RESEARCH SUMMARY

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1. Introduction

My main interest is in partial differential equations (PDEs) and methods of constructing solutions to these equations. I use a mixture of analytical and numerical techniques in analyzing PDEs. Frequently, I supplement these tools with experimental observations of the physical system in question. The kind of problems I am interested in range from more theoretical questions to very applied problems.

One of the major themes of my work is the exploitation of underlying structures for differential equations. Typically this involves taking advantage of the Hamiltonian structure and/or non-trivial conserved quantities of the PDE. Many inverse problems can be modeled as a map from one boundary condition of a PDE to another. The conserved quantities provide a means to regularize these maps, often suggesting suitable approximate models. Examples of this include the inverse bathymetry problem, boundary control of solutions to PDEs and the generation of water-waves in a wave-tank. See below for details.

Often times, a given problem requires new theoretical tools or suggests a unified framework to tackle multiple associated problems. In this sense, a number of the projects discussed below are related to one another. Below is one possible ordering of my previous and current work based on a division into broad themes.

2. Inverse problems in water waves

2.1. **The pressure problem.** The pressure problem consists of reconstructing the free surface of the water wave from measurements of the fluid pressure along the bottom boundary of the fluid. This problem is of foremost relevance to experimentalists and is frequently how surface elevation is measured. In [9], I derived a relationship between the pressure beneath a traveling wave and the surface elevation of the wave that takes into account the full nonlinearity of the water-wave equations. This relation permits the numerical reconstruction of the surface from pressure data. Further, it allows for the quick derivation of perturbation expansions, including not only those which are used currently but higher-order corrections to these formulas. Data from numerical simulations and physical experiments were used to validate the nonlocal relationship (typically with less than 1% error).

2.2. **The inverse bathymetry problem.** Using ideas from [1] as well as from the pressure problem [9], I investigated the problem of detecting the bathymetry of the ocean floor through surface measurements [11]. The analysis in [11] shows that if the surface profile η, the normal velocity η_t and the rate of change of the normal velocity η_{tt} are given as functions of the spatial variable at some instance, then it is possible to recover the bathymetry. In practice this means we require (at least) the surface elevation at three times $\eta(x, t_1)$, $\eta(x, t_2)$ and $\eta(x, t_3)$. The advantage of this

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formulation is the lack of any assumption on the nature of the fluid flow such as stationarity, traveling waves, *etc.* The formulation is derived from first principles. Numerical experiments indicate reasonable success. Primary challenges are experimental validation and a concrete theoretical understanding.The experiments for this problem are currently under way at the Pritchard Fluid Lab at Pennsylvania State University.

2.3. **Future directions.** A direction for further research involves obtaining approximate models for bathymetry detection. This is of particular interest when recovering bathymetry from experimental data, which typically does not have the smoothness of numerical data. Additionally, if one assumes the bathymetry and free surface are known, the detection algorithm detailed in [11] naturally leads to a method for sub-surface object detection.

3. Traveling water-waves and their stability

The overall goal in studying these special solutions is to form a framework for tackling inverse problems such as those in the previous section. These special solutions allow us to understand the effect of full nonlinearity and form the basis for comparing various asymptotic theories. To this end, I aim to extend the pressure problem mentioned above to the case of a constant vorticity. Of course, the physical relevance of these models is based on how stable these solutions are and so in collaboration with Katie Oliveras and Patrick Sprenger, I am currently studying the spectral stability of traveling waves with constant vorticity.

3.1. **A single equation for traveling water waves.** An unexpected consequence of my approach to the pressure problem was the discovery of a single scalar equation for the surface profile of a traveling water wave [8]. This formulation is valid for both 1-dimensional curves as well as for 2-dimensional surfaces. The key idea was to define the Normal→Tangential derivative operator H such that $H(\eta)G(\eta) \equiv \partial_x$, where $G(\eta)$ is the Dirichlet→Neumann operator. It is possible to define a Taylor series for this operator in manner similar to [3].

