maps are defined in the obvious way so that each row of the diagram forms a short exact sequence.

All vertical maps are quasi-isomorphism. They are defined in the obvious way with the exception the map \(g_{X}\) which is given as follows. By Proposition 4.10 we have a diagram

\[
\begin{align*}
C_*(p_1^*X, E_1) & \xrightarrow{k_{X, *}} C_*(p^*X, E) \\
C_*(X, B) \otimes p_1 X & \xrightarrow{\beta_{p_1 X}} C_*(X, B) \otimes p X \\
C_*(X, B) \otimes H_*(F_1) & \xrightarrow{k_{X}^\infty} C_*(X, B) \otimes H_*(F)
\end{align*}
\]
that commutes up to a chain homotopy. Using Lemma 7.2 we obtain from here a commutative diagram

\[
\begin{array}{ccc}
C_\ast(p_1^* X, E_1) & \longrightarrow & \text{Cyl}(k_{X\ast}) \\
\beta_{p,x} \downarrow & & \beta_{p,x} \\
C'_\ast(X, B) \otimes_{p_1 x} H_\ast(F_1) & \longrightarrow & C'_\ast(X, B) \otimes_{p_x} H_\ast(F)
\end{array}
\]

The map \(g_X\) is the composition of \(\bar{\beta}_{p,x}\) and the inclusion \(C'_\ast(X, B) \otimes_{p_1 x} H_\ast(F_1) \longrightarrow \text{Cyl}(k_{X\ast})\).

The lowest vertical edges in the diagram indicate a passage between chain complexes using additivity. This means the following construction. Since the map \(k_{X\ast}\) preserves the homological filtrations on \(C'_\ast(X, B) \otimes_{p_1 x} H_\ast(F_1)\) and \(C'_\ast(X, B) \otimes_{p_x} H_\ast(F)\) the complexes \(\text{Cyl}(k_{X\ast})\) and \(\text{Cone}(k_{X\ast})\) are endowed with induced filtrations. Each lowest vertical edge indicates a passage from the filtered chain complex to the direct sum of the filtration quotients. As usual, after passage to the induced maps \(A(B) \rightarrow K(Q)\) each additivity edge gives a homotopy obtained using Waldhausen’s additivity theorem. We note here that the additivity edges are related to one another as follows. The maps in the short exact sequence

\[
C'_\ast(X, B) \otimes_{p_1 x} H_\ast(F_1) \longrightarrow \text{Cyl}(k_{X\ast}) \longrightarrow \text{Cone}(k_{X\ast})
\]

preserve filtrations. Moreover, their restrictions to the filtration subcomplexes also form short exact sequences, and so do the induced maps of filtrations quotients. The bottom row of the diagram is the direct sum of the short exact sequences of these filtration quotients.

Each vertex of in the diagram (7-5) induces a map \(A(B) \rightarrow K(Q)\). All vertical edges define homotopies of such maps. Concatenation of homotopies defined by rightmost vertical edges gives a homotopy filling the diagram (7-2). This homotopy, together with the map \(f_1^A\) defines the map \(f_1^h: \text{Wh}_h^Q(B) \rightarrow \text{Wh}_h^Q(E)\).

The existence of a homotopy between the maps \(\text{Wh}_h^Q(p')\) and \(k_1 \cdot \text{Wh}_h^Q(p'_1) + f_1^h\) follows directly from the above construction. The map \(k_1 \cdot \text{Wh}_h^Q(p'_1)\) is defined by the map \(k_1 \cdot A(p'_1)\) and the homotopy induced by the leftmost vertical edges in the diagram (7-5). The map \(\text{Wh}_h^Q(p')\) is homotopic to the map defined by \(A(p')\) and the homotopy induced by the middle vertical edges in (7-5). As we have already noticed applying Waldhausen’s additivity theorem to the short exact sequence of functors (7-3) defines a homotopy between \(A(p')\) and \(k_1 \cdot A(p'_1) + f_1^A\). In order to lift this homotopy to a homotopy between \(\text{Wh}_h^Q(p')\) and \(k_1 \cdot \text{Wh}_h^Q(p'_1) + f_1^h\) it is enough to apply the additivity theorem to the horizontal short exact sequences in the diagram (7-5).
Construction of the map $f_2^h : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E_2)$ proceeds in the same way as the construction of $f_1^h$, with the cofibration $j : E_0 \to E_2$ used in place of $k_1$. By the same argument as above we obtain a homotopy $\text{Wh}_h^Q(p_1') = j_*\text{Wh}_h^Q(p_0') + f_2^h$. Finally, the fact that the maps $f_1^h$ and $k_2, f_2^h$ are homotopic can be verified directly by inspecting the construction of $f_1^h$ and $f_2^h$. 

A statement analogous to Theorem 7.1 holds for the smooth secondary transfer. Given a smooth bundle $p : E \to B$ whose fibers are manifolds with a boundary by the vertical boundary of $p$ we will understand the smooth bundle $\partial^n p : \partial^n E \to B$ obtained by restricting $p$ to the union of boundaries of its fibers. We have:

**7.3. Theorem.** Let $p : E \to B$ be a smooth bundle with closed fibers, and for $i = 0, 1, 2$ let $p_i : E_i \to B$ be unipotent subbundles of $p$ such that $p_0$ is the vertical boundary of both $p_1$ and $p_2$, and that $E = E_1 \cup_{E_0} E_2$. Then $p$ is a unipotent bundle and we have

$$[\text{Wh}_h^Q(p')] = [k_1_* \text{Wh}_h^Q(p_1)] + [k_2_* \text{Wh}_h^Q(p_2)] - [k_0_* \text{Wh}_h^Q(p_0)]$$

Here $k_i : \text{Wh}_h^Q(E_i) \to \text{Wh}_h^Q(E)$ is induced by the map $k_i : E_i \to E$.

**7.4. Note.** Recall that the construction of the smooth secondary transfer we are working with uses the combinatorial model of $Q(E_+)$ built using partitions (2.1). This model is functorial with respect to embeddings of submanifolds of codimension 0. As a consequence in the notation of Theorem 7.3 for $i = 1, 2$ the inclusion maps $k_i : E_i \to E$ induce maps $k_i : Q(E_{i+}) \to Q(E)$, which then lift to maps $k_i : \text{Wh}_h^Q(E_i) \to \text{Wh}_h^Q(E)$. The map $k_0 : Q(E_{0+}) \to Q(E_+)$ is constructed as follows. Let $b : E_0 \times [-1, 1] \to E$ be a fiberwise bicollar neighborhood of $E_0$. Thus, $b$ is a smooth embedding of bundles over $B$ such that $b(E_0 \times \{0\}) = E_0$, $b(E_0 \times [-1, 0]) \subseteq E_1$ and $b(E_0 \times \{0, 1\}) \subseteq E_2$. The inclusion $k_0 : E_0 \to E$ coincides with the composition

