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Part 1

Introduction
LECTURE 1

Introduction

1. What Is A Differential Equation?

A differential equation is an equation that contains derivatives of a function. On one hand, this seems obvious. On the other, it’s a remarkably subtle idea.

Why? Let’s think back to algebra. Algebraic equations had the form

\[ f(x) = c \]  

(1.1)

for some known function \( f(x) \) and some constant \( c \). \( f(x) \) may have been the derivative of another function, but we’d explicitly calculated it...the function was never unknown. Solving the equation meant finding values of the independent variable \( x \) that caused the function to have the desired value. This should be old news. But what does that have to do with anything at the moment?

Here’s the difference: the unknowns in a differential equation are functions, not numbers. This is an important conceptual jump, the central one in this course. We don’t plug in a value for \( x \) and see what \( f(x) \) evaluates to; we plug in expressions for a function and see what the whole mess simplifies to.

Another way (more abstract) of putting it is that a differential equation has the form

\[ L[y] = g(x) \]  

(1.2)

where \( y(x) \) is an unknown function (a variable in this case), \( g(x) \) is some other (known) function not involving \( y \), but rather only the independent variable, and \( L \) is a function defined on functions (what in mathematics is called an “operator”). Don’t worry too much about the concept of an operator, but hopefully you can see the parallel between Equations (1.1) and (1.2). They are extremely similar, except that Equation (1.1) involves numbers as inputs and solutions, while Equation (1.2) involves functions as inputs and solutions.

An example of an operator is \( \frac{d}{dx} \). If you plug in a function \( y \), you get \( \frac{dy}{dx} \), which is a (potentially) different function, just as if you plug a number \( x_0 \) into the function \( x^2 \), you get a (potentially) different number.

However, while differential equations and algebraic equations have a number of similarities, there are several consequences to the idea that we are solving equations for functions rather than numbers. First, it means that differential equations are hard. For this class, by “hard,” I mean “they take some getting used to.” This is likely the first time in your educational careers that this concept has been made explicit, though you’ve worked with relatively simple versions of them in calculus. We’re used to our equations having numerical solutions, and we understand exactly what it means for a number to solve an equation. Now, we’ll need to learn how to allow ourselves to abstract our notion of “equation.” In the real world, differential equations are actually difficult. Most can’t be solved, and some that can be solved require computers to do so.

Second, differential equations, when they have solutions, have infinitely many. Why? Let’s look at a differential equation you’ve all seen before:

\[ \frac{dy}{dx} = x. \]  

(1.3)

This is an equation you should have seen in calculus. What we want to find is a function \( y(x) \) that, when differentiated at a point \( x = x_0 \) gives us the same value \( x_0 \). We know how to solve this:
Differential Equations Lecture 1: Introduction

integrate.

\[ y(x) = \frac{x^2}{2} + c \quad (1.4) \]

Since differentiation ignores constants, when we integrate we have to account for this by adding in the constant of integration \( c \). This is where the infinity of solutions comes from, and this will come up every time we solve a differential equation. A solution of the form Equation (1.4) is called a \textit{general solution} because it can express every possible solution of the differential equation.

Usually, though, we’re interested in solving a particular initial value problem. Initial value problems are a combination of a differential equation and some initial conditions which fix a specific, or \textit{particular} solution. They have the form

\[ y^{(k)}(t_0) = y_k \]

for whichever derivatives of \( y \) are necessary. This is no different from an integration problem like Equation (1.3); we fixed a particular value of \( y \) to determine \( c \), and that gave us a unique function. So, when we are faced with an initial value problem, we’re doing two things: we’re finding all functions that satisfy the equation and then choosing the particular one that satisfies the given initial conditions.

Let’s say we have a differential equation. A function \( y \) is a solution to the differential equation if, when we take the relevant derivatives and plug them in, the equation “works”: everything simplifies the way it should.

**Example 1.1.** Consider

\[ 4x^2 y'' + 9xy' + y = 0. \]

I claim \( y(x) = x^{-\frac{1}{4}} \) is a solution to this equation. To check, we’ll need the first and second derivatives:

\[
\begin{align*}
y'(x) &= -\frac{1}{4} x^{-\frac{3}{4}} \\
y''(x) &= \frac{5}{16} x^{-\frac{5}{4}}
\end{align*}
\]

Then:

\[
\begin{align*}
4x^2 \left( \frac{5}{16} x^{-\frac{9}{4}} \right) + 9x \left( -\frac{1}{4} x^{-\frac{5}{4}} \right) + \left( x^{-\frac{1}{4}} \right) &= 0 \\
\frac{5}{4} x^{-\frac{7}{4}} - \frac{9}{4} x^{-\frac{1}{2}} + x^{-\frac{1}{4}} &= 0 \\
0 &= 0
\end{align*}
\]

So \( y(x) \) is in fact a solution. Here are some others:

\[
\begin{align*}
y(x) &= x^{-\frac{7}{4}} \\
y(x) &= 12x^{-\frac{1}{4}} \\
y(x) &= 6x^{-\frac{1}{4}} - 2x^{-\frac{7}{4}}
\end{align*}
\]

and so on. In fact, every solution to this differential equation is of the form

\[ y(x) = c_1 x^{-\frac{1}{4}} + c_2 x^{-\frac{7}{4}} \quad (1.5) \]

for some constants \( c_1 \) and \( c_2 \). So Equation (1.5) is the \textit{general solution} to the differential equation, and we could pick out a particular solution if we specified some initial conditions. \( \square \)
We refer to a solution such as in Example 1.1 as an **explicit solution** since it is of the form \( y = f(x); \) in other words, the function \( y(x) \) has been solved for with \( x \) as the only independent variable. We’ll see examples soon where we won’t be able to solve for \( y \) explicitly. Instead, in those cases we will have to be content to find **implicit solutions**.

**Example 1.2.** \( y^2 = t^2 + 1 \) is a implicit solution to

\[
y' = \frac{t}{y} \quad y(0) = 1
\]

**Exercise.** Check that, indeed, \( y^2 = t^2 - 3 \) is a solution to the initial value problem given in Example 1.2. Remember to use implicit differentiation.

**Example 1.3.** Let’s find the explicit solution to the problem given in Example 1.2. We know that \( y^2 = t^2 + 1 \) is a solution; now we want to solve for \( y \). Doing so yields \( y = \pm \sqrt{t^2 + 1} \). While this is an explicit solution for \( y \), it doesn’t satisfy the initial condition \( y(0) = 1 \) as \( \pm \sqrt{0^2 + 1} = \pm 1 \), not just 1, so we need to be careful. If we look more closely at \( y = \pm \sqrt{t^2 + 1} \), we see that this expression really contains two different functions, differing by sign. To figure out which one we want, we use the initial condition \( y(0) = 1 \). Only one of the two possibilities will satisfy the initial condition (this is not true in general, but it will be true in this example). Plugging in, we see that the positive one is correct, and

\[
y(t) = \sqrt{t^2 + 1}
\]

is the explicit solution to this initial value problem. □

Checking that the functions you calculate are actually solutions is very important. It’s a good habit to get into, because it’s very easy to make a sloppy mistake (such as a sign error) that can drastically change the solution.

The other issue we need to keep in mind is that solutions to differential equations are generally only valid on certain intervals. Consider Example 1.1. We know \( y(x) = x^{-\frac{1}{2}} \) is a solution to this equation... but for what values of \( x \)? It can’t be all of them, since \( y(x) \) isn’t even defined at \( x = 0 \). It turns out that this solution is valid for \( x > 0 \). In fact, looking at Equation (1.5), no solution to this differential equation exists at \( x = 0 \)! So even though \( y(x) \) symbolically satisfied the differential equation without any consideration about values of \( x \), the solution implied a certain restriction on the independent variable.

This is generally going to be the case: it’s very rare when a solution to a differential equation works for all values of the independent variable. The interval on which a solution is valid is called the **interval of validity**, and in some cases we’ll talk about being able to deduce some information about the interval of validity of a solution without actually having to find the solution.

### 2. Why do we care?

Good question! We care because differential equations can be used to model all sorts of continuous processes, such as radioactive decay, continuous interest, mixing problems, harmonic motion, pendulums, heat diffusion, and the movement of a string. Thus, from the perspective of a physicist or an engineer, differential equations are a foundational tool. Most of the math you will see further down the road in your major courses will involve differential equations.

Even if that weren’t the case, differential equations are a rich and interesting area. There’s a lot of interesting and unexpected behavior that can occur that we will unfortunately only be able to begin looking at.

### 3. Examples

Now let’s look at some examples of differential equations.
**Example 1.4 (Radioactive Decay).** Suppose we have a certain number of atoms $N(t)$ of some radioactive isotope, let’s say Carbon-14 (C-14). All such isotopes have a certain decay constant $\lambda$ associated to them which gives the rate at which a quantity of these atoms will decay. In the case of C-14, $\lambda = 1.21 \times 10^4$ yr$^{-1}$. The rate of change of $N$ is just $\frac{dN}{dt}$. So, our differential equation would look something like 

$$\frac{dN}{dt} = -\lambda N(t)$$

since we know the derivative will be negative. This is the principle behind carbon dating...if we can solve this equation, and we know how many C-14 atoms were present in a sample before it died and how many are there now (say $N_0$), we can solve $N(t) = N_0$ for $t$ and know how old the sample is. □

**Example 1.5 (Newton’s Law of Cooling).** In 1701, Newton published the following thermodynamic observation:

“The time rate of change of temperature in a body immersed in a constant temperature environment is proportional to the temperature difference between the body and the environment.”

Let’s say $B(t)$ is the body’s temperature. We’ll refer to the constant external temperature as $E$. Newton’s Law of Cooling, in mathematical form, becomes

$$\frac{dB}{dt} = \kappa (E - B(t))$$

where $\kappa$ is some proportionality constant that will depend on the material of the body. □

**Example 1.6 (Heat Equation).** Suppose we have a (one-dimensional) rod of length $l$. We want to know how the temperature distribution on the rod evolves. We’ll denote the temperature at position $x$ and time $t$ by $u(x,t)$. The heat equation (which we’ll look at more closely towards the end of the semester) says

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

where $k$ is some constant that indicates how quickly heat propagates through the material. What this says is that if we consider how the temperature is changing at a fixed point, it’s proportional to the concavity of the temperature at that time around that point. □

**Example 1.7 (Schrödinger Equation).** This one’s a doozy, so we won’t worry about interpreting it, but I will mention it again later in the class:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial x^2} + V(x)\Phi(x,t)$$

where $\Phi$ is the wave function, $\hbar$ is Planck’s constant divided by $2\pi$, and $V(x)$ is a potential function. □

### 4. Basic Definitions

Before we move on, we need to get some definitions out of the way so that we can talk about categories of equations in one shot.

**Definition 1.8.** If a differential equation involves the derivatives of a function of a single variable, it is an *ordinary differential equation*. If it involves the partial derivatives of a function of several variables, it is a *partial differential equation*. 
Newton’s Law of Cooling and Radioactive Decay were ordinary differential equations, while the Heat Equation and Schrodinger’s Equation were partial differential equations. If you haven’t had multivariable calculus, don’t worry: we’ll talk more about partial derivatives when they become more relevant.

**Definition 1.9.** The *order* of a differential equation is the highest derivative that occurs in the equation.

For example, both of the Newton’s Law of Cooling and Radioactive Decay equations are first order. The Heat Equation and the Schrodinger Equation are second order.

\[ y^{(4)} + 10y''' - 3y = 3t \]

is a fourth order ordinary differential equation.

**Definition 1.10.** An ordinary differential equation is called *linear* if it has the form

\[ a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = g(t) \]

for any functions \( a_i(t), g(t) \). For a partial differential equation, substitute in the relevant partial derivatives for the ordinary derivatives given above. A differential equation is *nonlinear* if it is not linear.

**Remark.** Notice that linearity of the differential equation has nothing to do with how the coefficient functions depend on \( t \) (namely, a linear differential equation can have nonlinear coefficient functions), only how the equation treats the unknown function \( y \). All we do is take derivatives of \( y \), multiply them by functions depending only on \( t \), and add them together. If anything else is done, the equation is nonlinear.

So, for example,

\[ y'' + \frac{1}{\sin t}y' = 2t^2 \]

is a linear equation, but

\[ y' y^2 = 2t \]

is nonlinear.

**5. General Strategy**

Here are the questions to constantly keep in mind as we discuss various types of differential equations:

1. **Does a solution exist to a given an initial value problem?**
   
   Not all initial value problems have solutions. Given an initial value problem, we’d like to know ahead of time if a solution might exist without having to waste hours banging our heads against the wall. This is the question of the *existence* of solutions to an initial value problem.

2. **If a solution exists, how many are there?**
   
   It’s possible that there are several, or even infinitely, many solutions. This is usually a bad result (particularly for applications), since if we subject five identical steel rods to the same conditions, they should all have the same temperature distributions. But if our initial value problem doesn’t have a unique solution, this wouldn’t be the case, which doesn’t make physical sense. So we’d like to know when a unique solution exists to make sure that we are working with the correct equation.

   This is the question of the *uniqueness* of solutions to an initial value problem.
(3) **How can we find a solution, if one exists?**

As this is an introductory course, we will spend almost all of our time working on methods for finding solutions to initial value problems for which existence and uniqueness are known. However, it is crucial that you understand that in general, an initial value problem ought not to have a unique solution, or even that a solution exist. The problems that we will be working with in this course are highly non-representative of the world of differential equations, which, believe it or not, is what makes them fun!
Part 2

First Order Equations
Lecture 2

Separable Equations

The most straightforward first order equations to solve are *separable equations*, which have the form

\[
\frac{dy}{dx} = f(y)g(x)
\]  

(2.1)

It’s critical that a separable equation has this form. If you can’t write the equation as the product of a function only depending on \( y \) and a function only depending on \( x \), the equation is not separable.

1. Solution Method

How do we solve them? The name should be revealing: the first step is to separate variables.

\[
\frac{1}{f(y)}dy = g(x)dx
\]

\[
\int \frac{1}{f(y)}dy = \int g(x)dx
\]

\[
F(y) = G(x) + c
\]

This gives an implicit solution for \( y(x) \) (assuming we were able to compute both the integrals, which is not a given, as you should be aware from calculus), which we may be able to solve for explicitly, but in general we shouldn’t expect to. At this point, if we have an initial condition we can plug it in to get the particular solution.

**Remark.** Do not forget the constant of integration in this solution process. In the form above, I’ve combined the two constants into \( c \). Strictly speaking, after integration we would get

\[
F(y) + c_1 = G(x) + c_2,
\]

but if we just subtract \( c_1 \) from both sides we don’t have to worry about writing down that extra step, since we just end up with some constant on one side anyway.

There’s one other point to keep in mind: division by zero is never ok. This brings up a slight complication to our solution method, since we won’t be accounting for those cases (if they occur in a given problem). The fix is fairly easy, though. If \( f(y_0) = 0 \), then \( \frac{dy}{dx}(y_0, x) = 0 \) for all \( x \); hence if \( y(x_0) = y_0 \) for any \( x_0 \), then \( y \) is the constant function \( y(x) = y_0 \). Such a solution is called an *equilibrium solution*. Hence we have several solutions: \( y(x) = y_k \) for any \( y_k \) satisfying \( f(y_k) = 0 \) and the general solution given by \( F(y) = G(x) + c \) as given above. When just asked to “solve” the differential equation, you ought to take note of all of them; when looking for a particular solution, see whether the initial \( y_0 \) is a zero of \( f(y) \) and go from there.

2. Examples
Example 2.1 (Newton’s Law of Cooling). Earlier, we saw Newton’s Law of Cooling and wrote it as the differential equation

\[
\frac{dB}{dt} = \kappa (E - B).
\]

This is definitely separable: using our earlier notation, \( f(B) = E - B \) (since \( E \) is a constant) and \( g(t) = \kappa \). Now suppose we have some initial condition \( B(0) = B_0 \). We first note that if \( B_0 = E \), this is an equilibrium solution, and in fact it is the only one. This makes sense physically, since if the body is at the same temperature as the environment, surely no heat transfer will occur. Let’s find the non-equilibrium solutions:

\[
\int \frac{dB}{E - B} = \int \kappa \, dt
\]

\[- \ln ||E - B|| = \kappa t + c \]

\[E - B = e^{-\kappa t + c} \]

\[= Ae^{-\kappa t} \]

\[B(t) = E - Ae^{-\kappa t}.\]

In this case we were able to solve for \( B(t) \) explicitly. Now if we use our initial condition:

\[B_0 = E - A \]

\[A = E - B_0 \]

\[B(t) = E - \frac{E - B_0}{e^{\kappa t}}. \]

□

Example 2.2.

\[
\frac{dy}{dx} = 2y^2 x \quad y(1) = \frac{1}{3}.
\]

This is certainly in the form of Equation (2.1) with \( f(y) = y^2 \) and \( g(x) = 2x \). First we look for the roots of \( f(y) \), which tells us that the only equilibrium solution is \( y(x) = 0 \), but this isn’t the particular solution we’re looking for as the initial condition doesn’t match up.

\[
\int \frac{dy}{y^2} = \int 2x \, dx
\]

\[- \frac{1}{y} = x^2 + c \]

\[- \frac{1}{y} = (1^2) + c \Rightarrow c = -4 \]

\[- \frac{1}{y} = x^2 - 4, \]

or

\[y(x) = \frac{1}{4 - x^2}. \]

What is the interval of validity for the solution to this initial value problem? The only problems occur when \( 4 - x^2 = 0 \), or when \( x = \pm 2 \). This solution, then, has three possible intervals of validity: \((-\infty, -2), (-2, 2), \) \text{ and } \((2, \infty)\). We want to choose the one containing our initial value of \( x \), which in this case is \( x = 1 \), and this means that the interval of validity for the solution is \(-2 < x < 2. \) □
EXERCISE. In Example 2.2, verify that an initial condition of
\[ y(-4) = -\frac{1}{12} \]
yields the same solution with an interval of validity of \((-\infty, -2)\). How about \( y(3) = -\frac{1}{5} ? \)

EXAMPLE 2.3.
\[ y' = \frac{3x^2 + 2x - 6}{2y - 2} \quad y(1) = 4 \]
There are no equilibrium solutions here, since there is no value of \( y \) that makes \( \frac{1}{2y-2} = 0 \), so:
\[
\int 2y - 2 \, dy = \int 3x^2 + 2x - 6 \, dx
\]
\[ y^2 - 2y = x^3 + x^2 - 6x + c. \]
In a situation like this, it’s convenient to compute the constant of integration before going on:
\[ 4^2 - 8 = 1 + 1 - 6 + c \]
\[ 4 = -4 + c \Rightarrow c = 8. \]
Now, we want to solve for \( y(x) \) explicitly. We can do this either by completing the square or by using the quadratic formula:
\[ y^2 - 2y + 1 = x^3 + x^2 - 6x + 8 + 1 \]
\[ (y - 1)^2 = x^3 + x^2 - 6x + 9 \]
\[ y(x) = 1 \pm \sqrt{x^3 + x^2 - 6x + 9}. \]
This expression contains two solutions; we need to choose the appropriate one. To do that, we use the initial condition, which is that \( y(1) = 4 \). We will need to choose the positive solution rather than the negative one, since the square root is always positive and \( 4 > 1 \).
\[ y(x) = 1 + \sqrt{x^3 + x^2 - 4x + 6}. \]
Computing the interval of validity is a bit more complicated, since it would require us to find the roots of \( x^3 + x^2 - 6x + 9 \), which unfortunately are not integers. But the principle is straightforward enough: we need to avoid those values of \( x \) such that \( x^3 + x^2 - 6x + 9 < 0 \). Once we know what those values are, we find the appropriate interval containing \( x = 1 \) and that will be the interval of validity. \[ \square \]

EXAMPLE 2.4.
\[ \frac{dy}{dx} = \frac{xy^3}{1 + x^2} \quad y(0) = 1 \]
We only have one equilibrium solution, namely, \( y(x) = 0 \), which isn’t our case, so let’s separate:
\[
\int \frac{dy}{y^3} = \int \frac{x}{1 + x^2} \, dx
\]
\[ -\frac{1}{2y^2} = \frac{1}{2} \ln (1 + x^2) + c \]
Use the initial condition:
\[ -\frac{1}{2} = c. \]
So:

\[ y^2 = \frac{1}{1 - \ln(1 + x^2)} \]
\[ y(x) = \frac{1}{\sqrt{1 - \ln(1 + x^2)}}. \]

Now all that’s left is to determine the interval of validity. We can’t divide by zero, nor can the quantity inside the square root be negative. Accordingly, our requirement is

\[
\ln (1 + x^2) < 1
\]
\[ 1 + x^2 < e \]
\[ x^2 < e - 1 \]

So the desired interval of validity is \(-\sqrt{e - 1} < x < \sqrt{e - 1}\). □

**Example 2.5.**

\[ x^2 \frac{dy}{dx} = y - 1 \quad y(0) = 1 \]

Putting this into separable form, we get

\[ \frac{dy}{dx} = \frac{y - 1}{x^2}. \]

There’s one equilibrium solution: \( y = 1 \). And our initial condition is that \( y(0) = 1 \), so in this case, the desired solution is just the constant function \( y(x) = 1 \). □

**Example 2.6.**

\[ \frac{dy}{dt} = e^{y-t} \sec(y) \left(1 + t^2\right) \quad y(0) = 0 \]

Before we can separate this, we first need to rewrite it slightly, so it’s in separable form, and we’ll see we have no equilibrium solutions, so we need to continue:

\[ \frac{dy}{dt} = \frac{e^y e^{-t}}{\cos(y)} (1 + t^2) \]
\[ \int e^{-y} \cos(y) \, dy = e^{-t} (1 + t^2) \, dt. \]

Both sides require integration by parts, and after doing that we’ll get an implicit solution.

\[ \frac{e^{-y}}{2} (\sin(y) - \cos(y)) = -e^{-t} (t^2 + 2t + 3) + c \]

Now we’ll use the initial condition.

\[ \frac{1}{2} (-1) = -3 + c \Rightarrow c = \frac{5}{2} \]
\[ \frac{e^{-y}}{2} (\sin(y) - \cos(y)) = -e^{-t} (t^2 + 2t + 3) + \frac{5}{2}. \]

In this case, we won’t be able to find an explicit solution, so we have to leave it in implicit form. It’s generally very difficult to find the interval of validity when we have an implicit solution, so we won’t bother for this problem. □
Differential Equations

LECTURE 3

Linear First Order Equations

Recall that an \( n \)th-order linear ordinary differential equation has the following form:

\[
a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \ldots + a_1(t) y'(t) + a_0(t) y(t) = g(t)
\]

for some continuous functions \( a_n(t), g(t) \). In other words, a differential equation is linear if it looks like a polynomial (with differentiation instead of exponentiation).