3.2. **Traveling waves with constant vorticity.** Recently, there has been an increased interest in the analysis and computation of traveling wave solutions with constant vorticity [2, 7, 14]. The case of constant vorticity is the simplest model of a wave-current interaction. In collaboration with Katie Oliveras, I have extended the formulation of [4] to the case of constant vorticity to obtain a single equation describing traveling water waves. In a recent publication [13], I derived a direct relationship between the surface of a traveling wave, and the pressure at the bottom of the fluid, including the effects of constant vorticity without approximation. The analysis revealed interesting features of the pressure, in particular, negative (below atmospheric) pressures beneath the crest.

4. Boundary-value problems for linear PDEs

Using ideas from integrable systems, Fokas has developed a method to efficiently analyze boundary-value problems for linear PDEs [6]. For a brief review of the method see [5]. I have extended Fokas' method to dispersion relations with a pole-type singularity (as in the case of the Linear Benjamin-Bona-Mahony equation). One of my future goals is to extend the method for more general types of dispersion relations, including fractional Laplacian type relations, piece-wise defined relations and more general pseudo-differential operators.

4.1. **Well-posedness of the Linear Benjamin-Bona-Mahony Equation.** In a paper with Bernard Deconinck, I extended Fokas' method to PDEs with mixed partial derivatives. In particular, I analyzed the well-posedness of linear BBM [12]. For a particular Robin boundary condition, a well-posed problem requires both initial and boundary data to be related to one another. This peculiar nature is a generic feature of equations with mixed partial derivatives and is due to the loss of invertibility of the differential operator present inside the time derivative for specific boundary conditions.

4.2. **The Stefan problem for the heat equation.** In joint work with Jonatan Lenells, Beatrice Pelloni and Natalie Sheils, I revisited the classical Stefan problem for the heat equation, in one spatial variable, within the framework of Fokas' method. The end result of the analysis is a single nonlinear equation for the position of the ice-water interface as a function of time. In a certain sense, this work adapts the AFM approach to water-waves [1] to the problem of melting ice.

4.3. **Boundary control of solutions to PDEs.** During my time at the Pritchard Fluids Laboratory at Penn State, I came to realize the importance of control problems in the context of experimental investigations. In the lab we often wish to simulate particular phenomena. However, it is usually the case the phenomenon of interest is incompatible with the constraints of the experimental set-up. For instance, the generation of solitary waves in a wave tank of finite extent. The Uniform Transform Method introduced by Fokas and collaborators permits one to relate the solution of an integrable PDE to its boundary values. Given a target solution, we attempt to solve for the appropriate boundary condition. As a result we we arrive at a natural control problem in a manner consistent with the underlying PDE.

5. Numerical studies

5.1. **A method to find the zeros of an analytic function.** The method consists of solving an associated ordinary differential equation. The required zero is the long-time limit of the solution to the differential equation, *i.e.* the zero is an attractor for the dynamical system. This method converges to the zero starting from any point in the domain of analyticity.

5.2. **Computing the Dirichlet-Neumann operator for the water-wave problem.** The most important aspect of an accurate numerical solver of the water-wave equations is the implementation of the Dirichlet-Neumann operator. In joint work with John Wilkening (University of California - Berkeley), I compared the effectiveness of solving Dirichlet–Neumann problems using five popular methods. The first three methods, inspired by theoretical developments in water-waves, involve highly ill-conditioned intermediate calculations that can be overcome using multiple-precision arithmetic. The remaining methods, boundary-integral method and transformed field expansions, avoided catastrophic cancellation of digits in intermediate results, and are much better suited to numerical computation.

5.3. **Numerical simulation of turbulence in density stratified fluids.** In a project prior to working on my thesis, I studied the problem of turbulent energy transfer in density stratified fluids [10]. A direct numerical simulation of the Navier-Stokes equations under the Boussinesq approximation, for high Reynolds numbers, lead to a detailed picture of the energy transfer between individual Fourier modes. All computations were performed using highly parallelized code run on the TeraGrid (a computing infrastructure that integrated high-performance computers, data resources and tools).