$$E_0 \to E_0 \times \{0\} \subseteq E_0 \times [-1, 1] \xrightarrow{b} E$$

For a partition $P \subseteq E_0 \times I$ the submanifold $P \times [-1, 1] \subseteq E_0 \times [-1, 1]$ defines (modulo permutation of coordinates) a partition of $(E_0 \times [-1, 1]) \times I$. This assignment induces a map $Q(E_{0+}) \to Q(E_0 \times [-1, 1])$. Furthermore, since $b$ is an embedding of codimension 0 it gives a map $b_* : Q(E_0 \times [-1, 1]) \to Q(E_+)$. Composing these two maps we obtain a map $k_0 : Q(E_{0+}) \to Q(E_+)$ which lifts to

$$k_0 : \text{Wh}_h^Q(E_0) \to \text{Wh}_h^Q(E)$$

**Proof of Theorem 7.3.** The basic scheme of the proof is the same as that of the proof of Theorem 7.1: it suffices to show that there exist maps $f_i^h : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E)$ and $f_i^h : \text{Wh}_h^Q(B) \to \text{Wh}_h^Q(E_2)$ such that

$$[\text{Wh}_h^Q(p')] = [k_1_* \text{Wh}_h^Q(p_1)] + [f_i^h] \quad \quad [\text{Wh}_h^Q(p_2')] = [j_* \text{Wh}_h^Q(p_0')] + [f_2^h]$$
where \( j: E_0 \to E_2 \) is the inclusion map, and that \([f_1^i] = [k_2, f_2^i].\) Moreover, the construction of the maps \( f_1^i, f_2^i \) and verification that they satisfy the above identities also mimics the arguments we used in the proof of Theorem 7.1. Recall that the map \( f_1^1 \) in that proof was defined using a map \( f_1^A : A(B) \to A(E) \) and a preferred homotopy \( \lambda E f_1^1 \simeq \chi(\text{Cone}(k_1 | F_1 \sigma)) \lambda B. \) Furthermore, in order to obtain a homotopy \( \text{Wh}^{Q_1}(p') \simeq k_1, \text{Wh}^{Q_1}(p') + f_1^Q \) it suffices to construct a homotopy

\[(7-6) \quad Q(p') \simeq k_1, Q(p') + f_1^Q \]

together with an appropriate homotopy of homotopies.

The construction of the map \( f_1^Q \) can be simplified in two ways. First, let \( S(B) \) denote the simplicial set of smooth singular simplices of \( B. \) It will suffice to construct a map \( f_1^Q : |S(B)| \to Q(E_+). \) Since \( B \simeq |S(B)| \) this map will extend to a map of infinite loop spaces \( f_1^Q : Q(B_+) \to Q(E_+). \) Second, the construction of the map \( f_1^Q : |S(B)| \to Q(E_+) \) can be reduced to the following combinatorial construction. Let \( \mathcal{T}_1 \mathcal{P}_k(E \times I) \) denote the simplicial category given by the first stage of the \( \mathcal{T}_* \) -construction. Thus for \( k \geq 0 \) the objects of \( \mathcal{T}_1 \mathcal{P}_k(E \times I) \) are pairs \((P_0, P_1)\) where \( P_1 \) is a bundle of partitions of \( E \times I \) over the standard simplex \( \Delta^k, \) and \( P_0 \) is a subbundle of \( P_1. \) Considering \( S(B) \) as a (discrete) simplicial category it will suffice to give a functor \( F_1^Q : S(B) \to \mathcal{T}_1 \mathcal{P}_k(E \times I). \) Such functor will determine the map \( f_1^Q : |S(B)| \to Q(E_+) \) (cf. [2, 2.5]).

In order to describe the functor \( F_1^Q \) we will use the setup of [2, 4.3]. Using the notation introduced there for a \( k \)-simplex \( \sigma \in S(B) \) by \( \sigma^* TB_\varepsilon \) we denote the disc bundle over \( \Delta^k \) induced by \( \sigma \) from the disc bundle of the tangent bundle of \( B. \) The exponential map defines a map of bundles

\[
\begin{array}{ccc}
\sigma^* TB_\varepsilon & \xrightarrow{\exp} & B \times \Delta^k \\
\downarrow & & \downarrow \\
\Delta^k & & B \times \Delta^k
\end{array}
\]

which is a fiberwise embedding. Let \( \frac{1}{3} < a < b < 1. \) Define

\[ P_\sigma := \sigma^* TB_\varepsilon \times [a, b] \cup (B \times \Delta^k) \times [0, \frac{1}{3}] \]

This space admits a map \( P_\sigma \to B \times \Delta^k \times I \) which lets us consider it (modulo permutation of factors) as a bundle of partitions of \( B \times I \) over \( \Delta^k. \) For \( \sigma \in S(B) \) we set

\[ F_1^Q(\sigma) := ((p_1 \times \text{id})^{-1}(P_\sigma) \cup E \times \Delta^k \times [0, \frac{1}{3}], (p \times \text{id})^{-1}(P_\sigma)) \]

In order to describe the homotopy \((7-6)\) we will use Waldhausen’s pre-additivity theorem [1, 3.6]. Take \( \mathcal{T}_2 \mathcal{P}_\varepsilon(E \times I) \) to be the second stage of the \( \mathcal{T}_* \)-construction on
It is a simplicial category with objects in the $k$-th simplicial dimension given by triples $(P_0, P_1, P_2)$ where $P_i$ is a bundle of partitions of $E \times I$ over $\Delta^k$, and $P_i$ is a subbundle of $P_{i+1}$. For $0 \leq i < j \leq k$ let $D_{ij} : \mathcal{T}_2 \mathcal{P}_\bullet(E \times I) \to \mathcal{T}_1 \mathcal{P}_\bullet(E \times I)$ denote the functor given by $D_{ij}(P_0, P_1, P_2) = (P_i, P_j)$. For a functor $H : \mathcal{S}(B) \to \mathcal{T}_2 \mathcal{P}_\bullet(E \times I)$ the preadditivity theorem gives a homotopy between the map $H(B) \to \mathcal{T}_1 \mathcal{P}_\bullet(E \times I)$ induced by the functor $D_{02}$ and the sum of maps induced by the functors $D_{01}$ and $D_{12}$. Consider the functor $H$ given by

$$H(\sigma) := (E \times \Delta^k \times [0, \frac{1}{2}], (p_1 \times \text{id})^{-1}(P_\sigma) \cup E \times \Delta^k \times [0, \frac{1}{2}], (p \times \text{id})^{-1}(P_\sigma))$$

Notice that $D_{12} = F^Q_1$. It remains to notice that by [2, §4] the maps induced by the functors $D_{01} H$ and $D_{02} H$ are, respectively, $k_1 Q(p'_1) \eta$ and $Q(p') \eta$ where $\eta : |\mathcal{S}(B)| \to Q(B_\ast)$ is the coaugmentation map.