This means that linear first order equations have a very simple form:

\[
a_1(t) y'(t) + a_2(t) y(t) = g(t).
\]

This isn’t a very suitable form to work with, however. What we want to do is to divide through by \( a_1(t) \) to obtain the following standard form:

\[
\frac{dy}{dt} + p(t) y(t) = q(t).
\]  

(3.1)

**Remark.** The linear equation must be in this form before we try to solve it. The following method is dependant on the coefficient of \( y' \) being 1 and \( p(t) \) having the correct sign.

Notice that if \( p(t) = 0 \) or \( q(t) = 0 \), Equation (3.1) is not only linear, but separable as well. However, if \( p(t) \neq 0 \) and \( q(t) \neq 0 \), separation of variables will not work. We’ll need to be a bit more clever to solve this equation.

1. Solution Method: Integrating Factors

The key to solving just about any first order equation is to put it in a form where we can integrate both sides, as when we separated variables for a separable equation. There’s no direct way to do that in this case, however. What we’ll do is hazard a guess as to something that might help, plug in, and then figure out specifics. This will be a recurring theme in this class.

Let’s suppose we multiply Equation (3.1) by some function \( \mu(t) \):

\[
\mu(t) \frac{dy}{dt} + \mu(t) p(t) y(t) = \mu(t) q(t).
\]

(3.2)

The left hand side of Equation (3.2) will hopefully look familiar; it resembles the equation for the product rule from calculus:

\[
y(t) \mu(t) + \mu'(t) y(t) = \frac{d}{dt} [\mu(t) y(t)].
\]

(3.3)

If we could choose our function \( \mu(t) \) such that this were the case, the left hand side of Equation (3.2) would just be the derivative of some product with respect to \( t \). Since the right hand side is nothing more than some function of \( t \), we could then integrate both sides and solve for \( y(t) \).

So what does \( \mu(t) \) need to be? For the left hand side of Equation 3.2 to be equivalent to the left hand side of Equation (3.3), we need

\[
\mu'(t) = \mu(t) p(t).
\]

---

1Remember that a function \( f(x) \) is continuous at \( x = x_0 \) if \( \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0) \), so the two-sided limit exists and is equal to the value of the function. Intuitively, this means are no breaks or jumps in the graph of \( f(x) \) in some interval around \( x = x_0 \).
Differential Equations

Lecture 3: Linear First Order Equations

This is a separable equation, and it’s not hard to see that the solution is

$$\mu(t) = e^{\int p(t) \, dt}. \quad (3.4)$$

**Exercise.** Check the above assertion.

With this choice of \( \mu(t) \), we can rewrite Equation (3.2) as

$$\frac{d}{dt} \left[ \mu(t) y(t) \right] = \mu(t) q(t)$$

and solve for \( y(t) \):

$$\mu(t) y(t) = \int \mu(t) q(t) \, dt$$

$$y(t) = \frac{\int \mu(t) q(t) \, dt}{\mu(t)}.$$

**Remark.** You may (and should!) notice that there is no constant of integration showing up on the left hand side despite the integration. What we’ve implicitly done is to combine it with whatever constant of integration would come up when we compute \( \int \mu(t) q(t) \, dt \), as you’ll see in the examples to come. We’ve also done this with the constant that ought to appear when we find \( \mu(t) \). It’s imperative not to forget the constant at the final stage, however, or your answer will be very, very wrong. If you find that you lose track of the constant because you’re not writing it down on both sides, there is no problem with doing so.

This method is known as the method of integrating factors and \( \mu(t) = e^{\int p(t) \, dt} \) as the integrating factor of Equation (3.1). My recommendation is not to memorize the last line in the solution, but rather to compute \( \mu(t) \), multiply through, and go from there, which is what I will do in the examples below. Also, notice that we always get an explicit solution using this method.

Before we do some examples, let’s summarize the previous discussion. The steps to solve a first order linear differential equation are:

1. Put the equation in the correct form
   $$y' + p(t)y = q(t).$$

2. Calculate the integrating factor,
   $$\mu(t) = e^{\int p(t) \, dt}.$$

3. Multiply both sides of the equation by \( \mu(t) \).
4. Integrate both sides, being careful not to forget the constant of integration.
5. Solve for \( y(t) \), using the initial condition (if applicable) to calculate the constant of integration.

**2. Examples**

**Example 3.1.**

$$\cos(x)y' + \sin(x)y = 2 \cos^3(x) \sin(x) - 1 \quad y \left( \frac{\pi}{4} \right) = -\sqrt{2}, 0 \leq x < \frac{\pi}{2}$$

We first need to put this equation in the correct form, so divide through by \( \cos(x) \).

$$y' + \tan(x)y = 2 \cos^2(x) \sin(x) - \sec(x)$$
Comparing this to Equation 3.1, we see that \( p(x) = \tan(x) \), so our integrating factor is
\[
\mu(x) = e^{\int \tan(x) \, dx} = e^{\ln|\sec(x)|} = e^{\ln\sec(x)} = \sec(x),
\]

using the facts that \( \sec(x) \) is non-negative on the interval \([0, \frac{\pi}{2}]\) and that \( e^{\ln f(x)} = f(x) \). So, we multiply our differential equation through by \( \sec(x) \) and solve for \( y(x) \).

\[
\sec(x)y' + \sec(x) \tan(x)y = 2 \cos(x) \sin(x) - \sec^2(x)
\]

\[
(\sec(x)y)' = 2 \cos(x) \sin(x) - \sec^2(x)
\]

\[
\sec(x)y(x) = \int 2 \cos(x) \sin(x) - \sec^2(x) \, dx
\]

\[
= \int \sin(2x) - \sec^2(x) \, dx
\]

\[
= -\frac{1}{2} \cos(2x) - \tan(x) + c
\]

\[
y(x) = -\frac{1}{2} \cos(2x) \cos(x) - \sin(x) + c \cos(x).
\]

Now we use our initial condition to compute \( c \).

\[
-\sqrt{2} = -\frac{1}{2} \cos\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) + c \cos\left(\frac{\pi}{4}\right)
\]

\[
= -\frac{\sqrt{2}}{2} + c \frac{\sqrt{2}}{2} \Rightarrow c = -1
\]

Thus our particular solution is

\[
y(x) = -\frac{1}{2} \cos(2x) \cos(x) - \sin(x) - \cos(x).
\]

**Example 3.2.**

\[ ty' + 3y = t^2 - t \quad y(1) = 1 \]

Once again, we start by putting this into standard form, which requires division by \( t \):

\[ y' + \frac{3}{t}y = t - 1 \]

Now we compute the integrating factor:

\[
\mu(t) = e^{\int \frac{3}{t} \, dt} = e^{3\ln t} = e^{\ln(t^3)} = t^3
\]

We continue by multiplying through by \( \mu(t) \) and using the product rule.

\[
(t^3y)' = t^4 - t^3
\]

\[
t^3y = \int t^4 - t^3 \, dt
\]

\[
= \frac{1}{5} t^5 - \frac{1}{4} t^4 + c
\]

\[
y(t) = \frac{t^2}{5} - \frac{t}{4} + \frac{c}{t^3}.
\]
Using our initial condition to find $c$ gives

\[ 1 = \frac{1}{5} - \frac{1}{4} + c \Rightarrow c = \frac{21}{20} \]

and our particular solution is

\[ y(t) = \frac{t^2}{5} - \frac{t}{4} + \frac{21}{20t^3}. \]

Example 3.3.

\[ 2y' - y = 4 \sin(3t) \quad y(0) = -1 \]

First, divide by 2 to put this in the correct form.

\[ y' - \frac{y}{2} = 2 \sin(3t) \]

Next, compute $\mu(t)$.

\[ \mu(t) = e^{\int -\frac{1}{2} dt} = e^{-\frac{t}{2}} \]

Multiply through by $\mu(t)$ and write the left hand side as a product.

\[ \left( e^{-\frac{t}{2}} y \right)' = 2e^{-\frac{t}{2}} \sin(3t) \]

Integrate both sides and solve for $y(t)$.

\[ e^{-\frac{t}{2}} y(t) = \int 2e^{-\frac{t}{2}} \sin(3t) \, dt \]

\[ = -\frac{24}{37} e^{-\frac{t}{2}} \cos(3t) - \frac{4}{37} e^{-\frac{t}{2}} \sin(3t) + c \]

\[ y(t) = -\frac{24}{37} \cos(3t) - \frac{4}{37} \sin(3t) + ce^\frac{t}{2} \]

Finally, we compute the constant of integration.

\[ -1 = -\frac{24}{37} + c \Rightarrow c = -\frac{13}{37} \]

Thus our particular solution is

\[ y(t) = -\frac{1}{37} \left( 24 \cos(3t) + 4 \sin(3t) + 13e^{\frac{t}{2}} \right). \]

Example 3.4.

\[ ty' - 2y = t^4 \cos(t) + t^3 - 3t^5 \quad y(\pi) = 2 \]

First, divide by $t$ to put this in the correct form.

\[ y' - \frac{2}{t} y = t^3 \cos(t) + t^2 - 3t^4 \]

Next, compute $\mu(t)$.

\[ \mu(t) = e^{\int -\frac{2}{t} dt} = e^{-2 \ln(t)} = t^{-2} \]

Multiply through and write the left hand side as a product.

\[ \left( t^{-2} y \right)' = t \cos(t) + 1 - 3t^2 \]

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Integrate both sides and solve for $y(t)$.

\[
t^{-2}y = \int t \cos(t) + 1 - 3t^2 \, dt
\]
\[
= t \sin(t) + \cos(t) + t - t^3 + c
\]
\[
y(t) = t^3 \sin(t) + t^2 \cos(t) + t^3 - t^5 + ct^2
\]

Plug in the initial condition to compute $c$.

\[
2 = \pi^3 \sin(\pi) + \pi^2 \cos(\pi) + \pi^3 - \pi^5 + c\pi^2
\]
\[
c\pi^2 = 2 + \pi^5 - \pi^3 + \pi^2
\]
\[
c = \frac{2 + \pi^5 - \pi^3 + \pi^2}{\pi^2}
\]

Our particular solution is then

\[
y(t) = t^3 \sin(t) + t^2 \cos(t) + t^3 - t^5 + \frac{2 + \pi^5 - \pi^3 + \pi^2}{\pi^2}t^2.
\]
LECTURE 4

Modeling With First Order Equations

Before we resume our discussion of how to solve certain types of first order equations, let’s move to some applications of linear and separable equations. Our goal is to get a rough introduction to the process of modeling: how do we write a differential equation to model (fairly basic) situations and what can the solution tell us?

We’ve already seen some examples of first order models. In Lecture 1, we saw two examples of differential equations meant to describe physical situations.

Example 4.1 (Radioactive Decay).
\[
\frac{dN}{dt} = -\lambda N(t),
\]
where \(N(t)\) is the number of atoms of a radioactive isotope and \(\lambda > 0\) is the decay constant. This equation is separable, and it’s easy to see that, if the initial data is \(N(0) = N_0\), the solution is
\[
N(t) = N_0 e^{-\lambda t}
\]
So we can see that radioactive decay is exponential. □

Example 4.2 (Newton’s Law of Cooling). If we immerse a body in an environment with constant temperature \(E\), then if \(B(t)\) is the temperature of the body we have
\[
\frac{dB}{dt} = \kappa (E - B),
\]
where \(\kappa > 0\) is a constant related to the material of the body and how it conducts heat. This equation is also separable; we solved it with initial condition \(B(0) = B_0\) to get
\[
B(t) = E - \frac{E - B_0}{e^{\kappa t}}.
\]
□

How do we write down a model given a description of the situation? There are a few different approaches that we’ll see in this class:

1. Remember that we can think of the derivative of a function as its rate of change. It’s possible that the description of the problem tells us directly what the rate of change is. Newton’s Law of Cooling was an example of this; the original quote told us that the rate of change of the body’s temperature was proportional to the difference in temperature between the body and the environment. All we had to do was set the relevant terms equal.

2. There are also going to be cases where we’re not explicitly given the formula for rate of change in the problem, but if we can glean how a function should change from its physical description, we can set the derivative equal to that. The basic premise that we’ll be using here is derivative = increase - decrease. Next, we’ll see some more in-depth examples of this process. It should be noted that this can only apply to first order equations, since higher order equations can’t just be interpreted as “the rate of change is equal to something.”
(3) We may just be adapting a known differential equation to a particular situation. Newton’s Second Law, \( F = ma \), is a common example of this. It’s either a first or second order differential equation (for velocity or position, respectively). If we compile the list of forces acting on the object in question, we can just plug them in for \( F \) to yield the differential equation for a particular situation. We’ll see this approach later this lecture with regard to falling bodies, but it will also appear later in the course concerning harmonic motion and pendulums.

(4) The last possibility is that we are able to determine (via physical principles) two different expressions for some quantity, one or both of which involve derivatives; setting them equal yields the desired differential equation. We’ll see this when we discuss partial differential equations later.

It’s important that, when faced with a problem, we begin by determining which of these approaches (or other ones that may be out there) is most applicable before proceeding. This is a skill that will prove valuable in the future.

One other very important point: in general, your differential equation should not depend on the initial condition. The initial data will tell you the starting point, but the way the system evolves (which is given by the differential equation) ought not to depend on that. Now, let’s look at some particular examples.

1. Interest

These are among the most straightforward problems to write down. Suppose we’ve got a bank account (or a mortgage, or some other loan, or so on...the particulars don’t change much about the equation) that gives \( r \% \) interest per year. If I withdraw a constant \( w \) dollars per month, what is the differential equation modeling my account’s balance?

Let’s take our time unit for \( t \) to be years, and denote the balance after \( t \) years by \( B(t) \). \( B'(t) \) is the rate of change of my account balance from year to year, so it will be the difference between the amount that’s added to my account and the amount that’s withdrawn. The only income I’m getting is interest and we know how much I withdraw each year (it’s the monthly withdrawal times 12). Hence

\[
B'(t) = \frac{r}{100} B(t) - 12w.
\]

This is a linear equation, so we could solve it using integrating factors to know the account balance at any time \( t \). Make sure you pay attention to units: we’re not withdrawing \( w \) dollars per year, we’re withdrawing \( 12w \).

The same setup will work perfectly for something like the following problem.

**Example 4.3.** Bill wants to take out a 25 year loan to buy a house. He knows that he can afford, maximum, monthly payments of $400. If the going interest rate on housing loans is 4%, what is the largest loan Bill can take out that he will be able to pay off in time?

Let’s say the amount that Bill owes at time \( t \) (it’s convenient here to measure \( t \) in years, so we’ll do that) is \( B(t) \). We want \( B(25) = 0 \). We also know that the balance will gain 4% interest (i.e., the amount being added to the balance is \( 0.04B \)) and he can make payments totalling $4800 each year. So the relevant initial value problem should be

\[
B'(t) = .04B(t) - 4800 \quad B(25) = 0.
\]

This is a linear equation with standard form

\[
B'(t) - .04B(t) = -4800,
\]

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so the integrating factor is \( \mu(t) = e^{\int -0.4 \, dt} = e^{-0.4t} \).

\[
\left( e^{-\frac{4}{100}t} B(t) \right)' = -4800 e^{-\frac{4}{100}t} \\
e^{-\frac{4}{100}t} B(t) = -4800 \int e^{-\frac{4}{100}t} \, dt \\
= 120000 e^{-\frac{4}{100}t} + c \\
B(t) = 120000 + ce^{\frac{4}{100}t} \\
B(25) = 0 = 120000 + ce^{\frac{4}{100}25} \Rightarrow c = -120000 e^{-1} \\
B(t) = 120000 - 120000 e^{\frac{4}{100}t-1}.
\]

We want to know the size of the loan, which is the amount Bill owes to begin with, hence \( B(0) \).

\[ B(0) = 120000 - 120000 e^{-1} = 120000 \left( 1 - e^{-1} \right) \]

\[ \square \]

2. Mixing Problems

Suppose we have an efficient mixing tank that can instantly completely mix whatever is inside of it. The tank has some liquid inside of it. Contaminant is being added to the tank at some constant rate and the mixed solution is drained out at a (possibly different) rate. We can ask ourselves what the amount of contaminant is in the tank at any given time.

How do we write down a differential equation to model this process? Let \( P(t) \) be the amount of pollutant (note: amount of pollutant, not the concentration; this is critical) in the tank at time \( t \). What information are we given? We know (or are given enough information to compute) the amount of contaminant that is entering and leaving the tank each unit of time. So we can use the second approach from above:

\[
\text{Rate of Change of } P(t) = \text{Rate of entry of contaminant} - \text{Rate of exit of contaminant}.
\]

The rate of entry can be determined in a few different ways. We might be directly adding contaminant (think about using a pipette to add food coloring to a tank of water) or we might be adding solution with a known concentration of contaminant to the tank (in which case the amount would be the concentration times the volume). The problem will make this clear and then we can compute how much contaminant is being added to the tank each time unit (as you’ll see shortly). The important thing is that we determine the amount of stuff being added.

What’s the rate of exit? Suppose that we’re draining the tank at a rate of \( r_{\text{out}} \). The amount of contaminant leaving the tank will be the amount contained in the drained solution; this is given by rate \( \times \) concentration. We know the rate, so we need to compute the concentration. This will just be the concentration of the solution in the tank (hence the importance of having a uniformly mixed solution), which is in turn given by the amount of contaminant in the tank divided by the volume. So the expression we get is

\[
\text{Rate of exit of contaminant} = \text{Rate of drained solution} \times \frac{\text{Amount of contaminant}}{\text{Volume of tank}}
\]
or

\[
\text{Rate of exit of contaminant} = r_{\text{out}} \frac{P(t)}{V(t)}.
\]

\footnote{Note that the well-mixing assumption is very important for this simplified situation...it’s what allows us to use first order ordinary differential equations rather than the partial differential equations that would be required if the concentration were non-uniform.}
What is $V(t)$? To figure that out, we need to know how the volume is changing. Certainly, the volume is decreasing by $r_0$ each $t$. Is anything being added to the volume? That depends; if we’re adding some solution to the tank at a certain rate $r_{in}$, that will add to the in-tank volume. If we’re directly adding contaminant not in solution, it won’t. So we’ll need to make to make this determination when reading the problem, and in the first case, if our initial volume is given by $V_0$, we’ll get $V(t) = V_0 + t(r_{in} - r_{out})$, and in the second, $V(t) = V_0 - tr_{out}$.