6. Physical laboratory investigations

6.1. **Experimental study of hydraulic jumps in partially closed containers.** Hydraulic jumps are a ubiquitous phenomenon and yet are not completely understood. The hydraulic jump is a result of a complex interaction of viscosity, surface tension, vorticity, non-hydrostatic pressure and free surface flows. In addition, hydraulic jumps occur in a variety of shapes from circular to polygonal. In recent experimental work, my colleague Andrew Belmonte (Pennsylvania State University) and I uncovered a secondary jump to occur under certain circumstances. The location of the secondary jump varies depending on the far field boundary condition.

6.2. **Generation of traveling water waves in a wave tank.** One of the foremost problems in a fluids laboratory is the generation of specific water-wave profiles using a moving paddle. At the Pritchard Fluid Lab in Penn State, I implemented a new paddle control system for shallow water-waves. The paddle motion is a consequence of an integral constraint of fluid flow, namely conservation of mass. Currently we are implementing a dual paddle control system to study the interaction of shallow and deep waves.

6.3. **Spin up and spin down in axisymmetric containers.** In experimental work performed as part of my masters thesis, I studied the effect of topography on the spin up/spin down of a rotating fluid. The project involved using a variety of visualization techniques to study instabilities and decay-rates in the fluid flow.

7. Other work

7.1. **Non-hydrostatic extensions to the shallow water equations.** The classical shallow water equations are derived under the assumption of hydrostatic pressure. These effects are particularly relevant when considering sudden changes in the bathymetry. Non-hydrostatic effects are introduced using ideas related to the pressure problem [9]. The bottom pressure acts as a forcing in the classical hyperbolic shallow water equations. In effect, this introduces a dispersive character to the original equations. Joint work with Katie Oliveras (Seattle University) and Kyle Mandli (Columbia University).

References

[1] M. J. Ablowitz, A. S. Fokas, and Z. H. Musslimani, *On a new non-local formulation of water waves*, J. Fluid Mech. **562** (2006), 313–343.

[2] A. Constantin and W. Strauss, *Exact steady periodic water waves with vorticity*, Communications on Pure and Applied Mathematics **57** (2004), 0481–0527.

[3] W. Craig and C. Sulem, *Numerical simulation of gravity waves.*, J. Comp. Phys. **108** (1993), 73–83.

[4] B. Deconinck and K. Oliveras, *The instability of periodic surface gravity waves*, J. Fluid Mech. **675** (2011), 141–167.

[5] B. Deconinck, T. Trogdon, and V. Vasan, *Solving linear partial differential equations*, Accepted for publication in SIAM-Review (2013), 1–21.

[6] A. S. Fokas, *A unified approach to boundary value problems*, CBMS-NSF regional conference series in applied mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[7] Joy Ko and Walter Strauss, *Large-amplitude steady rotational water waves*, European Journal of Mechanics-B/Fluids **27** (2008), no. 2, 96–109.

[8] K. Oliveras and V. Vasan, *A new equation describing for traveling waves*, J. Fluid Mech **717** (2013), 514–522.

[9] K. Oliveras, V. Vasan, B. Deconinck, and D. Henderson, *Recovering the water-wave profile from pressure measurements*, SIAM J. of Applied Mathematics **72** (2012), 897–918.

[10] J. Riley and V. Vasan, *Spectral energy transfer in strongly stratified flows*, Proceedings of EUROMECH Colloquium 512: Small Scale Turbulence and Related Gradient Structures (2009), 94–96.

[11] V. Vasan and B. Deconinck, *The inverse water wave problem of bathymetry detection*, Journal of Fluid Mechanics **714** (2013), 562–590.

[12] V. Vasan and B. Deconinck, *Well-posedness of boundary-value problems for the linear Benjamin-Bona-Mahony equation*, Disc. Cont. Dyn. Sys. Ser. A **33** (2013), 3171–3188.

[13] V. Vasan and K. L. Oliveras, *Pressure beneath a traveling wave with constant vorticity*, Disc. & Cont. Dyn. Sys. Ser. A (2013), To appear.

[14] E. Wahlen, *Steady periodic capillary waves with vorticity*, Ark. Mat. **44** (2006), 367–387.