The above constructions of the map $f^Q_1$ and the homotopy $Q(p') \simeq Q(p'_1) + f^Q_1$ are obtained by replicating the constructions of the maps $f^h_1$ and the homotopy $A(p') \simeq A(p'_1) + f^h_1$ from the proof of Theorem 7.1, using partitions and the $T\bullet$-construction in place of retractive spaces and the $S\bullet$-construction. The arguments that show that the map $f^Q_1$ admits a lift to $f^1_1 : \text{Wh}_\mathcal{S}(B) \to \text{Wh}_\mathcal{S}(E)$ and that the homotopy $(7-6)$ lifts to a homotopy $\text{Wh}_\mathcal{S}(p') \simeq k_1 \text{Wh}_\mathcal{S}(p'_1) + f^Q_1$ involve constructions on the level of chain complexes that duplicate the constructions from the proof of Theorem 7.1. The map $f^Q_2$ and the homotopy $\text{Wh}_\mathcal{S}(p'_1) = j_\ast \text{Wh}_\mathcal{S}(p'_0) + f^Q_2$ are obtained in an analogous way.

\[\square\]

8. Composition of unipotent fibrations

Our next goal is to give a proof of Theorem 1.3. Recall that this theorem states that the homotopy secondary transfer preserves compositions of unipotent fibrations, and the smooth secondary transfer preserves compositions of unipotent bundles.

The statement of Theorems 1.3 relies on the fact that the composition of unipotent bundles (or unipotent fibrations) is again unipotent. While this property is implicitly present in the work of Igusa [11] we give its proof below for completeness, and also because its main ingredient, Lemma 8.1, will be needed later on.

8.1. Lemma ([11, Lemma 8.9]). Let

$$F \to E \to B$$

\[\square\]
be a unipotent fibration. There is a finite sequence of unipotent fibrations

\[
\begin{array}{c}
E_0 \\
n \\
E_0 \\
E_1 \\
E_1 \\
\vdots \\
E_k \\
E_k \\
B \\
B \\
p_0 \\
p_1 \\
p_k \\
p_k
\end{array}
\]

such that

(i) \(E_0 = \Sigma^n BE\) for some \(n \geq 0\);
(ii) \(p_k : E_k \to B\) is a rational homotopy equivalence;
(iii) for every \(i\) we have a cofibration sequence over \(B\):

\[
B \times S^{n_i} \xrightarrow{\alpha_i} E_i \to E_{i+1}
\]

(i.e. \(E_{i+1} = E_i \cup_{B \times S^{n_i}} B \times D^{n_i+1}\)).

We denote here by \(\Sigma^n BE \to B\) the \(n\)-fold fiberwise suspension of the fibration \(p\), while \(B \times S^{n_i} \to B\) and \(B \times D^{n_i+1} \to B\) are, respectively, a product sphere bundle and a product disc bundle.

8.2. **Theorem** (Igusa). If \(q : D \to E\) and \(p : E \to B\) are unipotent bundles (resp. unipotent fibrations) then the composition \(pq : D \to B\) is also a unipotent bundle (resp. a unipotent fibration).

**Proof.** Let \(F_p, F_q, F_{pq}\) denote the fibers of \(p, q, pq\), respectively. Since the only non-trivial property we need to verify is that the action of \(\pi_1 B\) on \(H_*(F_{pq})\) is unipotent it is enough to show that the statement of the theorem holds for unipotent fibrations. We will split our argument into a few steps.

**Step 1.** The fibration \(pq\) is unipotent for an arbitrary unipotent fibration \(p\) and any product fibration \(q : E \times F_q \to E\).

Indeed, in this case we have an isomorphism of \(\pi_1 B\)-modules

\[
H_*(F_{pq}) \cong H_*(F_p) \otimes H_*(F_q)
\]

where the action of \(\pi_1 B\) on the right hand side is given by \(\alpha(x \otimes y) = \alpha x \otimes y\).

**Step 2.** The fibration \(pq\) is unipotent for an arbitrary unipotent fibration \(p : E \to B\) and any fibration \(q : D \to E\) with fiber \(F_q\) such that \(\tilde{H}_*(F_q) = 0\).

This holds since the map \((q|_{F_{pq}})_* : H_*(F_{pq}) \to H_*(F_p)\) is in this case an isomorphism of \(\pi_1 B\)-modules.
Step 3. For \( i = 0, 1, 2 \) let \( p_i: E_i \to B \) be a fibration with a path connected base space \( B \) and fiber \( F_i \) of a finite homotopy type. Assume that we have maps of fibrations

\[
E_1 \xleftarrow{f} E_0 \xrightarrow{i} E_2
\]

where \( i \) is a cofibration over \( B \). Let \( p: E_1 \cup_{E_0} E_2 \to B \) be the pushout map. If three of the fibrations \( p_0, p_1, p_2, p \) are unipotent then so is the fourth.

This follows from the Mayer-Vietoris sequence for the homology of the fibers and the fact that unipotent \( \pi_1 B \)-modules form a Serre category.

Step 4. As an application of Step 3 we obtain that if \( p, q \) are unipotent fibrations and \( \Sigma Eq: \Sigma E D \to E \) is a fiberwise suspension of \( q \) then \( pq \) is a unipotent fibration if and only if \( p\Sigma Eq \) is unipotent.

Step 5. Assume now that \( p \) and \( q \) are arbitrary unipotent fibrations. Applying Lemma 8.1 to \( q \) we obtain a sequence of fibrations

\[
\begin{array}{c}
D_0 \xrightarrow{q_0} D_1 \xrightarrow{q_1} \ldots \xrightarrow{q_i} D_k \\
E \xrightarrow{q_i} E \xrightarrow{q_i}
\end{array}
\]

We will show that \( pq_i \) is a unipotent fibration for all \( i \). In case of \( i = k \) this is a consequence of Step 2. Arguing inductively, assume that \( pq_{i+1} \) is unipotent for some \( i \). Since \( pq_{i+1} \) is the pushout in the diagram of fibrations over \( B \)

\[
E \times D^{n+1} \xleftarrow{E \times S^n} D_i
\]

and the fibrations \( E \times D^{n+1} \to B, E \times S^n \to B \) are unipotent by Step 1, we obtain using Step 3 that \( pq_i \) is also a unipotent fibration. As a consequence we get that \( pq_0 \) is unipotent. Since \( q_0 \) is an iterated fiberwise suspension of \( q \) by Step 4 we obtain that \( pq \) is unipotent as well. \( \square \)

9. Composition of secondary transfers

In this section we prove Theorems 1.3 as well as its analog for the homotopy secondary transfer:

9.1. Theorem. If \( p: E \to B \) and \( q: D \to E \) are unipotent fibrations then

\[
\text{Wh}_n^\sigma((pq)^i) \simeq \text{Wh}_n^\sigma(q^i) \circ \text{Wh}_n^\sigma(p^i)
\]