At the moment, this may seem complicated; it’s actually quite straightforward once we’ve got a problem in hand. So let’s consider some examples.

**Example 4.4.** Suppose a 120 gallon well-mixed tank initially contains 90 lb. of salt mixed with 90 gal. of water. Salt water (with a concentration of 2 lb/gal) comes into the tank at a rate of 4 gal/min. The solution flows out of the tank at a rate of 3 gal/min. How much salt is in the tank when it is full?

We can immediately write down the expression for the volume $V(t)$. How much liquid is entering the tank each minute? 4 gallons. How much is leaving during the same minute? 3 gallons. So each minute, the volume increases by 1, and we have $V(t) = 90 + (4 - 3)t = 90 + t$. This tell us that the tank will be full at $t = 30$.

We let $P(t)$ be the amount of salt (in pounds) in the tank at time $t$. Ultimately, we want to determine $P(30)$, since this is when the tank will be full. To write down a differential equation, we need to determine the rates at which salt is leaving and entering the tank each minute. How much salt is entering the tank? We have 4 gallons of salt water entering the tank each minute, and each of those gallons has 2 lb. of salt dissolved in it. Hence we’re adding 8 lbs. of salt to the tank each minute. How much is exiting the tank? We’re draining the solution at a rate of 3 gallons each minute, and we’ve seen that the concentration of each of those gallons is $P(t)/V(t)$.

$$
\frac{dP}{dt} = (4 \text{gal/min})(2 \text{lb/gal}) - (3 \text{gal/min}) \left( \frac{P(t)}{V(t)} \right) \\
= 8 - \frac{3P(t)}{90 + t}.
$$

This is the differential equation for the amount of salt in the tank; what we need now is an initial condition. That’s easy: $P(0) = 90$, as given in the problem. Now we’ve got our initial value problem; let’s solve. Our equation is linear (i.e. of the form $y’ + p(t)y = q(t)$), with

$$
p(t) = \frac{3}{90 + t} \\
q(t) = 8.
$$

Thus we’ll want to use the method of integrating factors.

$$
\mu(t) = e^{\int \frac{3}{90 + t} dt} = e^{\ln(90 + t)} = (90 + t)^3
$$

$$
\frac{d}{dt} \left[(90 + t)^3 P(t)\right] = 8(90 + t)^3
$$

$$(90 + t)^3 P(t) = \int 8(90 + t)^3 dt
$$

$$
= 2(90 + t)^4 + c
$$

$$
P(t) = 2(90 + t) + \frac{c}{(90 + t)^3}
$$

$$
P(0) = 90 = 2(90) + \frac{c}{90^3} \Rightarrow c = -(90)^4
$$

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So we end up with
\[ P(t) = 2(90 + t) - \frac{90^4}{(90 + t)^3}. \]

What did we originally want to know? We wanted to know \( P(30) \), which is the amount of salt in the tank when the tank is full. So we can evaluate that:
\[ P(30) = 240 - \frac{90^4}{120^3} = 240 - 90 \left( \frac{3}{4} \right)^3 = 240 - 90 \left( \frac{27}{64} \right). \]

Notice that we could’ve asked for the concentration at the overflow time, or for that matter, how much salt was in the tank at any particular time before overflow. Our final computation would have been the only difference.

\[ \square \]

**Exercise.** What is the concentration of the tank in Example 4.4 when the tank is full?

For future examples, I won’t be quite so verbose when explaining the various steps. The process is basically the same for every mixing problem you’re going to encounter; the only differences are the tedium involved in solving the equation and what exactly ought to be computed at the final step.

**Example 4.5.** A full 20 liter tank has 20 grams of yellow food coloring dissolved in it. If a yellow food coloring solution (with a concentration of 2 grams/liter) is piped into the tank at a rate of 3 liters/minute while the well mixed solution is drained out of the tank at a rate of 3 liters/minute, what is the limiting concentration of yellow food coloring solution in the tank?

Our differential equation in this case will be
\[
\frac{dP}{dt} = (3\text{l/min})(2\text{g/l}) - (3\text{l/min}) \frac{P(t)g}{V(t)l} = 6 - \frac{3P}{20}
\]

Notice that the volume is constant, since we are draining the tank at the same rate that we’re adding solution.

The integrating factor in this case is
\[ \mu(t) = e^{\int \frac{3}{20} dt} = e^{\frac{3}{20}t} \]

and our initial condition is \( P(0) = 20 \), so we solve:
\[
\frac{d}{dt} \left[ e^{\frac{3}{20}t} P(t) \right] = 6e^{\frac{3}{20}t} \\
e^{\frac{3}{20}t} P(t) = \int 6e^{\frac{3}{20}t} dt \\
= 40e^{\frac{3}{20}t} + c \\
P(t) = 40 + \frac{c}{e^{\frac{3}{20}t}} \\
P(0) = 20 = 40 + c \Rightarrow c = -20
\]

and we conclude
\[ P(t) = 40 - \frac{20}{e^{\frac{3}{20}t}}. \]

Now, we want to know what is going to happen to the concentration in the limit, or as \( t \to \infty \). We know the volume will always be 20 liters.
\[
\lim_{t \to \infty} \frac{P(t)}{V(t)} = \lim_{t \to \infty} \frac{40 - 20e^{-\frac{3}{20}t}}{20} = 2.
\]
So the limiting concentration is 2 g/l. Why does this make physical sense? □

It turns out that the same process will work (but be slightly more complicated to solve) if the concentration of the incoming solution is variable.

**Example 4.6.** A 150 gallon tank has 60 gallons of water with 5 pounds of salt dissolved in it. Water with a concentration of \(2 + \cos(t)\) lbs/gal comes into the tank at a rate of 9 gal/hour. If the well mixed solution leaves the tank at a rate of 6 gal/hour, how much salt is in the tank when it overflows?

The only difference between this and our previous examples is that the incoming concentration is variable. We now write down our differential equation, given that the volume starts at 600 gal and increases at a rate of 3 gal/min.

\[
\frac{dP}{dt} = 9 \left(2 + \cos(t)\right) - \frac{6P}{60 + 3t}
\]

Our initial condition is \(P(0) = 5\) and our integrating factor will be

\[
\mu(t) = e^{\int \frac{6}{60+3t} \, dt} = e^{2 \ln(20 + t)} = (20 + t)^2.
\]

\[
\frac{d}{dt} \left[(20 + t)^2 P(t)\right] = 9 \left(2 + \cos(t)\right) (20 + t)^2
\]

\[
(20 + t)^2 P(t) = \int 9 \left(2 + \cos(t)\right) (20 + t)^2 \, dt
\]

\[
= 9 \left(\frac{2}{3} (20 + t)^3 + (20 + t)^2 \sin(t) + 2(20 + t) \cos(t) - 2 \sin(t)\right) + c
\]

\[
P(t) = 9 \left(\frac{2}{3} (20 + t) + \sin(t) + \frac{2 \cos(t)}{20 + t} - \frac{2 \sin(t)}{(20 + t)^2}\right) + \frac{c}{(20 + t)^2}
\]

Now we apply the initial condition to get the constant \(c\).

\[
P(0) = 5 = 9 \left(\frac{2}{3} (20) + \frac{2}{20}\right) + \frac{c}{400}
\]

\[
= 120 + \frac{9}{10} + \frac{c}{400}
\]

\[
c = -46360.
\]

We want to know how much salt is in the tank when it overflows. This happens when the volume hits 150, or at \(t = 30\).

\[
P(30) = 300 + 9 \sin(30) + \frac{18 \cos(30)}{50} - \frac{18 \sin(30)}{2500} - \frac{46360}{2500}
\]

\[
\approx 272.63 \text{ pounds}
\]

We could make a more complicated problem by assuming, e.g., that there will be a change in the situation if the solution ever reached a critical concentration. The process would still be the same; we would just need to solve two different but linked initial value problems.
Differential Equations

LECTURE 5

More First Order Modeling

1. Falling Bodies

Let’s now consider some object falling to the ground. This body will obey Newton’s Second Law of Motion, which we’ve seen we can write as a first order differential equation for velocity.

\[ m \frac{dv}{dt} = F(t, v) \]

where \( m \) is the object’s mass and \( F \) is the net force acting on the body. We’ll deal with a simplified situation where the only forces in play are gravity and air resistance.

It’s important here to be careful with signs. The convention that we’re going to work with throughout the course is that downward displacements and forces are positive. Hence the force due to gravity is given by \( F_G = mg \), where \( g \approx 10 \text{m/s}^2 \) is the gravitational constant.

Air resistance, on the other hand, acts against velocity. If the object is moving up, the air resistance force will act downwards, and vice versa. In general, we’ll assume that air resistance is linearly dependant on velocity (i.e., \( F_A = -\alpha v \), where \( F_A \) is the force due to air resistance). This isn’t realistic, but it will make the problem simpler. So we end up with \( F(t, v) = F_G + F_A = 10m - \alpha v \), and our differential equation for velocity is

\[ m \frac{dv}{dt} = 10m - \alpha v. \]

**Example 5.1.** A 30 kg object is shot from a cannon straight up with an initial velocity of 10 m/s off the very tip of a bridge. If the air resistance is given by \( 6v \), determine the velocity of the mass at any time \( t \) and compute the rock’s terminal velocity.

Strictly speaking, there are two phases to this problem: the one where the object is moving upwards, and the one where it’s moving downwards. If we look at the forces, though, it turns out we get the same differential equation

\[ 30v' = 300 - 6v. \]

The initial condition is \( v(0) = -10 \), since we shot the object upwards. Our differential equation is linear; in the standard form it’s written as

\[ v' + \frac{1}{5}v = 10. \]

So we’ll use integrating factors. Our integrating factor is \( \mu(t) = e^{\frac{t}{5}} \).

\[ \frac{d}{dt} \left( e^{\frac{t}{5}} v(t) \right) = 10 e^{\frac{t}{5}} \]

\[ e^{\frac{t}{5}} v(t) = \int 10 e^{\frac{t}{5}} dt \]

\[ = 50e^{\frac{t}{5}} + c \]

\[ v(t) = 50 + \frac{c}{e^{\frac{t}{5}}} \]

\[ v(0) = -10 = 50 + c \Rightarrow c = -60 \]
So the velocity at any time $t$ is given by

$$v(t) = 50 - \frac{60}{e^{5t}}.$$ 

What is the terminal velocity of the rock? The terminal velocity is given by the limit of the velocity as $t \to \infty$, which is 50.

Notice that if we had been given the height of the bridge, we also could have computed the velocity of the rock when it hit the ground or the time when the rock hit the ground, since then we could have integrated velocity to get position.

\[\square\]

**Example 5.2.** A 75 kg skydiver jumps out of a plane with no initial velocity. Assuming the magnitude of air resistance is given by $0.8|v|$, what is the appropriate initial value problem modeling his velocity?

Air resistance is an upward force here (since the velocity is moving downwards), while gravity is acting downwards (towards the earth). So our force function should be

$$F(t, v) = mg - .8v.$$ 

Thus our initial value problem is

$$75v' = 75g - .8v \quad v(0) = 0.$$ 

\[\square\]
Exact Equations

The final category of first order differential equations we will consider are the so-called exact equations. These nonlinear equations have the form

\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0 \] (6.1)

where \( y = y(x) \) is a function of \( x \) and

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \]

where these two derivatives are partial derivatives, discussed more below.

We’ll begin this lecture by briefly discussing how to compute the partial derivatives of a multivariable function. Then we’ll see a motivating example of how to solve an exact equation, which will in turn lead to a discussion of why what we did in the example worked. Finally, we’ll see some examples done with the solution method in mind.

1. Multivariable Differentiation

Suppose we have a function \( f(x, y) \) depending on two variables. What does it mean to differentiate this function?

Recall that if we have a function of a single variable \( g(x) \), we can interpret the derivative as the rate of change of the output of \( g(x) \) as \( x \) increases. We’d like to have a similar notion of differentiation for our multivariable function \( f(x, y) \).

This situation is, of course, more complicated. We no longer just have to worry about what happens as \( x \) increases; we have to worry about how to measure the rate of change of \( f(x, y) \) as \( y \) increases or even as \( x \) and \( y \) vary in different ways. Geometrically, we can think of the graph of \( f(x, y) \) as specifying some surface in three-dimensional space, so talking about “the slope” at a point is vague in this way.

So what do we do? It turns out (and this is a topic that is explored more in a multivariable calculus class) that if we can account for how \( f(x, y) \) varies as only \( x \) or only \( y \) vary, we don’t have to worry about mixtures of the two. Let’s consider how we want to express the derivative of \( f(x, y) \) at a point \( (x_0, y_0) \). Without loss of any generality, we can consider what happens only as \( x \) varies (since the \( y \) situation will be analogous, and we’ve already punted on considering mixing the two to a class that is more suited to discussing that). Fixing \( y = y_0 \) reduces our function of two variables \( f(x, y) \) to a function of a single variable \( g(x) = f(x, y_0) \). So we define the partial derivative of \( f \) with respect to \( x \) at \( (x_0, y_0) \), denoted by

\[ \frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0), \]

to be \( g'(x_0) \) with \( g(x) = f(x, y_0) \). Geometrically, this is the slope of the curve on the graph of \( f \) corresponding to the line \( y = y_0 \) at the point \( x = x_0 \).

What does this mean, computationally? Our function \( g \) is defined by treating \( y \) as the constant \( y_0 \). So its derivative will be equivalent to the derivative of \( f \) taken with respect to \( x \) while treating \( y \) as a constant. The following examples should clarify the above discussion.
Example 6.1. Let \( f(x, y) = x^2 y + y^2 \). Then
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2xy \\
\frac{\partial f}{\partial y} &= x^2 + 2y.
\end{align*}
\]

Example 6.2. Let \( f(x, y) = y \sin(x) \).
\[
\begin{align*}
\frac{\partial f}{\partial x} &= y \cos(x) \\
\frac{\partial f}{\partial y} &= \sin(x).
\end{align*}
\]

We will also need to be able to recognize the multivariable chain rule. The relevant version of this says that if we have a function \( \Phi(x, y(x)) \) depending on some variable \( x \) and a function \( y \) depending on \( x \), then
\[
\frac{d\Phi}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{dy}{dx} = \Phi_x + \Phi_y y'.
\]

2. Exact Equations

Before we talk in more detail about how to solve an exact equation, let’s work one example to get a feel for what an exact equation is and why the solution method works. We’ll rework this later after a more general discussion to reveal all the details.

Example 6.3. Consider
\[
2xy - 6x^2 + (y + x^2 + 3) \frac{dy}{dx} = 0.
\]

The first step in solving an exact equation is to find a certain function \( \Phi(x, y) \). We’ll see how to find this function later (and in fact this computation is where most of the work lies), but for this example it will turn out that the desired function is
\[
\Phi(x, y) = \frac{1}{2} y^2 + (x^2 + 3) y - 2x^3.
\]

Notice that if we compute the partial derivatives of \( \Phi \), we obtain
\[
\begin{align*}
\Phi_x(x, y) &= 2xy - 6x^2 \\
\Phi_y(x, y) &= y + x^2 + 3.
\end{align*}
\]

Looking back at the differential equation, we can see that it can be rewritten as
\[
\Phi_x + \Phi_y \frac{dy}{dx} = 0.
\]

By Equation 6.2, we see that we can rewrite this as
\[
\frac{d\Phi}{dx} = 0.
\]

This tells us that our function \( \Phi \) must be equal to some constant, since its total derivative (the ordinary derivative obtained by replacing the all instances of the dependant variable \( y \) with the appropriate expression involving the independent variable \( x \)) is zero, and thus the general solution is
\[
\frac{1}{2} y^2 + (x^2 + 3) y - 2x^3 = c
\]
for some constant $c$, which is just the constant of integration. If we had an initial condition, at this point we could solve for $c$ and get the particular solution to the initial value problem. □

Let’s think about what we saw in the previous example. An exact equation has the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

with $M_y(x, y) = N_x(x, y)$. The key to finding the solution to an exact equation is to construct a function $\Phi(x, y)$ such that the differential equation turns into

$$\frac{d\Phi}{dx} = 0$$

by using the multivariable chain rule (6.2), as we did above. Thus we require that $\Phi$ satisfy

$$\Phi_x(x, y) = M(x, y)$$
$$\Phi_y(x, y) = N(x, y).$$

Remark. One of the standard facts of multivariable calculus is that mixed partial derivatives commute. This is why we want $M_y = N_x$: $M_y = \Phi_{xy}$ and $N_x = \Phi_{yx}$, and these should be identical if such a function $\Phi$ can exist. It’s imperative to check that a function is actually exact (by computing $M_x$ and $N_y$ and seeing that they coincide) before proceeding with the solution process, or there’s no way it can work.

Once we have this function $\Phi$, then we know $\frac{d\Phi}{dx} = 0$, and hence

$$\Phi(x, y) = c,$$

yielding an implicit general solution to the differential equation. The name “exact” equation refers to the fact that this class of equations can be expressed in terms of $\frac{d\Phi}{dx} = 0$ for some function $\Phi$.\(^1\)

So, we can see that the real work involves computing $\Phi(x, y)$. How can we do it? Let’s revisit Example 6.3, this time filling in all the details. We’ll also add an initial condition so that we can work a full initial value problem.

**Example 6.4.** Solve the initial value problem

$$2xy - 6x^2 + (y + x^2 + 3) \frac{dy}{dx} = 0 \quad y(0) = 2$$

Let’s begin, as we always should, by checking that this differential equation is actually exact. Comparing the equation to Equation 6.1, we have

$$M(x, y) = 2xy - 6x^2$$

and

$$N(x, y) = y + x^2 + 3.$$  

Then $M_y = 2x$ and $N_x = 2x$, hence the equation is exact.

\(^1\)There is a deep connection between this notion of exact equations, the method of integrating factors that we learned about in Lecture 3, and thermodynamics. The general concept of an integrating factor can be extended to inexact differential equations: find a function $\mu(x, y)$ that, when multiplied through an inexact equation, makes it exact (our earlier integrating factor method is a special case of this procedure), and then solve the exact equation. While we won’t discuss this general approach in this course, the development of modern thermodynamics can be mathematically viewed as a search for integrating factors and the solution to the resulting equation. For example, Clausius determined that the integrating factor for reversible heat is the reciprocal of temperature and the resulting solution was called entropy; indeed, this is how entropy was defined until advances in quantum mechanics gave it a more intuitive definition.
Now, how do we find $\Phi(x, y)$? We have that $\Phi_x = M$ and $\Phi_y = N$. Thus we could compute $\Phi$ in one of two ways:

$$\Phi(x, y) = \int M \, dx \quad \text{or} \quad \Phi(x, y) = \int N \, dy.$$ 

Note that it doesn’t usually matter which of $M$ or $N$ you choose to integrate to get $\Phi$. There are some examples in which one is clearly easier, but it’s a judgement call: integrate whichever seems easier to you. In some cases, you don’t have a choice.

In this case, they’re equally easy, so let’s integrate $M$.

$$\Phi(x, y) = \int 2xy - 6x^2 \, dx = x^2y - 2x^3 + h(y).$$

Notice the presence of the function $h(y)$ in the integral. This is the equivalent of the constant of integration that we obtain when we integrate a single variable function, since if we differentiate $\Phi$ with respect to $x$, any function that just depends on $y$ will vanish. So instead of tacking on a constant to the end, we need to have a function of $y$.

If we had integrated $N$ with respect to $y$ to get $\Phi$, we would have needed an equivalent function of $x$, something like $h(x)$, to play the role of the “constant of integration.” It’s very important that this not be forgotten.

Now all we need to do is to find $h(y)$ and we’ll have $\Phi$. How do we do this? We know that if we differentiate $\Phi$ with respect to $x$, $h$ will vanish, which is utterly unhelpful. However, if we differentiate $\Phi$ with respect to $y$, we’re in good shape, since $h'$ will appear and we know that $\Phi_y = N$. So we’ll compute $\Phi_y$, and any terms in $N$ that aren’t in $\Phi_y$ must be $h'(y)$. Analogously, if we had started by integrating $N$ with respect to $y$ to find $\Phi$, we would want to differentiate $\Phi$ with respect to $x$ and compare with $M$.