Our strategy will be as follows. First, in (9.2) and (9.3) we show that Theorem 9.1 holds in two special cases, and then we will use an argument of Igusa to show that its general statement follows from these special cases. Finally, we will show that essentially the same reasoning can be used to obtain a proof of Theorem 1.3.
9.2. Lemma. Let \( p: E \to B, q: D \to E \) be unipotent fibrations with fibers \( F_p \) and \( F_q \) respectively. If \( \tilde{H}_s(F_q) = 0 \) then

\[
\text{Wh}_s^Q((pq)^\dagger) \simeq \text{Wh}_s^Q(q^\dagger) \circ \text{Wh}_s^Q(p^\dagger)
\]

Proof: Let \( F_{pq} \) denote the fiber of \( pq \). Recall (5.3) that the map \( \text{Wh}_s^Q((pq)^\dagger) \) is defined by the diagram

\[
\begin{array}{ccc}
A(B) & \xrightarrow{A((pq)^\dagger)} & A(D) \\
\downarrow & & \downarrow \\
K(Q) & \xrightarrow{\chi(F_{pq})} & K(Q)
\end{array}
\]

which commutes up to the homotopy \( \eta_{pq}^h \). One the other hand, the composition \( \text{Wh}_s^Q(q^\dagger)\text{Wh}_s^Q(p^\dagger) \) is induced by the diagram

\[
\begin{array}{ccc}
A(B) & \xrightarrow{A(p^\dagger)} & A(E) & \xrightarrow{A(q^\dagger)} & A(D) \\
\downarrow & & \downarrow & & \downarrow \\
K(Q) & \xrightarrow{\chi(F_p)} & K(Q) & \xrightarrow{\chi(F_q)} & K(Q)
\end{array}
\]

The left square in this diagram commutes up to the homotopy \( \eta_p^h \), and the right square commutes up to the homotopy \( \eta_q^h \). It follows that the outer square, defining the map \( \text{Wh}_s^Q(q^\dagger)\text{Wh}_s^Q(p^\dagger) \), commutes up to the homotopy obtained by concatenating \( \eta_q^h \circ (A(p^\dagger) \times \text{id}_I) \) with \( \chi(F_q)\eta_p^h \).

By (11.2) in order to obtain a homotopy between \( \text{Wh}_s^Q(q^\dagger)\text{Wh}_s^Q(p^\dagger) \) and \( \text{Wh}_s^Q((pq)^\dagger) \) it is enough to construct the following data:

1) a homotopy \( H_A(p, q): A(B) \times I \to A(D) \) between \( A((pq)^\dagger) \) and \( A(q^\dagger)A(p^\dagger) \);

2) a homotopy \( H_K: K(Q) \times I \to K(Q) \) between the maps \( \chi(F_{pq}) \) and \( \chi(F_q)\chi(F_p) \);
3) a homotopy of homotopies that fills the following diagram:

\[
\begin{array}{ccc}
\lambda^b_D A(\eta^1)A(p^1) & \xrightarrow{\lambda^b_D H_A(p, q)} & \lambda^b_D A((pq)^1) \\
\eta_{pq} & & H_A(\lambda^b_B z \times \text{id}_I) \\
\chi(F_p)\lambda^b_E A(p^1) & \xrightarrow{\chi(F_q)\lambda^b_B} & \chi(F_p)\lambda^b_B
\end{array}
\]

(9-1)

Each vertex of this diagram represents a map \(A(B) \to K(Q)\) and edges represent homotopies of such maps.

1) Construction of \(H_A(p, q)\). The map \(A((pq)^1)\) comes from the functor \(R^{fd}(B) \to R^{fd}(D)\) that assigns to a retractive space \(X\) the space \((pq)^*X\), while the composition \(A(q^1)A(p^1)\) comes from the functor that sends \(X\) to \(q^*p^*X\). The canonical isomorphisms

\[(pq)^*X \cong q^*p^*X\]

define a natural transformation of functors, and so they induce a homotopy between \(A((pq)^1)\) and \(A(q^1)A(p^1)\). This is the homotopy \(H_A(p, q)\).

2) Construction of \(H_K\). Recall that the maps \(\chi(F_p), \chi(F_q), \text{ and } \chi(F_{pq})\) are induced by functors \(\mathcal{C}h^{fd}(Q) \to \mathcal{C}h^{fd}(Q)\) than tensor a chain complex \(C\) by, respectively, \(H_*(F_p), H_*(F_q), \text{ and } H_*(F_{pq})\). As a consequence the map \(\chi(F_q)\chi(F_p)\) is induced by the functor that tensors \(C \in \mathcal{C}h^{fd}(Q)\) by the chain complex \(H_*(F_p) \otimes H_*(F_q)\). Since \(H_*(F_q) = 0\) we have an isomorphism

\[H_*(F_{pq}) \xrightarrow{(q^*p^*)_h} H_*(F_p) \otimes H_*(F_q)\]

This induces a natural isomorphism of functors

\[- \otimes H_*(F_{pq}) \Rightarrow - \otimes (H_*(F_p) \otimes H_*(F_q))\]

which, in turn, defines the homotopy \(H_K\).

3) Construction of the homotopy of homotopies. In order to show that the diagram (9-1) can be filled by a homotopy of homotopies we will first replace it by the
Each vertex of this diagram represents a functor $R^d(B) \to \mathcal{C}^{d}(\mathbb{Q})$. The edges represent natural weak equivalences, with the exception of the lowest vertical edges where the passage between functors is obtained using additivity. The outer edges of this diagram correspond to the homotopies in the diagram (9-1). Since $\tilde{H}^*(F_q) = 0$ the twisted tensor product of the fibration $q^* p^* X \to p^* X$ is just the untwisted tensor product $C^*(p^* X, E) \otimes H_*(F_q)$. In effect the homotopy $\eta^h_q \circ (A(p^1) \times \text{id}_I)$ in (9-1) is induced simply by the the quasi-isomorphisms

\[
C_*(q^* p^* X, D) \xrightarrow{\beta_q} C'_*(p^* X, E) \otimes H_*(F_q) \cong C_* (p^* X, E) \otimes H_*(F_q)
\]

without using additivity.

In order to show that the diagram (9-1) can be filled by a homotopy of homotopies it is enough to show that each of the subdiagrams in the above diagram of functors can be filled by a homotopy of homotopies. In the case of the subdiagrams (1–4) such homotopies of homotopies exist since subdiagrams (1) and (3) commute strictly and subdiagrams (2) and (4) commute up to natural chain homotopies. Homotopy commutativity of the subdiagram (2) follows from [5, (7.4)]. The subdiagram (4) is homotopy commutative since it is obtained by applying Proposition
4.10 to the map of fibrations

\[
\begin{array}{c}
D \\
\downarrow q \\
\downarrow pq \\
B \\
\downarrow p \\
E
\end{array}
\]

Finally, the subdiagram (5) can be filled by a homotopy of homotopies since the maps

\[
C_*(X, B) \otimes_{(pq)_x} H_*(F_{pq}) \xrightarrow{q^\otimes} C_*(X, B) \otimes_{p_x} H_*(F_p) \xrightarrow{\cong} C_*(X, B) \otimes_{p_x} H_*(F_p) \otimes H_*(F_q)
\]

preserve the homological filtrations and induce isomorphisms on the filtration quotients. □

9.3. Lemma. Let \( p: E \to B \) be a unipotent fibration and let \( q: E \times S^0 \to E \) be the product fibration with fiber \( S^0 \). Then

\[
\text{Wh}_h^Q((pq)^!) \cong \text{Wh}_h^Q(q^!) \circ \text{Wh}_h^Q(p^!)
\]

Proof. The basic outline of our argument is the same as in the proof of Lemma 9.2. The same construction as in the proof of that lemma gives a homotopy \( H_A(p, q) \) between the maps \( A((pq)^!) \) and \( A(p^!)A(q^!) \). Next, we need a homotopy \( H_K \) between the maps \( \chi(F_p \times S^0) \) and \( \chi(S^0)\chi(F_p) \). The first of these maps comes from the functor \( C \mapsto C \otimes H_*(F_p \times S^0) \) while the second map is induced by the functor \( C \mapsto C \otimes H_*(F_p) \otimes H_*(S^0) \). The isomorphism \( H(F_p) \otimes H_*(S^0) \cong H_*(F_p \times S^0) \) induces a natural isomorphism of the above functors, which in turn defines the homotopy \( H_K \).

As the result of these constructions we obtain a diagram of homotopies (9-1) (with \( F_q = S^0 \) and \( F_{pq} = F_p \times S^0 \)). The remaining step is to show that this diagram can be filled by a homotopy of homotopies. Existence of such a homotopy of homotopies can be verified in a straightforward manner using the fact that for \( X \in R^{id}(B) \) we have a commutative diagram

\[
C_*(q^*p^*X, E \times S^0) \xrightarrow{\cong} C(p^*X, E) \otimes H_*(S^0) \\
\beta_{pq} \downarrow \quad \beta_p \otimes \text{id} \\
C_*(X, B) \otimes_{(pq)_x} H(E \times S^0) \xrightarrow{\cong} (C(X, B) \otimes_{p_x} H_*(F_p)) \otimes H_*(S^0)
\]

□

9.4. Igusa’s argument. In [11] Igusa used Lemma 8.1 to show that two higher torsion invariants of unipotent bundles coincide. We will adapt this argument to prove Theorems 1.3 and 9.1. The main idea of Igusa’s proof is to define a “difference torsion”, which is new invariant that measures the difference between the given two invariants of bundles, and then to show that this difference torsion vanishes for all
unipotent bundles. The essential properties of Igusa’s difference torsion are encapsulated in Definition 9.5. Proposition 9.6 spells out the conditions that guarantee its vanishing.

9.5. Definition. Let $B$ be a space of the homotopy type of a finite CW-complex, and let $\Lambda$ be an abelian group. An additive homotopy $B$-invariant of unipotent fibrations with values in $\Lambda$ is an assignment $\Phi$ that associates to each unipotent fibration $p: E \to B$ an element $\Phi(p) \in \Lambda$ and that satisfies the following conditions:

(\textbf{additivity}) given maps of fibrations over $B$

$$
\begin{array}{c}
E_1 & \overset{j}{\leftarrow} & E_0 & \rightarrow & E_2 \\
\downarrow{p_1} & \ & \downarrow{p_0} & \ & \downarrow{p_2} \\
B & & & & \\
\end{array}
$$

where $p_i$ is a unipotent fibration for $i = 0, 1, 2$, and $j$ is a cofibration we have

$$
\Phi(p_1 \cup_p_0 p_2) = \Phi(p_1) + \Phi(p_2) - \Phi(p_0)
$$

(\textbf{homotopy invariance}) if unipotent fibrations $p_i: E_i \to B$ ($i = 1, 2$) are fiberwise homotopy equivalent then $\Phi(p_1) = \Phi(p_2)$.

9.6. Proposition. Let $\Phi$ be an additive homotopy $B$-invariant with values in $\Lambda$. Assume that

- $\Phi(p) = 0$ for the product fibration $p: B \times S^0 \to B$;
- $\Phi(p) = 0$ if the map $p: E \to B$ is a rational homotopy equivalence.

Then $\Phi(p) = 0$ for all unipotent fibrations $p: E \to B$.

Proof: First notice that $\Phi(p) = 0$ if $p: B \times F \to B$ is a product fibration where $F$ is either a disc or a sphere. Indeed, in the first case $p$ is a rational homotopy equivalence. If $F$ is a sphere we can argue inductively starting with $F = S^0$ and using additivity.

Next, let $p: E \to B$ be an arbitrary unipotent fibration. Applying Lemma 8.1 to $p$ we obtain a sequence of fibrations

$$
\begin{array}{c}
E_0 & \rightarrow & E_1 & \rightarrow & \ldots & \rightarrow & E_k \\
\downarrow{p_0} & & \downarrow{p_1} & & \downarrow{p_k} \\
B & & & & & & \\
\end{array}
$$

(9-2)
Since $p_k$ is a rational homotopy equivalence we have $\Phi(p_k) = 0$. Next, since $p_i$ is obtained as a pushout of $p_{i-1}$ and product fibrations with disc and sphere fibers we can use additivity of $\Phi$ to get that $\Phi(p_i) = \Phi(p_{i-1})$. This gives

$$0 = \Phi(p_k) = \Phi(p_{k-1}) = \cdots = \Phi(p_0)$$

Finally, since $p_0$ is an $n$-fold fiberwise suspension of $p$ we can use additivity of $\Phi$ again to get

$$\Phi(p) = (-1)^n\Phi(p_0) = 0$$

\[\square\]

9.7. Note. Igusa uses a variant of Proposition 9.6 that will be also useful to us later on. Namely, for a smooth compact manifold $B$ consider an assignment $\Phi$ that satisfies additivity and homotopy invariance properties as in Definition 9.5, but is defined only for unipotent bundles over $B$. Then the statement of Proposition 9.6 still holds: if $\Phi$ vanishes on the product bundle $B \times S^0 \to B$ and on bundles that are given by a rational homotopy equivalence then it vanishes on all unipotent bundles. The proof of this fact is essentially the same as the proof of Proposition 9.6 with two additional observations:

- the homotopy invariance of $\Phi$ lets us define this invariant on all smoothable unipotent fibrations over $B$, i.e. all fibrations that are fiberwise homotopy equivalent to a unipotent bundle.

- if $p: E \to B$ is a unipotent bundle then all fibrations appearing in the diagram (9-2) are smoothable [11, Lemma 8.6].