Ok, so we can see that $\Phi_y = x^2 + h'(y)$ and $N = x^2 + y + 3$. So since these are equal, we must have $h'(y) = y + 3$, and so

$$h(y) = \int h'(y) \, dy = \frac{1}{2}y^2 + 3y.$$ 

Remark. We’re going to be a little careless with constants. In general, we’ll drop the constant of integration from our computation of $h$, since it will end up combining with the constant $c$ that we get as part of the solution process. As always, if you’re uncomfortable with this or find that you forget all constants of integration, you can always write them down at every step.

Thus, we have

$$\Phi(x, y) = x^2y - 2x^3 + \frac{1}{2}y^2 + 3y = \frac{1}{2}y^2 + (x^2 + 3)y - 2x^3,$$

which is precisely the $\Phi$ that we used in Example 6.3. From here, the problem is identical: we observe that the differential equation is just

$$\frac{d\Phi}{dx} = 0$$

and thus $\Phi(x, y) = y^2 + (x^2 + 3)y - 2x^3 = c$ for some constant $c$. To compute $c$, we’ll use our initial condition $y(0) = 2$.

$$2^2 + 6 = c \Rightarrow c = 10$$

and so we have a particular solution of

$$y^2 + (x^2 + 3)y - 2x^3 = 10.$$
This is a quadratic equation in $y$, so as we’ve seen we can complete the square or use the quadratic formula to get an explicit solution, which we want to do when we can.

\[
y^2 + (x^2 + 3)y - 2x^3 = 10
\]

\[
y^2 + (x^2 + 3)y + \frac{(x^2 + 3)^2}{4} = 10 + 2x^3 + \frac{(x^2 + 3)^2}{4}
\]

\[
\left(y + \frac{x^2 + 3}{2}\right)^2 = \frac{x^4 + 8x^3 + 6x^2 + 49}{4}
\]

\[
y(x) = \frac{-(x^2 + 3) \pm \sqrt{x^4 + 8x^3 + 6x^2 + 49}}{2}
\]

Now we use the initial condition to figure which whether we want the $+$ or the $-$ in that $\pm$. Since $y(0) = 2$, we have

\[
2 = y(0) = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2} = 2, -5.
\]

Thus we see that we’ll want the $+$ solution, and our particular solution is

\[
y(x) = \frac{-(x^2 + 3) + \sqrt{x^4 + 8x^3 + 6x^2 + 49}}{2}
\]

\[
\square
\]

**Example 6.5.** Solve the initial value problem

\[
2xy^2 + 2 = 2(3 - x^2y)y' \quad y(2) = 1.
\]

First, we need to put the equation into the form of Equation 6.1:

\[
2xy^2 + 2 - 2(3 - x^2y)y' = 0.
\]

Now, we have $M(x, y) = 2xy^2 + 2$ and $N(x, y) = -2(3 - x^2y)$ (incorporating the minus into $N$ is very important: otherwise, the derivatives won’t work out). Thus $M_y = 4xy = N_x$ and the equation is exact.

The next step is to compute $\Phi(x, y)$. In this example, it’s equally easy to integrate either $M$ or $N$, so let’s use $N$:

\[
\Phi(x, y) = \int N \, dy = \int 2x^2y - 6 \, dy = x^2y^2 - 6y + h(x).
\]

To find $h(x)$, we compute $\Phi_x = 2xy^2 + h'(x)$ and notice that for this to be equal to $M$, $h'(x) = 2$. Hence $h(x) = 2x$ and we have an implicit solution of

\[
x^2y^2 - 6y + 2x = c.
\]

Now, we use the initial condition $y(2) = 1$:

\[
4 - 6 + 4 = c \Rightarrow c = 2.
\]

So our implicit solution is

\[
x^2y^2 - 6y + 2x - 2 = 0.
\]

Again, we can solve for the explicit solution by completing the square or using the quadratic formula:

\[
y(x) = \frac{6 \pm \sqrt{36 - 4x^2(2x - 2)}}{2x^2} = \frac{3 \pm \sqrt{9 - 2x^3 + 2x^2}}{x^2}
\]
and using the initial condition, we see we want the “-” solution, so the explicit particular solution is
\[ y(x) = 3 - \sqrt{9 - 2x^3 + 2x^2} \]

\[ \Box \]

**Exercise.** Calculate the interval of validity for the particular solution obtained in Example 6.5. You will need to find a numerical solution to \( 9 - 2x^3 + 2x^2 = 0 \).

**Example 6.6.** Solve the IVP
\[ \frac{2ty}{t^2 + 1} - 2t - (4 - \ln(t^2 + 1)) y' = 0 \quad y(2) = 0 \]
and find the solution’s interval of validity.

This is already in the form of Equation 6.1, so let’s start by checking if it’s exact. \( M(t, y) = \frac{2ty}{t^2 + 1} - 2t \) and \( N(t, y) = \ln(t^2 + 1) - 4 \), so \( M_y = \frac{2t}{t^2 + 1} = N_t \). Thus the equation is exact. Now, let’s compute \( \Phi \). In this case, it will be easiest to integrate \( M \):
\[ \Phi = \int M \, dt = \int \frac{2ty}{t^2 + 1} - 2t \, dt = y \ln(t^2 + 1) - t^2 + h(y). \]
\[ \Phi_y = \ln(t^2 + 1) + h'(y) = \ln(t^2 + 1) - 4 = N \]
so we conclude \( h'(y) = -y \) and thus \( h(y) = -4y \). So our implicit solution is then
\[ y \ln(t^2 + 1) - t^2 - 4y = c \]
and we use our initial condition to compute \( c = -4 \). Thus the particular solution is
\[ y \ln(t^2 + 1) - t^2 - 4y = -4, \]
and this is very easy to solve explicitly. Doing so, we obtain
\[ y(x) = \frac{t^2 - 4}{\ln(t^2 + 1) - 4}. \]

Now, let’s find the interval of validity. We don’t have to worry about the logarithm since \( t^2 + 1 > 0 \) for all \( t \). Thus we only need to avoid division by zero, so we need to avoid the following points.
\[ \ln(t^2 + 1) - 4 = 0 \]
\[ \ln(t^2 + 1) = 4 \]
\[ t^2 = e^4 - 1 \]
\[ t = \pm \sqrt{e^4 - 1} \]

So there are three possible intervals of validity:
\[ (-\infty, -\sqrt{e^4 - 1}), (-\sqrt{e^4 - 1}, \sqrt{e^4 - 1}), \text{ and } (\sqrt{e^4 - 1}, \infty). \]
The middle one contains \( t = 2 \), so our interval of validity is \( (-\sqrt{e^4 - 1}, \sqrt{e^4 - 1}). \)
\[ \Box \]
Example 6.7. Solve

\[3y^3 e^{3xy} - 1 + (2ye^{3xy} + 3xy^2 e^{3xy}) y' = 0 \quad y(1) = 2.\]

We have \(M(x, y) = 3y^3 e^{3xy} - 1\) and \(N(x, y) = 2ye^{3xy} + 3xy^2 e^{3xy}\), so

\[M_y = 9y^2 e^{3xy} + 9xy^3 e^{3xy} = N_x.\]

Thus the equation is exact. We'll integrate \(M\), since it's a bit easier.

\[\Phi = \int M \, dx = \int 3y^3 e^{3xy} - 1 = y^2 e^{3xy} - x + h(y)\]

and

\[\Phi_y = 2ye^{3xy} + 3xy^2 e^{3xy} + h'(y).\]

Comparing \(\Phi_y\) to \(N\), we see that they are already identical, so we must have \(h'(y) = 0\) and hence \(h(y) = 0\) (since we're ignoring constants in \(h\)). So we have

\[y^2 e^{3xy} - x = c,\]

and using the initial condition gives \(c = 4e^6 - 1\). Thus our implicit particular solution is

\[y^2 e^{3xy} - x = 4e^6 - 1,\]

and we're done because we won’t be able to solve this explicitly. \(\square\)
LECTURE 7

Autonomous Equations and Population Dynamics

1. Autonomous Equations

First order differential equations relate the slope of a function to the values of the function and the independent variable. This means that, writing the differential equation as $y' = f(y, t)$, we can visualize solutions by plotting bits of slope at the appropriate $y - t$ coordinate points and trying to connect them.

This is, in general, (especially the final step – connecting the slope bits in a reasonably accurate way) a massive pain. The curves that correspond to certain slopes can be complicated, which can make it hard to look at a diagram and know which initial values correspond to which outcomes. However, for a certain class of equations, called autonomous equations, this process is greatly simplified. Autonomous equations don’t depend on $t$, i.e.,

$$y' = f(y),$$

which is nice because it means we don’t have to worry about being too precise with regard to the $t$ coordinate.

**Remark.** Notice that any autonomous equation is separable.

What we need to know to study these qualitatively is which values of $y$ make $y'$ zero, positive, or negative. The first group are called the equilibrium solutions of the differential equation. They are constant solutions, and we indicate them on the $y - t$ graph by horizontal lines.

After that, we can study the positivity of $f(y)$ on the intermediate intervals, which will tell us whether the equilibrium solutions attract nearby initial conditions (in which case they are called asymptotically stable), repel them (unstable), or some combination thereof (semi-stable).

**Example 7.1.** Consider $y' = y^2 - y - 2$. We’ll start by finding the equilibrium solutions, which, in this case, are the roots of $y^2 - y - 2 = (y - 2)(y + 1)$. So the equilibrium solutions are $y = -1$ and $y = 2$. These are constant solutions, indicated on the solution space by horizontal lines. We want to understand their stability. If we plot $y^2 - y - 2$ versus $y$ (see Figure 7.1), we can see that on the interval $(-\infty, -1)$, $f(y) > 0$ while on the interval $(-1, 2)$, $f(y) < 0$. So if we have a solution with initial condition $y(t_0) = y_0 < -1$, $y' = f(y) > 0$ and $y(t)$ will increase towards $-1$. If $-1 < y_0 < 2$, $y' < 0$, so the solution will decrease towards $-1$. Since solutions starting below $-1$ go to $-1$ and so do solutions above it, we conclude $y(t) = -1$ is an asymptotically stable equilibrium. Now, we can also see that for $y_0 > 2$, $f(y) > 0$, so we see that if we start with an initial condition near but not equal to 2, the solution will move away from 2. So $y(t) = 2$ is an unstable equilibrium.

**Example 7.2.** Now let’s take $y' = (y - 2)(y + 1)^2$. The equilibrium solutions are $y = -1$ and $y = 2$. To classify them, we graph $f(y) = (y - 2)(y + 1)^2$ (see Figure 7.3). If $y < -1$, we can see that $f(y) < 0$, so solutions starting below $-1$ will tend towards $-\infty$. If $-1 < y < 2$, $f(y) < 0$ again, so solutions starting in the relevant region will tend towards $-1$. So we have that $y = -1$ is a semi-stable solution. Finally, since for $y > 2$, $f(y) > 0$, solutions starting above 2 will asymptotically increase to $\infty$, and we have that $y = 2$ is unstable since no nearby solutions converge to it.
Figure 7.1. Plot of $y^2 - y - 2$ for Example 7.1. The zeros of this quadratic (at $y = -1$ and $y = 2$) are the equilibrium solutions to $y' = y^2 - y - 2$. Solutions with initial conditions corresponding to the region $y^2 - y - 2 < 0$ are decreasing, while those with initial conditions corresponding to the region $y^2 - y - 2 > 0$ are increasing.

Figure 7.2. Phase portrait for $y' = (y + 1)(y - 2)$ (Example 7.1) along with representative solutions. Compare the regions where solutions increase and decrease with the graph in Figure 7.1. Notice the behavior of the stable and unstable solutions.
Figure 7.3. Plot of \((y - 2)(y + 1)^2\) for Example 7.2. The zeros of this cubic (at \(y = -1\) and \(y = 4\)) are the equilibrium solutions to \(y' = (y - 4)(y + 1)^2\). Solutions with initial conditions corresponding to the region \((y - 4)(y + 1)^2 < 0\) are decreasing, while those with initial conditions corresponding to the region \((y - 4)(y + 1)^2 > 0\) are increasing.

Figure 7.4. Phase portrait for \(y' = (y + 1)^2(y - 2)\) (Example 7.2) along with representative solutions. Compare the regions where solutions increase and decrease with the graph in Figure 7.3. Notice the behavior of the semi-stable and unstable solutions.
2. Populations

Some good examples of autonomous equations come from population dynamics. The most naive population model is the so-called “population bomb” model:

\[ P'(t) = rP(t) \]

with \( r > 0 \). This differential equation is solved by \( P(t) = P_0 e^{rt} \), which indicates that the population would increase exponentially to infinity, since there’s always a net growth proportional to the population. This is not very realistic at all.

A better model is the so-called “logistic equation,” given by

\[ P'(t) = rP \left(1 - \frac{P}{N}\right) \]

where \( N > 0 \) is some constant (we’ll see what it means later). Looking at this model, we see that while we still have a birth rate of \( rP \), we also have a mortality rate proportional to \( P^2 \).

First, let’s solve this equation:

\[
\int \left( \frac{1}{P} + \frac{1}{P - N} \right) dP = \int r \, dt \\
\ln |P| - \ln \left| 1 - \frac{P}{N} \right| = rt + c \\
\frac{P}{1 - \frac{P}{N}} = Ae^{rt} \\
P = Ae^{rt} - \frac{1}{N} Ae^{rt} P \\
P(t) = \frac{Ae^{rt}}{1 + \frac{AN}{N} e^{rt}} \\
P(t) = \frac{AN}{Ne^{-rt} + A}
\]

and if \( P(0) = P_0 \), we solve and get \( A = \frac{P_0 N}{N-P_0} \) to yield

\[ P(t) = \frac{P_0 N}{(N-P_0) e^{-rt} + P_0}. \]

Great...but that solution doesn’t tell us a ton just by glancing at it. Let’s apply the methods from earlier this lecture to see what we can get.

Looking at the logistic equation, we can see that our equilibrium solutions are \( P = 0 \) and \( P = N \). Graphing \( f(P) = rP \left(1 - \frac{P}{N}\right) \), we see that for \( P < 0 \), \( f(P) < 0 \), for \( 0 < P < N \), \( f(P) > 0 \), and for \( P > N \), \( f(P) < 0 \). Thus 0 is unstable while \( N \) is asymptotically stable, and we can conclude that for any initial \( P_0 > 0 \),

\[ \lim_{t \to \infty} P(t) = N. \]

So what is \( N \)? It’s the carrying capacity of the environment. If the population exists, it will grow towards \( N \), but the closer it gets to \( N \) the slower the population will grow. If the population somehow starts off larger than the environment can support, it will die off until it reaches that
critical position. And if the population starts off at $N$, the births and deaths will balance out perfectly.

It’s also possible to construct similar models that have unstable equilibria above 0.

**Exercise.** Show that the equilibrium $P(t) = N$ is unstable for the autonomous equation $P'(t) = rP \left( \frac{P}{N} - 1 \right)$.
LECTURE 8

Existence and Uniqueness

At the beginning of the course, we made a point of noting three questions we wanted to keep in mind:

- Given an initial value problem, does a solution exist?
- If a solution exists, is it unique?
- If a solution exists, how do we find it?

So far, we’ve been mostly concerned with the last question. Today, we’ll discuss the first two: without solving an initial value problem, what information can we derive about existence and uniqueness of solutions? We’ll also see some strong differences between linear and nonlinear equations.

1. Linear Equations

While we’ll specifically be dealing with first order linear equations in this section, the same basic method works for higher order linear equations as well.

**Theorem 8.1 (Linear Fundamental Theorem of Existence and Uniqueness).** Consider the IVP

\[ y' + p(t)y = q(t), \quad y(t_0) = y_0. \]

If \( p(t) \) and \( q(t) \) are continuous functions on an open interval \( \alpha < t_0 < \beta \), then there exists a unique solution to the IVP defined on the interval \((\alpha, \beta)\).

**Remark.** The same result holds for general linear initial value problems: if we have the IVP

\[ y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t), \quad y(t_0) = y_0, y'(t_0) = y'_0, \ldots, y^{(n-1)}(t_0) = y^{(n-1)}_0 \]

then if \( a_i(t) \) (for \( i = 0, \ldots, n-1 \)) and \( g(t) \) are continuous on an open interval \( \alpha < t_0 < \beta \), there exists a unique solution to the IVP defined on the interval \((\alpha, \beta)\).

What exactly does Theorem 8.1 tell us? Two things:

(1) If the given linear differential equation is sufficiently “nice,” not only do we know that a solution exists, but we know it will be unique. In a lot of cases, knowing that a unique solution exists is more important than actually being able to solve the equation.

(2) If the interval \((\alpha, \beta)\) is the largest interval on which \( p(t) \) and \( q(t) \) are continuous, then \((\alpha, \beta)\) is the interval of validity to the unique solution guaranteed by the theorem. Thus, given an initial value problem involving a sufficiently nice linear differential equation, there is no need to solve the differential equation to get the interval of validity. The interval of validity also depends on \( t_0 \) (since the interval must contain it) but not at all on \( y_0 \), so all solutions corresponding to initial conditions of the form \( y(t_0) = y_0 \) will have the same interval of validity for every \( y_0 \).

**Example 8.1.** Without solving, determine the interval of validity for the solution to the following initial value problem.

\[ (t^2 - 4)y' + 4y = \ln|20 - 5t|, \quad y(3) = -3 \]
If we look at Theorem 8.1, we may notice that to use it, we’ll need to write the differential equation in the form given in the theorem (i.e., with the coefficient of $y'$ being 1). This gives us

$$y' + \frac{4}{t^2 - 4} = \frac{\ln|20 - 5t|}{t^2 - 4}.$$

Next, we want to find the points where either of the two coefficient functions are discontinuous. By throwing out those points, we can find all of the possible intervals of validity for solutions to the differential equation. Then we can see which one contains $t_0$.

Using the notation from Theorem 8.1, we have that $p(t)$ is discontinuous at $t = \pm 2$, since at those values we’re dividing by zero. $q(t)$ is discontinuous at $t = \pm 2$ for the same reason while also being discontinuous at $t = 4$, where we would be taking the logarithm of zero (notice we don’t need to worry about taking logarithms of negative numbers because of the absolute value signs).

This yields four intervals on which both $p(t)$ and $q(t)$ are continuous:

$$(-\infty, -2), (-2, 2), (2, 4), (4, \infty).$$

Notice that the endpoints of each of these intervals are points where either $p(t)$ or $q(t)$ are discontinuous. Hence we can guarantee that both will be nice on each interval.

Now we need to identify which of these intervals of the interval of validity for the solution to our particular initial value problem. The interval must contain $t_0 = 3$. Hence the interval of validity will be $(2, 4)$. \hfill \Box

**Remark.** The other intervals we found in the previous example could be intervals of validity for the same differential equation with a different initial condition. For example, if our initial condition had been $y(0) = -1$, we would have concluded that the unique solution to the initial value problem has an interval of validity of $(-2, 2)$.

What happens if the initial condition is at one of these bad values where either $p(t)$ or $q(t)$ are discontinuous? The answer is that we can’t conclude anything. If the hypotheses of the theorem aren’t met, then it’s not true that the conclusions of the theorem are false. We just can’t conclude anything. It may be that the solution doesn’t exist or that infinitely many solutions exist. We just don’t know and more problem-specific analysis would be required.

Let’s do one more example.

**Example 8.2.** Without solving, find the interval of validity to the following initial value problem.

$$\cos(x)\frac{dy}{dx} = \sin(x)y - \sqrt{x - 1}, \quad y\left(\frac{3}{2}\right) = 0$$

First, we need to put the equation into the proper form:

$$y' - \tan(x)y = -\frac{\sqrt{x - 1}}{\tan(x)}.$$

Using the notation from Theorem 8.1, we have that $p(t)$ is discontinuous at $x = \frac{n\pi}{2}$ for odd integers $n$ and $q(t)$ is discontinuous there and for any $x < 1$. Thus we can list the possible intervals of validity.

$$\left(1, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots, \left(\frac{(2n + 1)\pi}{2}, \frac{(2n + 3)\pi}{2}\right)$$

for all positive integers $n$. Since our interval of validity is $y\left(\frac{3}{2}\right) = 0$, we can conclude that the interval of validity is $(1, \frac{\pi}{2})$. \hfill \Box
2. Nonlinear Equations

Things get more tricky with nonlinear equations. In the linear case, every “nice enough” equation had a unique solution except for a few bad initial conditions. But even the innocent looking nonlinear equation
\[
\left( \frac{dy}{dx} \right)^2 + x^2 + 1 = 0
\]
has no real solutions.

Now, let’s look at the more general version of the theorem, which applies to nonlinear equations. It has a much weaker conclusion than the previous one, and we won’t be spending too much time on it, but it’s important to understand.

**Theorem 8.2.** Consider the IVP
\[
y' = f(t, y) \quad y(t_0) = y_0.
\]
If \( f \) and \( \frac{\partial f}{\partial y} \) are continuous functions on some rectangle \( \alpha < t < \beta, \gamma < y_0 < \delta \) containing the point \((t_0, y_0)\), then there is a unique solution to the initial value problem defined on some interval \((a,b)\) satisfying \( \alpha \leq a < t_0 < b \leq \beta \).

There are a few things to notice here.

1. Unlike Theorem 8.1, Theorem 8.2 doesn’t tell us the interval of validity for the guaranteed unique solution. Instead, it tells us the largest possible interval that the solution could exist on; we’d need to actually solve the IVP to get the interval of validity.
2. For nonlinear differential equations, the value of \( y_0 \) may affect the interval of validity, as we will see in a later example. The general strategy is to make sure that our initial condition doesn’t lie in or on the boundary of a “bad” region (where either \( f \) or its derivative are discontinuous) and then find the largest \( t \)-interval on the line \( y = y_0 \) containing \( t_0 \) where everything is nice and continuous.

**Example 8.3.** Determine the largest possible interval of validity for the initial value problem
\[
y' = x \ln(y) \quad y(2) = e.
\]
We have \( f(x, y) = x \ln(y) \), so \( \frac{\partial f}{\partial y} = \frac{x}{y} \). As far as \( y \) is concerned, the areas to be avoided are \( y \leq 0 \). Since our initial condition is \( y = e > 0 \), this is no problem. Now, there are no discontinuities involving \( x \) for either, so we will have a unique solution that exists somewhere inside \((−\infty, \infty)\). □

**Example 8.4.** Determine the largest possible interval of validity for the initial value problem
\[
y' = \sqrt{y - t^2} \quad y(0) = 1.
\]
\( f(t, y) = \sqrt{y - t^2} \) and \( \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-t^2}} \). The region of discontinuities is given by \( y \leq t^2 \). Our initial condition \( y(0) = 1 \) doesn’t lie in this region, so we can continue. The line \( y = 1 \) contains discontinuities at \(-1 < t < 1\), so our conclusion is that the interval of validity of the guaranteed unique solution is contained somewhere within \((-1,1)\). □

What can happen when the conditions of Theorem 8.2 aren’t met?

**Example 8.5.** Determine all possible solutions to the IVP
\[
y' = y^{\frac{1}{3}} \quad y(0) = 0.
\]
Let’s first note that this does not satisfy the conditions of the theorem, since \( \frac{\partial f}{\partial y} = \frac{1}{3y^\frac{2}{3}} \) is not continuous at the origin. Ok, so let’s solve this. It’s separable, and we first note that we have an equilibrium solution \( y(0) = 0 \). This is one that satisfies the initial conditions, but let’s continue on.
Using the initial condition gives $c = 0$, so we get that the solution is

$$y(t) = \pm \left( \frac{2}{3} t \right)^{\frac{3}{2}}.
$$

Notice that using our initial condition doesn’t allow us to choose between these possibilities, since both are satisfied. Thus we end up with three possible solutions to the initial value problem.

This is a good reminder that things don’t always work nicely in the world of differential equations. In this class, most of the problems we’ll work will be nice and have unique solutions, but this isn’t always the case when one actually works with differential equations. Now, let’s look at a case that illustrates the potential dependance on $y_0$ of the interval of validity to a solution in the nonlinear case.

**Example 8.6.** Determine the interval of validity for the initial value problem

$$y' = y^2 \quad y(0) = y_0.$$

First, we notice that this is a nonlinear equation. It’s also not hard to see that it meets the conditions of Theorem 8.2 for every value of $y_0$, and so regardless of the choice of $y_0$ there will be a unique solution. The theorem tells us that the solution will be defined somewhere inside $(-\infty, \infty)$, which is very helpful indeed.

So let’s solve. First, note that if $y_0 = 0$, the solution is the equilibrium solution $y(0) = 0$. Now, let’s assume $y_0 \neq 0$.

$$\int y^{-2} \, dy = \int dt$$

$$-\frac{1}{y} = t + c$$

Applying the initial condition gives

$$c = -\frac{1}{y_0}.$$

$$-\frac{1}{y} = t - \frac{1}{y_0}$$

$$y(t) = \frac{y_0}{1 - y_0 t}.$$

What is the interval of validity here? The only point of discontinuity is $t = \frac{1}{y_0}$. So the two possible intervals of validity are

$$\left( -\infty, \frac{1}{y_0} \right) \quad \left( \frac{1}{y_0}, \infty \right).$$

The correct choice will be the interval that contains $t_0 = 0$. But this will depend on $y_0$. If $y_0 > 0$, 0 will be contained in the interval $\left( -\infty, \frac{1}{y_0} \right)$ and so this is the interval of validity. On the other hand, if $y_0 < 0$, 0 is contained inside $\left( \frac{1}{y_0}, \infty \right)$ and so this is the interval of validity. Thus we have the following possible intervals of validity, depending on what $y_0$ is.
If \( y_0 > 0 \) \((-\infty, \frac{1}{y_0})\) is the interval of validity.

If \( y_0 = 0 \) \((-\infty, \infty)\) is the interval of validity.

If \( y_0 < 0 \) \((-\frac{1}{y_0}, \infty)\) is the interval of validity.

\[\square\]

3. Summary

What did we learn from this section? We saw the conditions for existence of a unique solution to an initial value problem involving a first order equation (and for higher order linear equations, in fact). Intervals of validity for linear equations don’t depend on the initial choice of \( y_0 \), while they may for nonlinear equations.

Second, we can find intervals of validity for solutions for linear equations without having to solve the equation, which is very useful. For a nonlinear equation, on the other hand, we would actually need to solve the equation. All we can potential find are places where the unique solution will definitely not be defined.
Part 3

Second Order Equations


LECTURE 9

Second Order Linear Differential Equations

1. Basic Concepts

We’ve seen one specific example of a second order linear differential equation before: Newton’s Second Law, which can be written as a second order equation for position \( s(t) \) as

\[
\frac{d^2s}{dt^2} = mF(t, s', s).
\]

One of the goals of this section of the course will be to derive differential equations for spring-mass systems that build on this differential equation. Before we can do that, however, we’ll need to develop the necessary mathematical tools.

One of the most basic second order differential equations is

\[
y'' = -y.
\]

By inspection, we might notice that this has two obvious nonzero solutions: \( y_1(t) = \cos(t) \) and \( y_2(t) = \sin(t) \). But what about \( y_3(t) = 9\cos(t) - 2\sin(t) \)? This is also a solution. So does any \( y(t) = c_1\cos(t) + c_2\sin(t) \), where \( c_1 \) and \( c_2 \) are constants. In fact, every solution to this differential equation is of this form.

**Example 9.1.** Find all of the solutions to \( y'' = 4y \).

We need to think of a function whose second derivative is 4 times the original function. What function comes back to a multiple of itself (without a sign change) after two derivatives? We might think of the exponential function, and checking exponents indicates that the following two functions are solutions: \( y_1(t) = e^{2t} \) and \( y_2(t) = e^{-2t} \). In fact, so are any functions of the form

\[
y(t) = c_1 e^{2t} + c_2 e^{-2t}.
\]

**Exercise.** Check that \( y_1(t) = e^{2t} \) and \( y_2(t) = e^{-2t} \) are in fact solutions to \( y'' = 4y \). By linearity, it follows that \( y(t) = c_1 e^{2t} + c_2 e^{-2t} \) is a solution for any \( c_1 \) and \( c_2 \). What about \( y_3(t) = \sin(2t) \) or \( y_4(t) = \cos(2t) \)?

The general form of a second order linear differential equation is

\[
p(t)y'' + q(t)y' + r(t)y = g(t).
\]

We call the equation homogeneous if \( g(t) = 0 \) and nonhomogeneous if \( g(t) \neq 0 \).

Why do we distinguish between the two cases? The first example discussed, \( y'' = -y \), is suggestive of why homogeneous equations are nice: we can see that there are two “different” solutions, and any linear combination of them is also a solution. This is in general true, and is important enough to have a name.

**Theorem 9.1 (Principle of Superposition).** If \( y_1(t) \) and \( y_2(t) \) are solutions to a second order linear homogeneous differential equation, then so is any linear combination

\[
y(t) = c_1y_1(t) + c_2y_2(t).
\]

This follows from homogeneity and the fact that the derivative is linear (so we can pull out the constants and split up derivatives of \( y(t) \) into the derivatives of \( y_1 \) and \( y_2 \)). So we know that if we have two solutions, we can find many more by taking linear combinations, but ultimately we would like to be able to write down a general solution to the differential equation, so that with some initial
conditions we could uniquely solve an initial value problem. The basic idea is that we want to find two “different” solutions \( y_1(t) \) and \( y_2(t) \), so that the general solution to the differential equation is \( y(t) = c_1 y_1(t) + c_2 y_2(t) \). What does “different” mean, exactly? We’ll hold off on answering that question for a few lectures. It’s not too hard to get a feel for when two solutions are “different” enough; the main point is that they shouldn’t be constant multiples.

Now, let’s think back to the example of \( y'' = -y \). We found two “different” solutions, \( y_1(t) = \cos(t) \) and \( y_2(t) = \sin(t) \) and saw that any solution can be written as a linear combination of these two solutions, \( y(t) = c_1 \cos(t) + c_2 \sin(t) \). We have two constants, so we’ll need two initial conditions to find a particular solution. We’ll generally give these conditions by specifying the values of \( y \) and \( y' \) at a particular \( t_0 \). So, for us, a second order linear homogeneous initial value problem might look like

\[
p(t)y'' + q(t)y' + r(t)y = 0 \quad y'(t_0) = y'_0, y(t_0) = y_0.
\]

Example 9.2. Find the particular solution to the initial value problem

\[
y'' + y = 0 \quad y(0) = 2, y'(0) = -1.
\]

We’re taking for granted for now that the general solution to this equation is

\[
y(t) = c_1 \cos(t) + c_2 \sin(t).
\]

To apply the initial conditions, we’ll need to know the derivative:

\[
y'(t) = -c_1 \sin(t) + c_2 \cos(t).
\]

Plugging in the initial conditions yields

\[
2 = c_1, \\
-1 = c_2,
\]

so the particular solution is

\[
y(t) = 2 \cos(t) - \sin(t).
\]

Sometimes applying initial conditions will mean we’ll have to solve a system of equations; other times it’s as easy as the previous example.

2. Homogeneous Equations With Constant Coefficients

To begin with, we’ll consider the easiest class of second order linear homogeneous equations, where the coefficient functions \( p(t), q(t), \) and \( r(t) \) are constants. In other words, the equations we’ll look at have the form

\[
ay'' + by' + cy = 0. \tag{9.1}
\]

How do we begin finding solutions to this equation? Somehow, the solution \( y(t) \) is linked to its first and second derivatives just by addition and multiplication by constants. If we think back to calculus, we can find a function that is linked to its derivative by a multiplicative constant: \( y(t) = e^{rt} \). To see if this is on the right track, let’s plug it into the differential equation 9.1 and see what we get. To do this, we’ll need the derivatives \( y'(t) = re^{rt} \) and \( y''(t) = r^2 e^{rt} \).

\[
a (r^2 e^{rt}) + b (re^{rt}) + ce^{rt} = 0 \\
e^{rt} (ar^2 + br + c) = 0
\]

What can we conclude? \( y(t) = e^{rt} \) is a solution to the differential equation if and only if \( e^{rt} (ar^2 + br + c) = 0 \). Since exponentials are never zero, we’re left with the condition that \( y(t) = e^{rt} \) will solve the differential equation as long as \( r \) is a solution to

\[
ar^2 + br + c = 0.
\]
This equation is called the \textit{characteristic equation} for 9.1.

Thus, to find solutions to a linear second order constant coefficient equation, we begin by writing down the characteristic equation. Then we find the roots $r_1$ and $r_2$ (note that these don’t necessarily have to be distinct, or even real). At this point, we’ll have some solutions to the differential equation:

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}.$$ 

Of course, it’s also possible these are the same, since we might have a repeated root. As a result, we’ll put off mentioning whether or not these two solutions are “different” enough to form the general solution to Equation 9.1. In fact, we have three cases, and in each case this question will be addressed differently.

Let’s look at an example.

\textbf{Example 9.3.} Find two solutions to the differential equation $y'' - 4y = 0$.

Notice that this is the same equation as in Example 9.1. The characteristic equation is $r^2 - 4 = 0$, and this has roots $r = \pm 2$. So we have two solutions $y_1(t) = e^{2t}$ and $y_2(t) = e^{-2t}$, which agree with our earlier guesses. \hfill $\square$

So what are the three cases that were mentioned earlier? They’re the same as the three possibilities for types of roots of quadratic equations:

1. Real, distinct roots $r_1 \neq r_2$.
2. Complex roots $r_1, r_2 = \alpha \pm \beta i$.
3. A repeated real root $r_1 = r_2 = r$.

We’ll look at each case more closely in the lectures to come.
**Distinct Roots of the Characteristic Equation**

1. Real, Distinct Roots

Recall from last time that a second order linear homogeneous differential equation with constant coefficients

\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (10.1)

is solved by \( y(t) = e^{rt} \), where \( r \) solves the characteristic equation

\[ ar^2 + br + c = 0. \]  \hspace{1cm} (10.2)

So, when the characteristic equation has two distinct real roots \( r_1 \neq r_2 \), we get two solutions \( y_1(t) = e^{r_1t} \) and \( y_2(t) = e^{r_2t} \). It will turn out that, in this case, \( y_1(t) \) and \( y_2(t) \) are “different” enough so that the general solution of Equation 10.1 is given by

\[ y(t) = c_1e^{r_1t} + c_2e^{r_2t}. \]

Then, given initial conditions, we can solve for \( c_1 \) and \( c_2 \).

**Example 10.1.** Solve the IVP

\[ y'' + 3y' - 18y = 0 \quad y(0) = 0, y'(0) = -1. \]

The characteristic equation is

\[ r^2 + 3r - 18 = 0 \]
\[ (r + 6)(r - 3) = 0\]

so the roots are \( r_1 = -6 \) and \( r_2 = 3 \). Thus the general solution and its derivative are

\[ y(t) = c_1e^{-6t} + c_2e^{3t} \]
\[ y'(t) = -6c_1e^{-3t} + 3c_2e^{3t} \]

and plugging in the initial conditions yields the system of equations

\[ 0 = c_1 + c_2 \]
\[ -1 = -6c_1 + 3c_2 \]

which has solution \( c_1 = \frac{1}{9} \) and \( c_2 = -\frac{1}{9} \). Thus the particular solution is

\[ y(t) = \frac{1}{9}e^{-6t} - \frac{1}{9}e^{3t}. \]

**Example 10.2.** Solve the IVP

\[ y'' - 7y' + 10y = 0 \quad y(0) = 3, y(0) = 2. \]

The characteristic equation is

\[ r^2 - 7r + 10 = 0 \]
\[ (r - 5)(r - 2) = 0. \]
The roots are $r_1 = 5$ and $r_2 = 2$, so the general solution and its derivative are
\[ y(t) = c_1 e^{5t} + c_2 e^{2t} \]
\[ y'(t) = 5c_1 e^{5t} + 2c_2 e^{2t}. \]
Plugging in the initial conditions yields the system of equations
\[ 3 = c_1 + c_2 \]
\[ 2 = 5c_1 + 2c_2 \]
which has solution $c_1 = -\frac{4}{3}$, $c_2 = \frac{13}{3}$, and so the particular solution is
\[ y(t) = -\frac{4}{3} e^{5t} + \frac{13}{3} e^{2t}. \]

**Example 10.3.** Solve the IVP
\[ 2y'' - 5y' + 2y = 0 \quad y(0) = -3, y'(0) = 3. \]
The characteristic equation is
\[ 2r^2 - 5r + 2 = 0 \]
\[ (2r - 1)(r - 2) = 0. \]
The roots are $r_1 = \frac{1}{2}$ and $r_2 = 2$. Thus the general solution and its derivative are
\[ y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{2t} \]
\[ y'(t) = \frac{c_1}{2} e^{\frac{1}{2}t} + 2c_2 e^{2t}. \]
Plugging in the initial conditions gives
\[ -3 = c_1 + c_2 \]
\[ 3 = \frac{c_1}{2} + 2c_2. \]
This system is solved by $c_1 = -6$ and $c_2 = 3$. So the particular solution is
\[ y(t) = -6e^{\frac{1}{2}t} + 3e^{2t}. \]

**Example 10.4.** Solve the IVP
\[ y'' + 5y' = 0 \quad y(0) = 2, y'(0) = -5. \]
The characteristic equation is
\[ r^2 + 5r = 0 \]
\[ r(r + 5) = 0 \]
and this has roots $r_1 = 0$ and $r_2 = -5$. The general solution and its derivative are
\[ y(t) = c_1 + c_2 e^{-5t} \]
\[ y'(t) = -5c_2 e^{-5t}. \]
Using the initial conditions yields
\[ 2 = c_1 + c_2 \]
\[ -5 = -5c_2 \]
which has solution $c_1 = 1$ and $c_2 = 1$, So the particular solution is
\[ y(t) = 1 + e^{-5t}. \]

Example 10.5. Solve the IVP
\[ y'' - 2y' - 8 = 0 \quad y(2) = 1, y'(2) = 0. \]
The characteristic equation is
\[ r^2 - 2r - 8 = 0 \quad (r - 4)(r + 2) = 0 \]
so the roots are $r_1 = 4$ and $r_2 = -2$. The general solution and its derivative are
\[ y(t) = c_1 e^{4t} + c_2 e^{-2t} \]
\[ y'(t) = 4c_1 e^{4t} - 2c_2 e^{-2t} \]
and plugging in the initial conditions yields
\[ 1 = c_1 e^{8} + c_2 e^{-4} \]
\[ 0 = 4c_1 e^{8} - 2c_2 e^{-4}. \]
The solution is $c_1 = \frac{2}{6} e^{-8}$ and $c_2 = \frac{4}{6} e^{4}$ and the particular solution is
\[ y(t) = \frac{1}{3} e^{8} e^{4t} + \frac{2}{3} e^{-2t}. \]

Example 10.6. Find the general solution to the DE
\[ y'' + y' - 3y = 0. \]
The characteristic equation is
\[ r^2 + r - 3 = 0 \]
which has roots
\[ r_{1,2} = \frac{-1 \pm \sqrt{13}}{2}. \]
Thus the general solution is
\[ y(t) = c_1 e^{\frac{-1 + \sqrt{13}}{2} t} + c_2 e^{\frac{-1 - \sqrt{13}}{2} t}. \]

2. Complex Roots

Now suppose the characteristic equation 10.2 has complex roots of the form $r_{1,2} = \alpha \pm i\beta$. By our earlier discussion, this means that we have the following two solutions to our original differential equation 10.1:
\[ y_1(t) = e^{(\alpha + i\beta)t} \quad y_2(t) = e^{(\alpha - i\beta)t}. \]
This is problematic, because both $y_1(t)$ and $y_2(t)$ are complex-valued. Since our original equation was both simple and had real coefficients, it would be ideal if we could find two real valued “different” enough solutions so that we can form a real-valued general solution. How do we do this?
Euler’s Formula. Euler’s Formula says
\[ e^{i\theta} = \cos(\theta) + i\sin(\theta). \]
In other words, we can write an imaginary exponential as a sin and a cos. How do we establish this? There are two nice ways of seeing this fact.