We are now ready to give

**Proof of Theorem 9.1.** Let $p: E \to B$, $q: D \to E$ be unipotent fibrations. We have commutative diagrams

$$
\begin{array}{ccc}
\text{Wh}_h^Q(B) & \xrightarrow{\text{Wh}_h^Q((pq)^i)} & \text{Wh}_h^Q(D) \\
\downarrow{i_N} & & \downarrow{i_D} \\
A(B) & \xrightarrow{A((pq)^i)} & A(D)
\end{array}
\quad
\begin{array}{ccc}
\text{Wh}_h^Q(B) & \xrightarrow{\text{Wh}_h^Q(q)\text{Wh}_h^Q(p)} & \text{Wh}_h^Q(D) \\
\downarrow{i_N} & & \downarrow{i_D} \\
A(B) & \xrightarrow{A(q)A(p)} & A(D)
\end{array}
$$

In the same way as in the proof of Lemma 9.2 we construct a homotopy $H_A(p, q)$ between the maps $A((pq)^i)$ and $A(q)A(p^i)$. By (11.3) this data defines a map $\varphi(p, q): \text{Wh}_h^Q(B) \to \Omega K(Q)$ such that $[\varphi(p, q)] = 0$ if and only if $H_A$ admits a lift to a homotopy between $\text{Wh}_h^Q((pq)^i)$ and $\text{Wh}_h^Q(q^i)\text{Wh}_h^Q(p^i)$.

Fix a fibration $p: E \to B$. Let $\Phi_p$ be the assignment that associates to a unipotent fibration $q: D \to E$ the homotopy class $[\varphi(p, q)]$. We claim that $\Phi_p$ is an additive
homotopy $E$-invariant with values in the group $[\text{Wh}^\mathbb{Q}_h(B), \Omega K(\mathbb{Q})]$ (9.5). In order to verify homotopy invariance of $\Phi_p$, assume that we have a fiberwise homotopy equivalence

$$D_1 \xrightarrow{f} D_2 \xleftarrow{q_1} E \xleftarrow{q_2} D_2$$

We need to check that the maps $\varphi(p, q_1)$ and $\varphi(p, q_2)$ are homotopic. By Proposition 6.1 we can construct a homotopy

$$H^\text{Wh(E)}_f: \text{Wh}^\mathbb{Q}_h(E) \times I \to \text{Wh}^\mathbb{Q}_h(D_2)$$

between the maps $f_*\text{Wh}^\mathbb{Q}_h(q_1^!)$ and $\text{Wh}^\mathbb{Q}_h(q_2^!)$. As a consequence the map

$$H^\text{Wh(B)}_f: \text{Wh}^\mathbb{Q}_h(B) \times I \to \text{Wh}^\mathbb{Q}_h(D_2)$$

is a homotopy between $f_*\text{Wh}^\mathbb{Q}_h((pq_1)^!)$ and $\text{Wh}^\mathbb{Q}_h((pq_2)^!)$. On the other hand $f$ is also a fiberwise homotopy equivalence of the fibrations $pq_1$ and $pq_2$, so we have a homotopy

$$H^\text{A(E)}_f: A(B) \times I \to A(D_2)$$

between the maps $f_*A((pq_1)^!)$ and $A((pq_2)^!)$. By the proof of Proposition 6.1 we also have a homotopy

$$H^\text{A(B)}_f: A(B) \times I \to A(D)$$

between the maps $f_*A((pq_1)^!)$ and $A((pq_2)^!)$. All these homotopies fit into commutative diagrams

Consider the diagram

$$\begin{array}{ccc}
\text{Wh}^\mathbb{Q}_h(B) \times I & \xrightarrow{H^\text{Wh(B)}_f} & \text{Wh}^\mathbb{Q}_h(D_2) \\
\downarrow i_B \times \text{id}_I & & \downarrow i_D_2 \\
A(B) \times I & \xrightarrow{H^\text{A(B)}_f} & A(D_2)
\end{array} \quad \begin{array}{ccc}
\text{Wh}^\mathbb{Q}_h(B) \times I & \xrightarrow{H^\text{Wh(E)}_f} & \text{Wh}^\mathbb{Q}_h(D) \\
\downarrow i_B \times \text{id}_I & & \downarrow i_D_2 \\
A(B) \times I & \xrightarrow{H^\text{A(E)}_f} & A(D)
\end{array}$$

(9-3)
Each vertex of this diagram represents a map $A(B) \to A(D_2)$ and edges represent homotopies of such maps. This diagram is induced by a diagram of functors $\mathbb{F}_d(B) \to \mathbb{F}_d(D_2)$ and natural equivalences of such functors. It is straightforward to check that this underlying diagram of functors commutes. This implies that the diagram (9-3) can be filled by a homotopy of homotopies. This homotopy of homotopies can be interpreted as a homotopy between $H_{A(B)}$ and $H_{A(D_2)}$. By (11.3) this homotopy defines a map $Wh^Q(B) \times I \to \Omega K(Q)$. One can check that this map determines a homotopy between $\varphi(p, q_1)$ and $\varphi(p, q_2)$. Additivity of $\Phi_p$ can be verified in a similar way, using additivity of secondary transfers.

By Lemma 9.2 and Lemma 9.3 we have $\Phi_p(q) = 0$ if $q$ is a product fibration or a rational homotopy equivalence. Proposition 9.6 implies then that $[\varphi(p, q)] = \Phi_p(q) = 0$ for any unipotent fibration $q: D \to E$. 

\textbf{Proof of Theorem 1.3.} Let $p: E \to B$ and $q: D \to E$ be unipotent bundles. Recall (3.2) that the map $Q(p^!)\sim$ is induced by functors between categories of partitions that assign to a partition $P \subseteq B \times I$ the partition $(p \times \text{id})^{-1}(P) \subseteq E \times I$. Since $(pq \times \text{id})^{-1}(P) = (q \times \text{id})^{-1}(p \times \text{id})^{-1}(P)$ we obtain a homotopy $H_Q(p, q)$ between the maps $Q((pq)^!)$ and $Q((q^!)Q(p^!))$. By (11.3) the maps $Wh^Q(Q^!)$ and the homotopy $H_Q(p, q)$ define a map $\psi(p, q): Wh^Q(B) \to \Omega K(Q)$ such that $Wh^Q(Q^!)Wh^Q(Q^!) = Wh^Q((pq)^!)$ if $[\psi(p, q)] = 0$.

It remains to show that $[\psi(p, q)] = 0$ for all unipotent bundles $p, q$. Recall (3.3) that by $\mu_p$ we denoted the homotopy between the maps $A(p^!)a_B$ and $a_EQ(p^!)$. Consider the diagram

\[
\begin{array}{ccc}
A_DQ((pq)^!) & \xrightarrow{a_DH_Q(p, q)} & A_DQ(q^!)Q(p^!)
\\
\downarrow_{\mu_{pq}} & & \downarrow_{\mu_q \circ (Q(p^!) \times \text{id}_I)}
\\
A((pq)^!)a_B & \xrightarrow{H_A(p, q) \circ (a_B \times \text{id}_I)} & A(q^!)A(p^!)a_B
\end{array}
\]

(9-4)

Vertices of this diagram corresponds to a map $Q(B_\pm) \to A(D)$ and edges correspond to homotopies of such maps. Each of these homotopies is induced by natural
isomorphisms between retractive spaces over \( D \) obtained from partitions \( P \subseteq B \times I \):

\[
\begin{array}{ccc}
(pq \times \text{id})^{-1}(P) & \rightarrow & (q \times \text{id})^{-1}(p \times \text{id})^{-1}(P) \\
\downarrow & & \downarrow \\
q^*(p \times \text{id})^{-1}(P) & \rightarrow & q^*p^*P
\end{array}
\]

Since this diagram commutes we obtain a homotopy of homotopies filling the diagram (9-4).

Let \( b_B : \text{Wh}_s^Q(B) \rightarrow \text{Wh}_h^Q(B) \) be the map induced by the map of fibrations

\[
\begin{array}{ccc}
\text{Wh}_s^Q(B) & \xrightarrow{b_B} & \text{Wh}_h^Q(B) \\
\downarrow & & \downarrow \\
Q(B_+) & \xrightarrow{a_B} & A(B) \\
\rightarrow & & \rightarrow
\end{array}
\]

The construction of the map \( \psi(p, q) \) described in (11.3) combined with existence of a homotopy of homotopies in the diagram (9-4) gives

\[
\psi(p, q) \simeq \varphi(p, q)b_B
\]

where \( \varphi(p, q) \) is the map defined in the proof of Theorem 9.1. In that proof we showed that \([\varphi(p, q)] = 0\), so also \([\psi(p, q)] = 0\). \(\square\)

10. Secondary transfer and smooth torsion

In [2] and [1] (joint with B. Williams and J. Klein) we described a homotopy theoretical construction of the smooth torsion of unipotent bundles and showed that it defines characteristic classes which coincide with the higher torsion invariants of Igusa and Klein. The construction of the smooth torsion of a bundle \( p : B \rightarrow E \) proceeds as follows. Let \( \eta_B : B \rightarrow Q(B_+) \) denote the coaugmentation map. By [2, Theorem 6.7] the map

\[
\lambda_E Q(p^! \eta_B) : B \rightarrow K(\mathbb{Q})
\]

is homotopic via a preferred homotopy \( \omega^p \) to a constant map. This defines a map \( \tau^s(p) : B \rightarrow \text{Wh}_s^Q(E) \) which is a lift of \( Q(p^! \eta_B) \). The map \( \tau^s(p) \) is the smooth torsion of the bundle \( p \).

The secondary transfer of unipotent bundles described in this paper can be used to construct another map \( \bar{\tau}^s(p) : B \rightarrow \text{Wh}_s^Q(E) \). Namely, since the identity map
Our final goal in this paper is to prove Theorem 1.4 which says that for any unipotent bundle \( p \) the maps \( \tau^i(p) \) and \( \overline{\tau}^i(p) \) are homotopic. We will also show that as a consequence the statement of Theorem 1.1 holds: for any pair of composable unipotent bundles \( p \) and \( q \) the higher torsion cohomology classes of \( p, q \) and \( pq \) are related by the formula (1-1).

The proof of Theorem 1.4 will use the same scheme as the proof of the composition formula of unipotent fibrations (Proposition 9.1). We will first show that this theorem holds when \( p \) is either a product bundle or it is a rational homotopy equivalence, and then we will use Igusa’s argument (9.4) to extend this result to all unipotent bundles.

10.1. Lemma. Let \( p: E \to B \) be a unipotent bundle. If \( p \) is either the product bundle with fiber \( S^0 \) or a rational homotopy equivalence then \( \tau^i(p) \simeq \overline{\tau}^i(p) \).

Proof. This follows essentially from [1, §6]. We proved there that if \( p: E \to B \) is a unipotent bundle that satisfies the assumptions of the Leray-Hirsch theorem then the quasi-isomorphisms

\[
C_*(p^*X) \xrightarrow{\sim} C_*(X) \otimes H_*(F_p)
\]

given for \( X \in R^{fd}(B) \) by that theorem define a map

\[
\text{Wh}^Q_{LH}(p^!): \text{Wh}^Q_{LH}(B) \to \text{Wh}^Q_{LH}(E)
\]

and that \( \tau^i(p) \simeq \text{Wh}^Q_{LH}(p^!)\tau^i(id_B) \). It is straightforward to check that if \( p \) is either a product bundle \( B \times S^0 \to B \) or a rational homotopy equivalence then \( \text{Wh}^Q_{LH}(p^!) \simeq \text{Wh}^Q_{LH}(p^! \cdot) \).

\[\square\]

Proof of Theorem 1.4. Both \( \tau^i(p) \) and \( \overline{\tau}^i(p) \) are defined as lifts of the map \( Q(p^!)\eta_B \). As a consequence they define a map \( \varrho(p): B \to \Omega K(Q) \) such that the homotopy class of the composition

\[
B \xrightarrow{\varrho(p)} \Omega K(Q) \to \text{Wh}^Q_{LH}(E)
\]

coincides with the element \([\tau^i(p)] - [\overline{\tau}^i(p)] \in [B, \text{Wh}^Q_{LH}(E)]\). It suffices to show that \([\varrho(p)] = 0 \in [B, \Omega K(Q)]\).

We claim that the assignment \( p \mapsto [\varrho(p)] \) is an additive homotopy \( B \)-invariant of unipotent bundles with values in \([B, \Omega K(Q)]\) (9.7). Indeed, additivity of this assignment follows essentially from the additivity of the secondary transfer (7.3) and the additivity of the smooth torsion [1, Theorem 5.1]. Homotopy invariance can be verified using the fact that the construction of \( \varrho(p) \) involves only chain
complexes associated to $p$. Using Lemma 10.1 we obtain that $[\varrho(p)] = 0$ if $p$ is either a product bundle or a rational homotopy equivalence. By Proposition 9.6 and (9.7) we get then that $[\varrho(p)] = 0$ for all unipotent bundles $p$.

Proof of Theorem 1.1. Combining Theorems 1.4 and 1.3 we obtain the statement of Corollary 1.5: for any unipotent bundles $p: E \to B$ and $q: D \to E$ there is a homotopy

$$\tau^s(pq) \simeq Wh_S^\mathbb{Q}(q^!)\tau^s(p)$$

In [1, Theorem 7.1] an analogous decomposition of the smooth torsion of $pq$ (in the case where $q$ is a Leray-Hirsch bundle) was the main ingredient in the proof of the fact that the cohomological torsion of $p$ and $q$ satisfies the formula (1-1). The same argument can be now used to obtain the formula (1-1) for arbitrary unipotent bundles $p$ and $q$.

□

11. Appendix A: Maps of homotopy fibers

Multiple arguments in this paper involve constructions of maps between homotopy fibers as well as constructions of homotopies between such maps. We summarize here the basic scheme of such constructions.