Differential Equations. First, we want to write \( e^{i\theta} = f(\theta) + ig(\theta) \). We also have
\[ f' + ig' = \frac{d}{d\theta} [e^{i\theta}] = ie^{i\theta} = if - g. \]
Thus \( f' = -g \) and \( g' = f \). Since \( e^0 = 1 \), we know that \( f(0) = 1 \) and \( g(0) = 0 \). We conclude that \( f(\theta) = \cos(\theta) \) and \( g(\theta) = \sin(\theta) \), so
\[ e^{i\theta} = \cos(\theta) + i\sin(\theta). \]

Taylor Series. Recall that the Taylor series for \( e^x \) is
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \]
while the Taylor series for \( \sin(x) \) and \( \cos(x) \) are
\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\]
If we set \( x = i\theta \) in the first series, we get
\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \ldots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \right)
\]
\[= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \cos(\theta) + i\sin(\theta). \]

So we can write our two complex exponentials as
\[ e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} (\cos(\beta t) + i\sin(\beta t)) \]
\[ e^{(\alpha-i\beta)t} = e^{\alpha t} e^{-i\beta t} = e^{\alpha t} (\cos(\beta t) - i\sin(\beta t)) \]
where the minus sign pops out of the sign in the second equation due to the oddness of \( \sin \). Notice that our new expressions are still complex. However, using the Principle of Superposition, we can obtain the following two solutions:
\[
y_1(t) = \frac{1}{2} (e^{\alpha t} (\cos(\beta t) + i\sin(\beta t)) + e^{\alpha t} (\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t} \cos(\beta t) \\
y_2(t) = \frac{1}{2i} (e^{\alpha t} (\cos(\beta t) + i\sin(\beta t)) - e^{\alpha t} (\cos(\beta t) - i\sin(\beta t))) = e^{\alpha t} \sin(\beta t). \\
\]
Differential Equations  Lecture 10: Distinct Roots of the Characteristic Equation

**Exercise.** Check that \( y_1(t) = e^{\alpha t} \cos(\beta t) \) and \( y_2(t) = e^{\alpha t} \sin(\beta t) \) are in fact solutions to the differential equation 10.1 when the roots of 10.2 are \( \alpha \pm i\beta \).

So now we have two solutions \( y_1(t) \) and \( y_2(t) \) which are real-valued. It turns out that they’re also “different” enough, so, if the roots to the characteristic equation are \( r_{1,2} = \alpha \pm i\beta \), we have the general solution

\[
y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t).
\]

Let’s do some examples.

**Example 10.7.** Solve the IVP

\[
y'' - 4y' + 9y = 0 \quad y(0) = 0, y'(0) = -2.
\]

The characteristic equation is

\[
r^2 - 4r + 9 = 0
\]

which has roots \( r_{1,2} = 2 \pm i\sqrt{5} \). Thus the general solution and its derivative are

\[
y(t) = c_1 e^{2t} \cos(\sqrt{5}t) + c_2 e^{2t} \sin(\sqrt{5}t)
y'(t) = 2c_1 e^{2t} \cos(\sqrt{5}t) - \sqrt{5}c_1 e^{2t} \sin(\sqrt{5}t) + 2c_2 e^{2t} \sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t} \cos(\sqrt{5}t).
\]

If we apply the initial conditions, we get

\[
0 = c_1
-2 = 2c_1 + \sqrt{5}c_2
\]

which is solved by \( c_1 = 0 \) and \( c_2 = -\frac{2}{\sqrt{5}} \). So the particular solution is

\[
y(t) = -\frac{2}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t).
\]

\( \square \)
LECUTRE 11

Complex and Repeated Roots

1. Complex Roots

Last time, we saw that when the characteristic equation has complex roots $r_{1,2} = \alpha \pm i\beta$, the general solution of

$$ay'' + by' + cy = 0$$

is

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t).$$

Let’s do some examples.

**Example 11.1.** Solve the IVP

$$y'' - 4y' + 9y = 0 \quad y(0) = 0, y'(0) = -2.$$  

The characteristic equation is

$$r^2 - 4r + 9 = 0$$

which has roots $r_{1,2} = 2 \pm i\sqrt{5}$. Thus the general solution and its derivative are

$$y(t) = c_1 e^{2t} \cos(\sqrt{5}t) + c_2 e^{2t} \sin(\sqrt{5}t)$$

$$y'(t) = 2c_1 e^{2t} \cos(\sqrt{5}t) - \sqrt{5}c_2 e^{2t} \sin(\sqrt{5}t) + 2c_2 e^{2t} \sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t} \cos(\sqrt{5}t).$$

If we apply the initial conditions, we get

$$0 = c_1$$

$$-2 = 2c_1 + \sqrt{5}c_2$$

which is solved by $c_1 = 0$ and $c_2 = -\frac{2}{\sqrt{5}}$. So the particular solution is

$$y(t) = -\frac{2}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t).$$

**Example 11.2.** Solve the IVP

$$y'' + 6y' + 16y = 0 \quad y(0) = 2, y'(0) = -1.$$  

The characteristic equation is

$$r^2 + 6r + 16 = 0,$$

which has roots $r_{1,2} = -3 \pm i\sqrt{7}$. Hence the general solution and its derivative are

$$y(t) = c_1 e^{-3t} \cos(\sqrt{7}t) + c_2 e^{-3t} \sin(\sqrt{7}t)$$

$$y'(t) = -3c_1 e^{-3t} \cos(\sqrt{7}t) - \sqrt{7}c_1 e^{-3t} \sin(\sqrt{7}t) - 3c_2 e^{-3t} \sin(\sqrt{7}t) + \sqrt{7}c_2 e^{-3t} \cos(\sqrt{7}t)$$

and plugging in initial conditions yields the system

$$2 = c_1$$

$$-1 = -3c_1 + \sqrt{7}c_2,$$
so we conclude $c_1 = 2$ and $c_2 = \frac{5\sqrt{7}}{7}$ and the particular solution is

$$y(t) = 2e^{-3t} \cos(\sqrt{7}t) + \frac{5\sqrt{7}}{7} e^{-3t} \sin(\sqrt{7}t).$$

□

**Example 11.3.** Solve the IVP

$$4y'' + 12y' + 10y = 0 \quad y(0) = -1, y'(0) = 3.$$ 

The characteristic equation is

$$4r^2 + 12r + 10 = 0,$$

which has roots $r_{1,2} = -\frac{3}{2} \pm \frac{1}{2} i$. So the general solution and its derivative are

$$y(t) = c_1 e^{\frac{3}{2} t} \cos \left( \frac{t}{2} \right) + c_2 e^{\frac{3}{2} t} \sin \left( \frac{t}{2} \right),$$

$$y'(t) = \frac{3}{2} c_1 e^{\frac{3}{2} t} \cos \left( \frac{t}{2} \right) - \frac{1}{2} c_1 e^{\frac{3}{2} t} \sin \left( \frac{t}{2} \right) + \frac{3}{2} c_2 e^{\frac{3}{2} t} \sin \left( \frac{3}{2} \right) + \frac{1}{2} c_2 e^{\frac{3}{2} t} \cos \left( \frac{t}{2} \right).$$

Plugging in the initial conditions yields

$$-1 = c_1,$$

$$3 = \frac{3}{2} c_1 + \frac{1}{2} c_2,$$

which has solution $c_1 = -1$ and $c_2 = 9$. The particular solution is

$$y(t) = -e^{\frac{3}{2} t} \cos \left( \frac{t}{2} \right) + 9 e^{\frac{3}{2} t} \sin \left( \frac{t}{2} \right).$$

□

**Example 11.4.** Solve the IVP

$$y'' + 4y = 0 \quad y \left( \frac{\pi}{4} \right) = -10, y' \left( \frac{\pi}{4} \right) = 4.$$ 

The characteristic equation is

$$r^2 + 4 = 0,$$

which has roots $r_{1,2} = \pm 2i$. The general solution and its derivative are

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t),$$

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t).$$

The initial conditions gives the system

$$-10 = c_2,$$

$$4 = -2c_1$$

so we conclude $c_1 = -2$ and $c_2 = -10$ and the particular solution is

$$y(t) = -2 \cos(2t) - 10 \sin(2t).$$

□
2. Repeated Roots

The last case we have to consider is when the characteristic equation has a repeated root \( r_1 = r_2 = r \). This is problematic, though: our usual method of finding solutions to constant coefficient equations would lead us to form the two solutions

\[
y_1(t) = e^{rt} \quad y_2(t) = e^{rt}.
\]

But these are the same, and definitely not different in any sense, let alone different enough to form a general solution. So we’re left needing a second solution which is “different” from \( y_1(t) = e^{rt} \). What do we do?

Let’s start by recalling that if the quadratic equation \( ar^2 + br + c = 0 \) has a repeated root \( r \), it must be \( r = -\frac{b}{2a} \). Thus our solution is, more precisely, \( y_1(t) = e^{-\frac{b}{2a}t} \). We know that any constant multiple of \( y_1 \) is a solution. These, though, won’t help us find a different enough second solution. The question that might come to mind is if it’s possible that we can find a solution of the form

\[
y_2(t) = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t},
\]

i.e., \( y_2 \) is the product of some other function of \( t \) and \( y_1 \).

In the past, when we’ve guessed at the form of a solution, we’ve checked our guess and derived specifics by plugging the guess into the differential equation. This is no different. So we’ll need to differentiate \( y_2(t) \):

\[
y_2'(t) = v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t}
\]

\[
y_2''(t) = v''(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}
\]

\[
= v''(t)e^{-\frac{b}{2a}t} - \frac{b}{a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}.
\]

For the rest of this calculation, I’m going to stop explicitly denoting the parameter of \( v \), but don’t forget that \( v \) is a function, not a constant. Now, plugging in:

\[
a \left( v''e^{-\frac{b}{2a}t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t} \right) + b \left( v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t} \right) + c \left( ve^{-\frac{b}{2a}t} \right) = 0
\]

\[
e^{-\frac{b}{2a}t} \left( av'' + (-b + b)v' + \left( \frac{b^2}{4a} - \frac{b^2}{2a} + c \right) v \right) = 0
\]

\[
e^{-\frac{b}{2a}t} \left( av'' - \frac{1}{4a} (b^2 - 4ac) v \right) = 0.
\]

Since we’re in the repeated root case, we know that the discriminant \( b^2 - 4ac = 0 \). As exponentials are never zero, we’re left with the condition

\[av'' = 0 \Rightarrow v'' = 0.\]

Notice that if \( a = 0 \), our equation would be first order, rather than second order, so by assumption, \( a \neq 0 \), and this is why \( v'' = 0 \). So, what form can \( v \) take? It must be that \( v \) is linear, i.e.

\[v(t) = ct + k\]

for some constants \( c \) and \( k \). Thus, for any such \( v(t) \), \( y_2(t) = v(t)e^{-\frac{b}{2a}t} \) will be a solution. The most general possible \( v(t) \) that will work for us is \( ct + k \). We can take \( c = 1 \) and \( k = 0 \) to get a specific \( v \) which is nice and simple, and then our second solution is

\[y_2(t) = te^{-\frac{b}{2a}t}\]

and the general solution is

\[y(t) = c_1e^{-\frac{b}{2a}t} + tc_2e^{-\frac{b}{2a}t}.\]
Remark. Here’s another way of looking at this choice of constants. Suppose we don’t make it. Then we have, for our general solution,

\[ y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 (ct + k)e^{-\frac{b}{2a}t} \]

\[ = c_1 e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t} + c_2 ke^{-\frac{b}{2a}t} \]

\[ = (c_1 + c_2 k)e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t}. \]

Since \(c_1, c_2, c,\) and \(k\) are just constants, we’ll just roll them together and write

\[ y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 cte^{-\frac{b}{2a}t}. \]

To summarize the previous discussion: if the characteristic equation has repeated roots \(r_1 = r_2 = r,\) the general solution is

\[ y(t) = c_1 e^{rt} + c_2 te^{rt}. \]

Let’s work some examples.
LECTURE 12

Repeated Root Examples and Reduction of Order

1. Repeated Roots, Part Deux

Last class, we saw that the differential equation
\[ ay'' + by' + cy = 0 \]
has general solution
\[ y(t) = c_1 e^{rt} + c_2 te^{rt} \]
when the characteristic equation \( ar^2 + br + c \) has a single repeated root \( r \). Now, let’s look at some examples.

Example 12.1. Solve the IVP
\[ y'' - 4y' + 4y = 0 \quad y(0) = -1, y'(0) = 6. \]
The characteristic equation is
\[ r^2 - 4r + 4 = 0 \]
\[ (r - 2)^2 = 0 \]
so we see that we have a repeated root \( r = 2 \). The general solution and its derivative are
\[ y(t) = c_1 e^{2t} + c_2 te^{2t} \]
\[ y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 te^{2t} \]
and plugging in initial conditions yields
\[ -1 = c_1 \]
\[ 6 = 2c_1 + c_2 \]
so we have \( c_1 = -1 \) and \( c_2 = 8 \). The particular solution is
\[ y(t) = -e^{2t} + 6te^{2t}. \]

Example 12.2. Solve the IVP
\[ 16y'' + 40y' + 25y = 0 \quad y(0) = -1, y'(0) = 2. \]
The characteristic equation is
\[ 16r^2 + 40r + 25 = 0 \]
\[ (4r + 5)^2 = 0 \]
and so we conclude that we have a repeated root \( r = -\frac{5}{4} \) and the general solution and its derivative are
\[ y(t) = c_1 e^{-\frac{5}{4}t} + c_2 te^{-\frac{5}{4}t} \]
\[ y'(t) = -\frac{5}{4}c_1 e^{-\frac{5}{4}t} + c_2 e^{-\frac{5}{4}t} - \frac{5}{4}c_2 te^{-\frac{5}{4}t}. \]
Plugging in the initial conditions yields

\[-1 = c_1 \]
\[2 = -\frac{5}{4} c_1 + c_2 \]

so \(c_1 = -1\) and \(c_2 = \frac{3}{4}\). The particular solution is

\[y(t) = -e^{-\frac{5}{4}t} + \frac{3}{4} te^{-\frac{5}{4}t}.\]

\[\square\]

2. Reduction of Order

We’ve just spent a few classes discussing second order linear homogeneous equations with constant coefficients, i.e., equations with the form

\[ay'' + by' + cy = 0.\]

Now, before we briefly discuss the theory of second order linear homogeneous equations, let’s consider what happens if our coefficients aren’t constant. In other words, we’re looking at equations of the form

\[p(t)y'' + q(t)y' + r(t)y = 0.\]

In general, finding solutions to these equations isn’t easy, but if we know (or can guess at) a solution, we can use the technique we used in the repeated roots case to find a second solution that is “different” enough from our original solution to give us a general solution. This method is known as reduction of order.

Let’s look at an example to begin with.

**Example 12.3.** Find the general solution to

\[2t^2y'' + ty' - 3y = 0 \tag{12.1}\]

given that \(y_1(t) = t^{-1}\) is a solution.

Let’s think back to the repeated roots case we discussed last class. We knew we had a solution \(y_1\) and needed to find a distinct solution. What did we do? We asked which nonconstant functions \(v(t)\) make \(y_2(t) = v(t)y_1(t)\) also a solution. Let’s do the same thing here.

Set \(y_2(t) = v(t)y_1(t)\). Then \(y_2\) and its derivatives are

\[y_2 = vt^{-1}\]
\[y_2' = v't^{-1} - vt^{-2}\]
\[y_2'' = v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.\]

The next step is to plug into Equation 12.1 so we can solve for \(v\):

\[2t^2 \left( v''t^{-1} - 2v't^{-2} + 2vt^{-3} \right) + t \left( v't^{-1} - vt^{-2} \right) - 3vt^{-1} = 0\]
\[2v''t - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} = 0\]
\[2tv'' - 3v' = 0.\]

Notice that the only terms left involve \(v''\) and \(v'\), not \(v\). This also happened in the repeated roots case (where the \(v'\) term also dropped out, but that in general won’t happen). The \(v\) term should always disappear at this point, so this is a good spot to make sure that we’ve done our differentiation and algebra are correct. If there’s a \(v\) term left, we haven’t.
Now, we know that if \( y_2 \) is a solution, the function \( v \) must satisfy
\[
2tv'' - 3v = 0.
\]
But this is a second order linear homogeneous equation with nonconstant coefficients. All we’ve
done is to replace our original one with this new one. So have we actually helped ourselves?

Yes, we have. Since there is no \( v \) term in the new differential equation, we can make the
substitution \( w(t) = v'(t) \) (which also gives \( u'(t) = v'(t) \)). By changing variables in this way, our
equation becomes
\[
2tw' - 3w = 0. \tag{12.2}
\]
This is a first order linear equation, which we know how to solve. This is precisely why the method
is called reduction of order: we’re taking a second order equation and using a known solution to
make it first order.

Alright, so let’s go ahead and solve Equation 12.2. First, we need to put it in the correct form
for integrating factors:
\[
w' - \frac{3}{2t}w = 0.
\]
So our integrating factor is
\[
\mu(t) = e^{\int -\frac{3}{2t} \, dt} = e^{-\frac{3}{2} \ln(t)} = t^{-\frac{3}{2}}.
\]
\[
\left[ t^{-\frac{3}{2}}w \right]' = 0
\]
\[
t^{-\frac{3}{2}}w = c
\]
\[
w(t) = ct^{-\frac{3}{2}}
\]
So we know what \( w(t) \) must be to solve Equation 12.2. But to solve Equation 12.1, our original
differential equation, we don’t need \( w(t) \), we need \( v(t) \). Since \( v'(t) = w(t) \), integrating \( w \) will give
our \( v \).
\[
v(t) = \int w(t) \, dt
\]
\[
= \int ct^{-\frac{3}{2}} \, dt
\]
\[
= \frac{2}{5}ct^{\frac{5}{2}} + k
\]
Now, this is the most general form for \( v(t) \): for any constants \( c, k \), \( y_2(t) = v(t)y_1(t) \) will solve
Equation 12.1. So, just like in the repeated roots case, we can choose \( c \) and \( k \) to pick one particular
such \( v(t) \) that’s nice and simple. The only constraint is that we can’t choose \( c = 0 \), since then \( v(t) \)
would be a constant and \( y_1 \) and \( y_2 \) would be essentially the same. A natural choice is to take \( c = \frac{5}{2} \)
and \( k = 0 \). Then \( v(t) = t^{\frac{3}{2}} \), so \( y_2(t) = v(t)y_1(t) = t^{\frac{5}{2}} \), and the general solution to Equation 12.1 is
\[
y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}}.
\]

Reduction of order is a very nice and powerful method for finding a second solution to a
differential equation when we don’t have any other method, but it does require us to start with a
given (or guessed at) solution. Sometimes finding this first solution is very difficult, but once we
have it, we can reduce the order to find the second solution and hence a general solution.

It’s also important to be careful with these problems: sometimes the algebra gets a little nasty,
and it’s easy to make sloppy mistakes. It’s important to make sure that the \( v \) term does actually
drop out of the equation after we plug in the derivatives of \( y_2 \) and it’s also important to check the
second solution we obtain in case we made an algebraic error somewhere.
Let’s do one more example.

**Example 12.4.** Find the general solution to

\[ t^2 y'' + 2ty' - 2y = 0 \]

given that

\[ y_1(t) = t \]

is a solution.

We start by setting \( y_2(t) = v(t)y_1(t) \). So we have

\[
\begin{align*}
y_2 &= tv \\
y_2' &= tv' + v \\
y_2'' &= tv'' + v' + v' = tv'' + 2v'.
\end{align*}
\]

Next, we plug in and arrange terms.

\[
\begin{align*}
t^2 (tv'' + 2v') + 2t (tv' + v) - 2tv &= 0 \\
t^3 v'' + 2t^2 v' + 2t^2 v' + 2tv - 2tv &= 0 \\
t^3 v'' + 4t^2 v' &= 0.
\end{align*}
\]

Notice that, as desired, the \( v \) term drops out. We make the change of variable \( w(t) = v'(t) \) to obtain

\[ t^3 w' + 4t^2 w = 0 \]

, which has integrating factor \( \mu(t) = t^4 \).

\[
\begin{align*}
[t^4 w]' &= 0 \\
t^4 w &= c \\
w(t) &= ct^{-4}
\end{align*}
\]

So we have

\[
v(t) = \int w(t) \, dt = \int ct^{-4} \, dt = -\frac{c}{3} t^{-3} + k.
\]

A nice choice for the constants is \( c = -3 \) and \( k = 0 \), so we get \( v(t) = t^{-3} \), which gives a second solution of \( y_2(t) = v(t)y_1(t) = t^{-2} \). So our general solution is

\[ y(t) = c_1 t + c_2 t^{-2}. \]

\[ \square \]
LECTURE 13

Fundamental Sets of Solutions

Over the past few lectures, our focus has been on constructing general solutions to certain linear differential equations. To do this, we’ve needed to find two “different” solutions \( y_1(t) \) and \( y_2(t) \) so that their linear combination is a general solution. But how do we know when \( y_1 \) and \( y_2 \) are “different” enough for this to be the case? What is the precise condition on \( y_1 \) and \( y_2 \) that makes them “different” enough to form a general solution?