11.1. Maps of homotopy fibers. For a space $X$ with a basepoint $x_0$ let $P_{x_0}X$ denote the space of paths in $X$ that start at $x_0$. By the homotopy fiber of a map $p: Y \to X$ over $x_0$ we understand the standard construction

$$\text{hofib}(p)_{x_0} := \{ (\omega, y) \in P_{x_0}X \times Y \mid \omega(1) = p(y) \}$$

We will denote by $i_Y: \text{hofib}(p)_{x_0} \to Y$ the map given by $i_Y(\omega, y) = y$.

Assume that we have a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & Y' \\
p \downarrow & & \downarrow p' \\
X & \xrightarrow{f} & X'
\end{array}
$$

such that $f(x_0) = x'_0$. Given a homotopy $h$ from $fp$ to $p'\tilde{f}$ we obtain a map $\tilde{f}^*: \text{hofib}(p)_{x_0} \to \text{hofib}(p')_{x'_0}$ given by

$$\tilde{f}^*(\omega, y) := (f\omega * h_y, \tilde{f}(y))$$

Here $h_y$ denotes the path in $X'$ defined by $h_y(t) = h(y, t)$, and $*$ indicates concatenation of paths. We have

$$i_{Y'}\tilde{f}^* = \tilde{f}i_Y$$
11.2. Homotopies of maps of homotopy fibers. Assume that we have two diagrams:

\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{f}_0} & Y' \\
p & \downarrow & p' \\
X & \xrightarrow{f_0} & X'
\end{array} \quad \begin{array}{ccc}
Y & \xrightarrow{\tilde{f}_1} & Y' \\
p & \downarrow & p' \\
X & \xrightarrow{f_1} & X'
\end{array}
\]

that commute up to homotopies \( h_0 \) and \( h_1 \), respectively, and such that \( f_0(x_0) = f_1(x_0) = x'_0 \). This gives two maps of the homotopy fibers:

\[
\tilde{f}_0, \tilde{f}_1 : \text{hofib}(p)_{x_0} \to \text{hofib}(p')_{x'_0}
\]

In order to obtain a homotopy between these maps it suffices to construct the following data:

1) a homotopy \( \tilde{H} : Y \times I \to Y' \) between \( \tilde{f}_0 \) and \( \tilde{f}_1 \);
2) a basepoint preserving homotopy \( H : X \times I \to X' \) between \( f_0 \) and \( f_1 \);
3) a homotopy of homotopies between the two homotopies \( p' \tilde{f}_0 \approx f_1 p' \): the one given as a concatenation of \( p' H \) with \( h_1 \), and the one obtained by concatenating \( h_0 \) with \( H(p \times \text{id}_I) \):

\[
\begin{array}{c}
f_0 p \xrightarrow{H(p \times \text{id}_I)} f_1 p \\
\downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

where

\[
\begin{array}{c}
h_0 \quad h_1 \\
\uparrow \quad \uparrow
\end{array}
\]

\[
\begin{array}{c}
p' \tilde{f}_0 \xrightarrow{p' H} p' \tilde{f}_1 \\
\end{array}
\]

Giving such a homotopy of homotopies is equivalent to giving a map

\[
\Theta : Y \times I \times I \to X'
\]

such that \( \Theta|_{Y \times \{i\} \times I} = h_i \) for \( i = 0, 1 \), \( \Theta|_{Y \times I \times \{0\}} = p' H \), and \( \Theta|_{Y \times I \times \{1\}} = H(p \times \text{id}_I) \).

The homotopy \( \tilde{H} \) between \( \tilde{f}_1 \) and \( \tilde{f}_2 \) is defined by:

\[
\tilde{H}((\omega, y), t) := (H_t \omega * \Theta_{y, t}, \tilde{H}(y, t))
\]

where \( H_t : X \to X' \) is given by \( H_t(x) = H(t, x) \) and \( \Theta_{y, t} \) is the path in \( X' \) given by \( \Theta_{y, t}(s) = \Theta(y, t, s) \).

11.3. An obstruction to lifting a homotopy. Assume again that we have two homotopy commutative squares (11-1) and that \( \tilde{f}_0, \tilde{f}_1 \) are the maps of homotopy fibers defined by these squares. Assume also that we have a homotopy \( \tilde{H} \) between \( \tilde{f}_0 \) and \( \tilde{f}_1 \), and that we want to determine whether there exists a homotopy \( \tilde{H} \) between \( \tilde{f}_1 \)
and \( \tilde{f}_2 \) that fits into a commutative diagram

\[
\begin{array}{c}
\text{hofib}(p)_{x_0} \times I \\
\downarrow i_Y \times \text{id}_I \\
Y \times I
\end{array} \xrightarrow{\tilde{H}} \begin{array}{c}
\text{hofib}(p')_{y_0}' \\
\downarrow i_{Y'} \\
Y'
\end{array}
\]

In (11.2) we described data that suffices to construct such \( \tilde{H} \), but here we are interested in a condition that is equivalent to the existence of this homotopy. We can describe such a condition as follows.\(^5\)

Let \( C_p : \text{hofib}(p)_{x_0} \times I \to X \) be the map given by \( C_p((\omega, y), t) = \omega(t) \). This is a homotopy between the constant map into \( x_0 \) and the map \( \pi_Y \). Consider the following diagram:

\[
\begin{array}{ccc}
p'\tilde{f}_0i_Y & p'\tilde{H}(i_Y \times \text{id}_I) & p'\tilde{f}_1i_Y \\
\downarrow \text{ho}(i_Y \times \text{id}_I) & \downarrow \tilde{h}_1(i_{Y'} \times \text{id}_{Y'}) & \\
f_0\pi_Y & \tilde{h}_1(i_{Y'} \times \text{id}_{Y'}) & f_1\pi_{Y'}
\end{array}
\]

Each vertex of this diagram represent a map \( \text{hofib}(p)_{x_0} \to X' \) and edges represent homotopies of such maps. Concatenating all homotopies appearing here we obtain a homotopy from the constant map into \( x_0' \) to itself, or equivalently a map

\[ \varphi : \text{hofib}(p)_{x_0} \to \Omega X' \]

It is straightforward to verify that the map \( \varphi \) is contractible if and only if there exists a homotopy \( \tilde{H} \) such that the diagram (11.2) commutes. In other words the homotopy class of \( \varphi \) is an obstruction to lifting the homotopy \( \tilde{H} \) to a homotopy defined on the level of the homotopy fibers.

**Note.** Let \( p' : (Y', y_0') \to (X', x_0') \) be a map of infinite loop spaces where \( x_0', y_0' \) are the trivial elements in \( X' \) and \( Y' \). In this case the map \( \varphi \) has a simpler interpretation. Namely, let \( j_{Y'} : \Omega X' \to \text{hofib}(p')_{x_0}' \) be the map given by \( j_{Y'}(\omega) = (\omega, y_0) \). The set of homotopy classes \( \{ \text{hofib}(p)_{x_0}, \text{hofib}(p')_{x_0}' \} \) has a structure of an abelian group and we have:

\[ j_{Y'}[\varphi] = [\tilde{f}_1] - [\tilde{f}_2] \]

\(^5\) See also [18, Lemma 4.1]
REFERENCES

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