1. Existence and Uniqueness

The first question we should ask is if, given an initial value problem involving a linear second order equation, a solution exists. We’ve already commented that the answer is yes (when discussing the first order case), but let’s just repeat:

**Theorem 13.1.** Consider the initial value problem

\[
y'' + p(t)y' + q(t)y = g(t) \quad y(t_0) = y_0, y'(t_0) = y'_0.
\]

If \( p(t) \), \( q(t) \), and \( g(t) \) are all continuous on some interval \((a, b)\) such that \( a < t_0 < b \), then the initial value problem has a unique solution defined on \((a, b)\).

2. The Wronskian

Let’s suppose we’re working with the following initial value problem

\[
p(t)y'' + q(t)y' + r(t)y = 0 \quad (13.1a)
\]

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (13.1b)
\]

and that we know two solutions \( y_1(t) \) and \( y_2(t) \). Since the differential equation \((13.1a)\) is linear and homogeneous, the Principle of Superposition says that any linear combination

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) \quad (13.2)
\]

is also a solution. We want to know when this is a general solution. For this to be the case, it must satisfy the general initial conditions \((13.1b)\). In other words, without any restrictions (beyond requiring \( t_0 \) to not be a point of discontinuity for the coefficient functions, so that Theorem 13.1 applies) we should be able to find constants \( c_1 \) and \( c_2 \) that work for the given initial conditions.

We start by differentiating Equation \((13.2)\) and plugging in the initial conditions.

\[
y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)
\]

\[
y'_0 = y'(t_0) = c_1 y'_1(t_0) + c_2 y'_2(t_0) \quad (13.3)
\]

Let’s solve this system of equations. We have that

\[
c_1 = \frac{y_0 - c_2 y_2(t_0)}{y_1(t_0)}.
\]
Thus:

\[ y' = \frac{y_0y_1'(t_0) - c_2y_2(t_0)y_1'(t_0)}{y_1(t_0)} + c_2y_2'(t_0) \]

\[ = \frac{y_0y_1'(t_0) - c_2y_2(t_0)y_1'(t_0) + c_2y_2'(t_0)y_1(t_0)}{y_1(t_0)} \]

and we compute

\[ c_2 = \frac{y_0y_1(t_0) - y_0y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)} \]

\[ c_1 = \frac{y_0y_2(t_0) - y_0y_2'(t_0)}{y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)}. \]

Notice that \( c_1 \) and \( c_2 \) have the same quantity in their denominators: \( y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0). \) So the only time we won’t be able to solve for \( c_1 \) and \( c_2 \) is when this quantity is zero.

**Definition 13.1.** The quantity

\[ W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \]

is called the Wronskian\(^1\) of \( y_1 \) and \( y_2 \) at \( t_0. \)

**Remark.**

1. When it’s clear what the two functions in question are, we will often just denote the Wronskian by \( W. \)

2. The notation we’ve used for the Wronskian seems to indicate that we can think of \( W(y_1, y_2)(t) \) as a function dependent on \( t. \) This is definitely the case. The “Wronskian of \( y_1 \) and \( y_2, \)” \( W(y_1, y_2)(t), \) is a function of \( t \) and can be evaluated at any \( t \) where \( y_1 \) and \( y_2 \) are defined. This is important, since for the two solutions \( y_1 \) and \( y_2 \) to satisfy the general initial conditions (13.1b), we’ll need \( W(y_1, y_2) \) to be nonzero at any value of \( t_0 \) where Theorem 13.1 applies.

3. We could have also solved the system of equations (13.3) by using Cramer’s Rule from linear algebra.

\[ c_1 = \begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix} \quad c_2 = \begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix} \]

\[ \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \]

\[ \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \]

Here

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]

is the determinant of the 2x2 matrix. It’s alright if you haven’t taken linear algebra; the formula is easy enough. Notice that the denominators of \( c_1 \) and \( c_2 \) here are exactly the same as before. So we also have

\[ W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}. \]

We will generally express the Wronskian in this determinant form.

---

\( ^1 \)The Wronskian is named after the Polish philosopher and mathematician Józef Hoëne-Wroński (1778-1853). Wroński focused on applying philosophy to mathematics. The Wronskian has its origin in Wroński’s attempt to supplant the use of infinite series representations of functions with his own ideas. For many years, Wroński’s ideas were given short shrift, primarily because his contemporaries felt that he had too high an opinion of himself and his research and his violent reaction to criticism. It’s now felt that while much of his work was wrong, there were flashes of brilliant insight in his papers. Shortly before his death, he reportedly said “God Almighty, there’s still so much more I wanted to say!”
To summarize, two solutions \( y_1(t) \) and \( y_2(t) \) will form a general solution to Equation (13.1a) if they satisfy the general initial conditions (13.1b). The above computation showed that this will be the case so long as

\[
W(y_1, y_2)(t_0) = \begin{vmatrix}
  y_1(t_0) & y_2(t_0) \\
  y'_1(t_0) & y'_2(t_0)
\end{vmatrix} = y_1(t_0)y'_2(t_0) - y_2(t_0)y'_1(t_0) \neq 0.
\]

If \( y_1(t) \) and \( y_2(t) \) are solutions to (13.1a) and \( W(y_1, y_2)(t) \neq 0 \), then the two solutions are said to be a fundamental set of solutions for (13.1a) and the general solution is

\[
y(t) = c_1y_1(t) + c_2y_2(t).
\]

In other words, two solutions are “different” enough to form a general solution if they are a fundamental set of solutions and their Wronskian is nonzero. Let’s check some of the claims we made earlier.

**Example 13.2.** If \( r_1 \) and \( r_2 \) are distinct real roots of the characteristic equation for \( ay'' + by' + cy = 0 \), check that

\[
y_1(t) = e^{r_1t} \quad \text{and} \quad y_2(t) = e^{r_2t}
\]

form a fundamental set of solutions.

To show this, we’ll need to compute the Wronskian and see that it isn’t zero.

\[
W = \begin{vmatrix}
  e^{r_1t} & e^{r_2t} \\
  r_1e^{r_1t} & r_2e^{r_2t}
\end{vmatrix} = r_2e^{(r_2+r_1)t} - r_1e^{(r_2+r_1)t} = e^{(r_2-r_1)t}(r_2 - r_1)
\]

Since exponentials are never zero and \( r_2 \neq r_1 \), we conclude that \( W \neq 0 \) and so, as claimed earlier, \( y_1 \) and \( y_2 \) form a fundamental set of solutions for the differential equation and the general solution is in fact

\[
y(t) = c_1y_1(t) + c_2y_2(t).
\]

□

**Example 13.3.** The first example we did during the Reduction of Order lecture was to find a second solution to

\[
2t^2y'' + ty' - 3y = 0
\]

given that \( y_1(t) = t^{-1} \) is a solution. Let’s show that \( y_1(t) \) and \( y_2(t) = t^{\frac{3}{2}} \) form a fundamental set of solutions. To do this, we compute the Wronskian.

\[
W = \begin{vmatrix}
  t^{-1} & t^{\frac{3}{2}} \\
  -t^{-2} & \frac{3}{2}t^{\frac{1}{2}}
\end{vmatrix} = \frac{3}{2}t^{-\frac{1}{2}} + t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}}
\]

Thus \( W \neq 0 \), so they are a fundamental set of solutions.

Notice that we can’t plug \( t = 0 \) into the Wronskian. This is alright, since we can’t plug \( t = 0 \) into the solutions, either.

Thus the general solution is, in fact,

\[
y(t) = c_1t^{-1} + c_2t^{\frac{3}{2}}.
\]

□
Example 13.4. The other reduction of order example we did involved the differential equation
\[ t^2y'' + 2ty' - 2y = 0. \]
We were given that \( y_1(t) = t \) was a solution and found that \( y_2(t) = t^{-2} \) was another solution. Again, let’s check that these two solutions in fact form a fundamental set of solutions.

\[
W = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.
\]
So the solutions are a fundamental set of solutions, and the general solution is
\[ y(t) = c_1t + c_2t^{-2}. \]

The final question we need to ask here is how we know if a fundamental set of solutions will exist for a given differential equations. This is a version of the existence question, but for general solutions to a differential equation instead of particular solutions to an initial value problem. The following theorem gives the answer.

**Theorem 13.2.** Consider the differential equation
\[ y'' + p(t)y' + q(t) = 0 \]
where \( p(t) \) and \( q(t) \) are continuous on some interval \((a, b)\). Suppose \( a < t_0 < b \). If \( y_1(t) \) is a solution satisfying the initial conditions
\[ y(t_0) = 1 \quad y'(t_0) = 0 \]
and \( y_2(t) \) is a solution satisfying
\[ y(t_0) = 0 \quad y'(t_0) = 1, \]
then \( y_1(t) \) and \( y_2(t) \) form a fundamental set of solutions.

In general, we won’t want to use this theorem to compute the fundamental set of solutions, since the particular solutions we get from those initial conditions might not be as nice as would like. The main importance it has is that it assures us that, as long as the coefficient functions \( p(t) \) and \( q(t) \) are continuous, a fundamental set of solutions will exist.

### 3. Linear Independence

The motivation for introducing the Wronskian was to study when a general linear combination of two solutions to the differential equation (13.1a) was the general solution. We found that if this is the case, the Wronskian will be nonzero. But it’s not yet clear why we should expect the Wronskian of two functions to be either zero or nonzero.

To do this, we need to introduce a couple of new notions. Suppose we have two functions \( f(t) \) and \( g(t) \). We can always consider the equation
\[ c_1f(t) + c_2g(t) = 0. \tag{13.4} \]
Notice that \( c_1 = 0 \) and \( c_2 = 0 \) always solve this equation, regardless of what \( f \) and \( g \) are.

**Definition 13.5.** If there are nonzero constants \( c_1 \) and \( c_2 \) such that Equation 13.4 is satisfied for all \( t \), then the functions \( f \) and \( g \) are said to be **linearly dependent**. On the other hand, if the only constants for which Equation 13.4 is true are \( c_1 = 0 \) and \( c_2 = 0 \), then \( f \) and \( g \) are said to be **linearly independent**.
Remark. Two functions are linearly dependant precisely when they are constant multiples of each other. If Equation 13.4 is satisfied by nonzero $c_1$ and $c_2$, we have

$$f(t) = -\frac{c_2}{c_1} g(t).$$

This is often useful in deciding whether or not two functions are linearly dependant or not.
Example 13.6. Determine if the following pairs of functions are linearly dependant or independant.

(1) \( f(x) = \cos(2x) \quad g(x) = 2\cos^2(x) - 2\sin^2(x) \)

(2) \( f(t) = t^2 \quad g(t) = t^4 \)

(1) Let’s start by writing down Equation 13.4 for these functions. We have

\[ c_1 \cos(2x) + 2c_2(\cos^2(x) - \sin^2(x)) = 0 \]

We’re looking for nonzero constants \( c_1 \) and \( c_2 \) that make this true. In this case, we have a trig identity: \( \cos(2x) = \cos^2(x) - \sin^2(x) \). Then our equation becomes

\[ (c_1 + 2c_2)\cos(2x) = 0. \]

This equation is true for any \( c_1 \) and \( c_2 \) satisfying \( c_1 + 2c_2 = 0 \). So we can take, for example, \( c_1 = 2 \) and \( c_2 = -1 \), though there are infinitely many different pairs that will work. We conclude that the two functions are linearly dependant.

(2) Once again, we start by writing down Equation 13.4.

\[ c_1t^2 + c_2t^4 = 0 \]

There’s no nice identity or formula we can use, so let’s start by observing that if this is true, we can differentiate both sides and also get a true equation. This gives the system of equations

\[ c_1t^2 + c_2t^4 = 0 \]
\[ 2c_1t + 4c_2t^3 = 0. \]

We will solve this system for \( c_1 \) and \( c_2 \). The second equation tells us that \( c_1 = -2c_2t^2 \). Plugging this into the first equation gives

\[ 2(-2c_2t^2)t^2 + c_2t^4 = 0 \]
\[ -3c_2t^4 = 0. \]

This is only true for all \( t \) if \( c_2 = 0 \) which in turn tells us that \( c_1 = 0 \). Hence the functions are linearly independent.

As you can see, this can be a fairly tedious process, and sometimes tricks are required to proceed. It would be nice to have a criterion for linear independence that might let us avoid these computations. This is where the Wronskian helps.

Theorem 13.3. Given two functions \( f(t) \) and \( g(t) \) which are differentiable on some interval \( (a,b) \),

(1) If \( W(f,g)(t_0) \neq 0 \) for some \( a < t_0 < b \), then \( f(t) \) and \( g(t) \) are linearly independent on \( (a,b) \) and

(2) If \( f(t) \) and \( g(t) \) are linearly dependent on \( (a,b) \), then \( W(f,g)(t) = 0 \) for all \( a < t < b \).

Remark. Be careful: the preceding theorem does not say that if \( W(f,g)(x) = 0 \) then \( f \) and \( g \) are linearly dependant. Two linearly independent functions may have a zero Wronskian.

Let’s use this theorem to double check our earlier example.
Example 13.7.

1. Let’s consider \( f(x) = \cos(2x) \) and \( g(x) = 2 \cos^2(x) - 2 \sin^2(x) \).

\[
W = \begin{vmatrix}
\cos(2x) & 2 \cos^2(x) - 2 \sin^2(x) \\
-\sin(2x) & -4 \cos(x) \sin(x) - 4 \cos(x) \sin(x)
\end{vmatrix}
= \begin{vmatrix}
\cos(2x) & 2 \cos(2x) \\
-\sin(2x) & -4 \sin(2x)
\end{vmatrix}
= -4 \cos(2x) \sin(2x) + 4 \cos(2x) \sin(2x) = 0
\]

We knew that \( f \) and \( g \) were linearly independent, so this result is consistent. Again, we could not have concluded this simply from the Wronskian calculation.

2. Now let’s take \( f(t) = t^2 \) and \( g(t) = t^4 \).

\[
W = \begin{vmatrix}
t^2 & t^4 \\
2t & 4t^3
\end{vmatrix}
= 4t^5 - 2t^5
= 2t^5
\]

The Wronskian will be nonzero so long as \( t \neq 0 \), which is sufficient to conclude that \( f \) and \( g \) are linearly independent on any interval that does not include \( t = 0 \).
LECTURE 14

More On The Wronskian and Nonhomogeneous Equations

1. The Wronskian, Part Deux

Last lecture, we introduced the Wronskian of two functions $y_1$ and $y_2$,

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = y_1(t)y'_2(t) - y'_1(t)y_2(t).$$

We saw that if $W(y_1, y_2)(t) \neq 0$, then $y_1$ and $y_2$ are linearly independant, i.e., the only constants $c_1$ and $c_2$ that satisfy

$$c_1y_1(t) + c_2y_2(t) = 0$$

are $c_1 = c_2 = 0$. In other words, two functions are linearly independent if they aren’t constant multiples of each other.

We also saw that, in the context where $y_1$ and $y_2$ are solutions to the linear homogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y = 0,$$

$W(y_1, y_2)(t) \neq 0$ is precisely the condition for the general solution of the differential equation to be

$$y(t) = c_1y_1(t) + c_2y_2(t),$$

i.e., for $y_1$ and $y_2$ to be a fundamental set of solutions. Thus, if $y_1$ and $y_2$ are a fundamental set of solutions, they are linearly independent. If we have two solutions $y_1$ and $y_2$ which are linearly dependent, on the other hand, then they cannot possibly be a fundamental set of solutions, as they have a zero Wronskian.

There was one point last lecture that we should clear up now. We know that if our initial data is at $t_0$, $y_1$ and $y_2$ will be a fundamental set of conditions if and only if $W(y_1, y_2)(t_0) \neq 0$. But this is a condition only at one point. What happens if $y_1$ and $y_2$ have a nonzero Wronskian only at $t_0$ but not at nearby points? This would be problematic, since then our condition would tell us that $y_1$ and $y_2$ are a fundamental set of solutions for initial data at $t_0$, but not for any initial data near $t_0$. The possibility of this should seem odd, since we know there should be a unique solution on any interval around $t_0$ where we have continuity. So how do we know that this can’t happen? The answer is something called Abel’s Theorem.

2. Abel’s Theorem

You may notice that throughout our entire discussion of the Wronskian, we have yet to actually use the differential equation in question (beyond deriving the formula for the Wronskian assuming that $y_1$ and $y_2$ satisfied some differential equation). Fortunately, when $y_1$ and $y_2$ are solutions to a linear homogeneous differential equation, we can say a bit more about their Wronskian.

**Theorem 14.1 (Abel’s Theorem).** Suppose $y_1(t)$ and $y_2(t)$ solve the linear homogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on some interval $(a, b)$. Then, for $a < t < b$, their Wronskian is given by

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x) \, dx},$$

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where \( t_0 \) is in \((a, b)\).

If \( W(y_1, y_2)(t_0) \neq 0 \) at some point \( t_0 \) in the interval \((a, b)\), then Abel’s Theorem tells us that the Wronskian can’t be zero for any \( t \) in \((a, b)\), since exponentials are never zero. This assures us that we can change our initial data (without crossing points of discontinuity of the coefficient functions) without worry that our general solution will change.

Another advantage of Abel’s Theorem is that it lets us compute the general form of the Wronskian of any two solutions to the differential equation without knowing them explicitly. This is useful, for example, with regard to reduction of order, where we only begin by knowing a single solution. The formulation given in the statement of the theorem isn’t so computationally useful, however, because we might not have a precise \( t_0 \) in mind, let alone knowing the value of the Wronskian there. But if we apply the Fundamental Theorem of Calculus, things simplify nicely.

\[
W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x) \, dx} = ce^{-\int p(t) \, dt}
\]

What is this constant \( c \)? Well, it doesn’t really end up mattering. If we know the value of the Wronskian at one point, we can compute it, but our general interest in the Wronskian mostly involves knowing its general form. As long as we know \( c \neq 0 \), that’s all that matters to us.

**Example 14.1.** Compute, up to a constant, the Wronskian of two solutions \( y_1 \) and \( y_2 \) of the differential equation

\[
t^4 y'' - 2t^3 y' - t^8 y = 0.
\]

First, we need to put the equation in the form specified in Abel’s Theorem. We do this by dividing by the leading coefficient.

\[
y'' - \frac{2}{t} y' - t^4 y = 0.
\]

So, Abel’s Theorem tells us

\[
W = ce^{-\int \frac{2}{t} \, dt} = ce^{\ln t} = ct^2.
\]

Ok, great...but the main virtue of this is that it gives us a second way to compute the Wronskian. A general rule in mathematics is that whenever you can compute something in two different ways, something good will happen. In this case, we know by Abel’s Theorem that

\[
W(y_1, y_2)(t) = ce^{-\int p(t) \, dt}.
\]

On the other hand, by definition,

\[
W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).
\]

Setting these equal, if we know one solution \( y_1(t) \), we’re left with a first order differential equation for \( y_2 \) that we can then solve.

Let’s see this with an example of reduction of order we did the traditional way.

**Example 14.2.** Suppose we want to find the general solution to \( 2t^2 y'' + ty' - 3y = 0 \) and we’re given that \( y_1(t) = t^{-1} \) is a solution. We need to find a second solution that will form a fundamental set of solutions with \( y_1 \). Let’s compute the Wronskian both ways.

\[
ct^{-\frac{1}{2}} e^{\frac{1}{2} \ln(t)} = W(t^{-1}, y_2)(t) = y_2' t^{-1} + y_2 t^{-2}
\]

\[
y_2' t^{-1} + y_2 t^{-2} = ce^{-\frac{1}{2}} \ln(t) = ct^{-\frac{1}{2}}
\]
This is a first order linear equation with integrating factor $\mu(t) = e^{\int t^{-1} \, dt} = e^{\ln(t)} = t$. Thus

$$[ty_2]' = \frac{t}{2}$$

$$ty_2 = \frac{2}{5}ct^\frac{3}{2} + k$$

$$y_2(t) = \frac{2}{5}ct^\frac{3}{2} + kt^{-1}$$

Now, we can choose constants $c$ and $k$. Notice that $k$ is the coefficient of $t^{-1}$, which is just $y_1(t)$. So we don’t have to worry about that term, and we can take $k = 0$. We can similarly take $c = \frac{5}{2}$, and so we’ll get $y_2(t) = t^\frac{3}{2}$, which is precisely what we had gotten when we did reduction of order the traditional way.

3. Nonhomogeneous Equations

We’ve finished discussing homogeneous equations. Now, let’s turn to nonhomogeneous equations. A second order linear nonhomogeneous equation has the form

$$p(t)y'' + q(t)y' + r(t)y = g(t), \quad (14.1)$$

where $g(t) \neq 0$. How do we get the general solution to these?

Suppose we have two solutions $Y_1(t)$ and $Y_2(t)$. We should note that the Principle of Superposition no longer holds, as our equation is nonhomogeneous. As a result, we can’t just take linear combinations of $Y_1(t)$ and $Y_2(t)$ and expect to get another solution. First, consider the equation

$$p(t)y'' + q(t)y' + r(t)y = 0, \quad (14.2)$$

which we will call the associated homogeneous equation to Equation (14.1). We then have the following result.

**Theorem 14.2.** Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to Equation (14.1) and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to Equation (14.2). Then $Y_1(t) - Y_2(t)$ is a solution to Equation (14.2) and has the form

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t).$$

This notation will be standard for us: we will use capital letters to denote solutions to Equation (14.1) and lower case letters to denote solutions to Equation (14.2).

Now, how can we verify this theorem? Let’s plug in $Y_1 - Y_2$ to Equation (14.2).

$$p(t)(Y_1 - Y_2)'' + q(t)(Y_1 - Y_2)' + r(t)(Y_1 - Y_2) = 0$$

$$\left(p(t)Y_1'' + q(t)Y_1' + r(t)Y_1\right) - \left(p(t)Y_2'' + q(t)Y_2' + r(t)Y_2\right) = 0$$

$$g(t) - g(t) = 0$$

since $Y_2$ and $Y_2$ solve (14.1)

$$0 = 0$$

So, we have that $Y_1(t) - Y_2(t)$ solves Equation (14.2) Now, we know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to Equation (14.2), and so any solution can be written as a linear combination of them. Thus, for some constants $c_1$ and $c_2$,

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t).$$

So the difference of any two solutions of Equation (14.1) is a solution to Equation (14.2). Let’s suppose we’ve got one solution to Equation (14.1), which we’ll denote by $Y_p(t)$. Let $Y(t)$ denote the general solution. We’ve just seen that we have

$$Y(t) - Y_p(t) = c_1y_1(t) + c_2y_2(t)$$
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or

\[ Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t) \]

where \( y_1 \) and \( y_2 \) are a fundamental set of solutions to \( Y(t) \). We will call

\[ y_c(t) = c_1 y_1(t) + c_2 y_2(t) \]

the *complementary solution* and \( Y_p(t) \) a *particular solution*. So, the general solution can be expressed as

\[ Y(t) = y_c(t) + Y_p(t). \]

Thus, to find the general solution to Equation 14.1, we’ll need to find the general solution to Equation 14.2 (which we always know how to do in the constant coefficient case) and then find *some* solution to Equation 14.1. Adding these two pieces together will give us the general solution to Equation 14.1.

This should make sense. If we vary a solution to Equation 14.1 by just adding in some solution to Equation 14.2, it should still solve Equation 14.1, since the new term will contribute nothing when plugged into the equation. So there’s complete freedom to change solutions by terms solving Equation 14.2, and so its general solution enters into the picture.

The outstanding question now is how to find some particular solution \( Y_p(t) \) to Equation 14.1. There are two methods to do this. One of them, called Undetermined Coefficients, we will discuss at length: it reduces the problem entirely to an algebraic one, but it only works in very select circumstances. The other, called Variation of Parameters, we will touch on: it’s a much more general method, but it requires integration which may not always be doable.
LECTURE 15

Undetermined Coefficients

Let’s start with the method of Undetermined Coefficients. One disadvantage of this method is that it is only really useful for constant coefficient differential equations, so we’ll restrict our attention to equations of the form

\[ ay'' + by' + cy = g(t) \]

for \( g(t) \neq 0 \). The other main disadvantage of this method is that it only works for a fairly small group of \( g(t) \)’s (though these functions include some of the most common ones).

Recall that what we are trying to do is to determine some particular solution \( Y_p(t) \) to Equation 15.1 (there are in fact many different solutions to Equation 15.1: we’re just after one). Then we can add that to the general solution to the associated homogeneous equation to get the general solution to Equation 15.1. The idea behind this method is that, for certain classes of nonhomogeneous terms, we’re able to make a good educated guess as to how \( Y_p(t) \) should look, up to some unknown coefficients (hence the name of the method). Then, we plug our guess into the differential equation and try to solve for the coefficients. If we can, our guess was correct and we have our \( Y_p(t) \). If we can’t solve for the coefficients, then we guessed incorrectly and we will need to try again.

1. The Basic Functions

There are three “groups” of basic types of nonhomogeneous terms \( g(t) \) that are susceptible to this method: exponentials, trig functions (specifically, sin and cos), and polynomials. Once we see how these functions work, we’ll be able to combine them in various ways.

1.1. Exponentials. Let’s walk through an example where \( g(t) \) is an exponential and see how the method works in this case.

Example 15.1. Determine a particular solution to

\[ y'' - 4y' - 12y = 2e^{4t}. \]

How can we make a guess as to the form of \( Y_p(t) \)? When we plug \( Y_p(t) \) into the equation, we should get \( g(t) = 2e^{4t} \). We know that exponentials don’t change (up to a constant) during differentiation, so a reasonable guess might be

\[ Y_p(t) = Ae^{4t} \]

for some coefficient \( A \).

If we plug into the differential equation, we get

\[ 16Ae^{4t} - 4(4Ae^{4t}) - 12Ae^{4t} = 2e^{4t} \]

\[ -12Ae^{4t} = 2e^{4t}. \]

For these to be equal (and thus for our guess to be the solution), we’ll need \( A \) to satisfy

\[ -12A = 2 \quad \Rightarrow \quad A = -\frac{1}{6}. \]
So with this choice of $A$, our guess works, and the particular solution is

$$Y_p(t) = -\frac{1}{6}e^{4t}.$$ 

Let’s do an example of a full initial value problem so that we can go through the whole process once.

**Example 15.2.** Solve the IVP

$$y'' - 4y' - 12y = 2e^{4t}, \quad y(0) = -\frac{13}{6}, \quad y'(0) = \frac{7}{3}.$$ 

We know that the general solution has the form

$$y(t) = y_c(t) + Y_p(t)$$

where the complimentary solution $y_c(t)$ is the general solution to the associated homogeneous equation

$$y'' - 4y' - 12y = 0$$

and $Y_p(t)$ is a particular solution to the original differential equation. From the previous example, we know that we can take

$$Y_p(t) = -\frac{1}{6}e^{4t}.$$ 

What is the complimentary solution? Our associated homogeneous equation has constant coefficients, so we need to find the roots of the characteristic equation.

$$r^2 - 4r - 12 = 0$$

$$(r - 6)(r + 2) = 0.$$ 

So we conclude that $r_1 = 6$ and $r_2 = -2$. These are distinct real roots, so the complimentary solution will be

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}.$$ 

We must be careful here: the initial conditions are for the original, nonhomogeneous equation, not for the associated homogeneous equation. We absolutely do not want to apply them at this stage to $y_c$, since that is not the (or in fact, a) solution to the original equation.

Our general solution is the sum of $y_c(t)$ and $Y_p(t)$. We’ll need it and its derivative to apply the initial conditions.

$$y(t) = c_1e^{6t} + c_2e^{-2t} - \frac{1}{6}e^{4t}$$

$$y'(t) = 6c_1e^{6t} - 2c_2e^{2t} - \frac{2}{3}e^{4t}.$$ 

Now we apply our initial conditions.

$$-\frac{13}{6} = y(0) = c_1 + c_2 - \frac{1}{6}$$

$$\frac{7}{3} = y'(0) = 6c_1 - 2c_2 - \frac{2}{3}$$

This system is solved by $c_1 = -\frac{1}{8}$ and $c_2 = -\frac{15}{8}$, so our solution is

$$y(t) = -\frac{1}{8}e^{6t} - \frac{15}{8}e^{-2t} - \frac{1}{6}e^{4t}.$$ 

---

**1.2. Trig Functions.** The second class of nonhomogeneous terms for which we can use this method are trig functions, specifically sin and cos.
Example 15.3. Find a particular solution for the following IVP:
\[ y'' - 4y' - 12y = 6 \cos(4t). \]

In the first example, our nonhomogeneous term was an exponential, and we know that when we differentiate exponentials they persist. In this case, we’ve got a cosine function. When we differentiate a cosine, we get a sine. So we might expect our initial guess to require a sine term in addition to a cosine. Let’s try it: set
\[ Y_p(t) = A \cos(4t) + B \sin(4t). \]

Next, we differentiate and plug in.
\[
-16A \cos(4t) - 16B \sin(4t) - 4 (-4A \sin(4t) + 4B \cos(4t)) - 12 (A \cos(4t) + B \sin(4t)) = 13 \cos(4t)
\]
\[
(-16A - 16B - 12A) \cos(4t) + (-16B + 16A - 12B) \sin(4t) = 13 \cos(4t)
\]
\[
(-28A - 16B) \cos(4t) + (16A - 28B) \sin(4t) = 13 \cos(4t)
\]

To solve for \( A \) and \( B \), we now set coefficients equal. Note that the coefficient of \( \sin(4t) \) on the right hand side is 0. So we get the system of equations
\[
\begin{align*}
\cos(4t) : & \quad -28A - 16B = 13 \\
\sin(4t) : & \quad 16A - 28B = 0.
\end{align*}
\]

This system is solved by \( A = -\frac{7}{20} \) and \( B = -\frac{1}{5} \). So a particular solution is
\[ Y_p(t) = -\frac{7}{20} \cos(4t) - \frac{1}{5} \sin(4t). \]

It’s worth noting here that the guess would have been the same if \( g(t) \) had been a sine instead of a cosine.
LECTURE 16

Undetermined Coefficients 2: Electric Boogaloo

1. Recap

Last class, we saw that if we have a nonhomogeneous equation

\[ p(t)y'' + q(t)y' + r(t)y = g(t), \]

the general solution has the form

\[ y(t) = y_c(t) + Y_p(t), \]

where \( y_c(t) = c_1y_1(t) + c_2y_2(t) \), the complimentary solution, is the general solution to the homogeneous equation

\[ p(t)y'' + q(t)y' + r(t)y = 0 \]

and \( Y_p(t) \), a particular solution, is some solution to the original equation.

When the coefficients of the homogeneous equation are constant, we can use the method of Undetermined Coefficients, which works for certain nonhomogeneous terms \( g(t) \). With this method, we make a guess at the form of a particular solution \( Y_p(t) \), leaving the coefficients undetermined. Then we plug this guess into the differential equation and do some algebra to calculate what the coefficients should be.

There are three basic types of nonhomogeneous terms that this method works for: exponentials, sines and cosines, and polynomials. If \( g(t) = ae^{\alpha t} \), we make the guess \( Y_p(t) = Ae^{\alpha t} \). If \( g(t) = a\sin(\alpha t) \) or \( a\cos(\alpha t) \), our guess is \( Y_p(t) = A\cos(\alpha t) + B\sin(\alpha t) \).

2. Polynomials

The third and final basic class of nonhomogeneous term we can use this method with are polynomials.

Example 16.1. Find a particular solution to

\[ y'' - 4y' - 12y = 3t^3 - 5t + 2. \]

In this case, \( g(t) \) is a cubic polynomial. When we differentiate a polynomial, its order decreases. So if our initial guess is a general cubic, we should be able to capture all of the terms that will arise from differentiating. We guess

\[ Y_p(t) = At^3 + Bt^2 + Ct + D. \]

Now, let’s differentiate and plug in.

\[
6At + 2B - 4\left(3At^2 + 2Bt + C\right) - 12\left(At^3 + Bt^2 + Ct + D\right) = 3t^3 - 5t + 2
\]

\[
-12At^3 + (-12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) = 3t^3 - 5t + 2
\]
We obtain a system of equations by setting coefficients equal.

\[ t^3 : \quad -12A = 3 \quad \Rightarrow \quad A = -\frac{1}{4} \]

\[ t^2 : \quad -12A - 12B = 0 \quad \Rightarrow \quad B = \frac{1}{4} \]

\[ t^1 : \quad 6A - 8B - 12C = -5 \quad \Rightarrow \quad C = \frac{1}{8} \]

\[ t^0 : \quad 2B - 4C - 12D = 2 \quad \Rightarrow \quad D = -\frac{1}{6} \]

So a particular solution is

\[ Y_p(t) = -\frac{1}{4} t^3 + \frac{1}{4} t^2 + \frac{1}{8} t - \frac{1}{6}. \]

To be more general, if \( g(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \), a polynomial of order \( n \), then our guess is that a particular solution is also an \( n \)th degree polynomial, i.e., \( Y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0 \).

Now, let’s summarize what we’ve seen so far.

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>form of ( Y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a e^{\alpha t} )</td>
<td>( A e^{\alpha t} )</td>
</tr>
<tr>
<td>( a \sin(\alpha t) )</td>
<td>( A \cos(\alpha t) + B \sin(\alpha t) )</td>
</tr>
<tr>
<td>( a \cos(\alpha t) )</td>
<td>( A \cos(\alpha t) + B \sin(\alpha t) )</td>
</tr>
<tr>
<td>( a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 )</td>
<td>( A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0 )</td>
</tr>
</tbody>
</table>

We’re now going to look at more complicated nonhomogeneous terms. For this method to work, they must always come from products and sums of the basic types.

### 3. Products

The basic idea is that the guess for the product is the product of the guesses. We’ll illustrate this through an example.

**Example 16.2.** Find a particular solution to

\[ y'' - 4y' - 12y = te^{4t}. \]

Let’s start by writing down the guesses for each of the individual pieces. In this example, \( g(t) \) is the product of a 1st order polynomial, \( t \), and an exponential, \( e^{4t} \). The guess for the polynomial is \( At + B \) while the guess for the exponential is \( Ce^{4t} \). This yields a guess of

\[ Ce^{4t}(At + B). \]

We want to minimize the number of constants, though, and we can see that

\[ Ce^{4t}(At + B) = e^{4t}(ACt + BC). \]

This gives us really two constants, so we’ll write the guess as

\[ Y_p(t) = e^{4t}(At + B) \]

Notice that this is just the guess for the \( t \) with an exponential tacked on.
Let’s differentiate and plug in.
\[
16e^{4t}(At + B) + 8Ae^{4t} - 4(4e^{4t}(At + B) + Ae^{4t}) - 12e^{4t}(At + B) = te^{4t}
\]
\[
(16A - 16A - 12A)te^{2t} + (16B + 8A - 16B - 4A - 12B)e^{4t} = te^{4t}
\]
\[-12Ate^{4t} + (4A - 12B)e^{4t} = te^{4t}
\]
Then we set coefficients equal.
\[
(te^{4t}) : \quad -12A = 1 \quad \Rightarrow \quad A = -\frac{1}{12}
\]
\[
(e^{4t}) : \quad (4A - 12B) = 0 \quad \Rightarrow \quad B = -\frac{1}{36}
\]
So, a particular solution for this differential equation is
\[
Y_p(t) = e^{4t} \left( -\frac{1}{12}t - \frac{1}{36} \right) = -\frac{e^{4t}}{36} (3t + 1).
\]

The basic rule of thumb when we have a product involving an exponential is that we write down the guess for the other part of the product, then multiply that by the exponential without any leading coefficient. This allows us to not have to explicitly multiply through as we did in the previous example.

Let’s do another.

**Example 16.3.** Find a particular solution to
\[
y'' - 4y' - 12y = 29e^{5t} \sin(3t).
\]
Using the rule of thumb from before, we write down the guess for \(\sin(3t)\) and tack on \(e^{5t}\). This gives a guess of
\[
Y_p(t) = e^{5t}(A \cos(3t) + B \sin(3t)).
\]
So, let’s differentiate and plug in.
\[
25e^{5t}(A \cos(3t) + B \sin(3t)) + 30e^{5t}(-A \sin(3t) + B \cos(3t)) +
9e^{5t}(-A \cos(3t) - B \sin(3t)) - 4(5e^{5t}(A \cos(3t) + B \sin(3t)) +
3e^{5t}(-A \sin(3t) + B \cos(3t))) - 12e^{5t}(A \cos(3t) + B \sin(3t)) = 29e^{5t} \sin(3t)
\]
Next, gather like terms.
\[
(-16A + 18B)e^{5t} \cos(3t) + (-18A - 16B)e^{5t} \sin(3t) = 29e^{5t} \sin(3t)
\]
Set coefficients equal.
\[
(e^{5t} \cos(3t)) : \quad -16A + 18B = 0
\]
\[
(e^{5t} \sin(3t)) : \quad -18A - 16B = 29
\]
This is solved by \(A = -\frac{9}{10}\) and \(B = -\frac{4}{5}\). So a particular solution to this differential equation is
\[
Y_p(t) = e^{5t} \left( -\frac{9}{10}t - \frac{4}{5} \right) = -\frac{e^{5t}}{10} (9t + 8).
\]
LECTURE 17

Undetermined Coefficients Beyond Thunderdome

1. Recap (again!)

The point of the method of Undetermined Coefficients is to make a guess at the form of a particular solution \(Y_p(t)\) of a nonhomogeneous equation based on the form of the nonhomogeneous term \(g(t)\). This method works for exponentials (if \(g(t) = ae^{\alpha t}\), we guess \(Y_p(t) = Ae^{\alpha t}\)), trig functions (if \(g(t) = a\cos(\alpha t)\) or \(a\sin(\alpha t)\), we guess \(Y_p(t) = A\cos(\alpha t) + B\sin(\alpha t)\)), and polynomials (if \(g(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0\), we guess \(Y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0\)).

We saw last class that if \(g(t)\) is a product of these basic types, our guess for \(Y_p(t)\) is the product of the guesses for these basic types, with care taken to make sure we don’t have any extraneous coefficients.

Let’s practice writing down the guesses for a couple of these.

Example 17.1. Write down the form of the particular solution to \(y'' - 4y' - 12y = g(t)\) for the following \(g(t)\)s:

1. \(g(t) = (t^3 - 45t) \cos(4t)\)
   
   Here we’ve got the product of a quadratic and a cosine. The guess for the quadratic is \(At^3 + Bt^2 + Ct + D\) and the guess for the cosine is \(E \cos(4t) + F \sin(4t)\).
   
   Multiplying the two guesses gives
   \[
   (At^3 + Bt^2 + Ct + D)(E \cos(4t)) + (At^3 + Bt^2 + Ct + D)(F \sin(4t))
   \]
   \[
   (AEt^3 + BEt^2 + CEt + DE) \cos t + (AFt^3 + B Ft^2 + CFt + DF) \sin t.
   \]
   
   Each of our coefficients here is a product of two constants, which is just another constant. So, as before, to simplify everything, we’ll replace each of those with a single constant to yield the following guess.
   \[
   Y_p(t) = (At^3 + Bt^2 + Ct + D) \cos t + (Et^3 + Ft^2 + Gt + H) \sin t.
   \]
   
   This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiplied by a cosine, then add to that the guess for the polynomial (with different constants!) multiplied by a sine.

2. \(g(t) = e^{2t}(2 - 5t) \cos(6t)\)
   
   This nonhomogeneous term has all three things. So, combining the two general rules from before, we get
   \[
   Y_p(t) = e^{2t}(At + B) \cos(6t) + e^{2t}(Ct + D) \sin(6t).
   \]
2. Sums

We have the following important fact. If $Y_1$ satisfies
\[ p(t)y'' + q(t)y' + r(t)y = g_1(t) \]
and $Y_2$ satisfies
\[ p(t)y'' + q(t)y' + r(t)y = g_2(t) \]
, then $Y_1 + Y_2$ satisfies
\[ p(t)y'' + q(t)y' + r(t)y = g_1(t) + g_2(t). \]
This means that if our nonhomogeneous term $g(t)$ is a sum of terms we know how to deal with, we can write down the guesses for each of those terms and add them together for our guess. We may just need to be careful about redundant terms.

**Example 17.2.** Find a particular solution to
\[ y'' - 4y' - 12y = e^{7t} + 12. \]
Our nonhomogeneous term $g(t) = e^{7t} + 12$ is the sum of an exponential $g_1(t) = e^{7t}$ and a 0th degree polynomial $g_2(t) = 12$. The guess for $g_1(t)$ is
\[ Ae^{7t} \]
while the guess for $g_2(t)$ is
\[ B. \]
Adding these together gives
\[ Ae^{7t} + B. \]
This can’t be simplified in any way, so based on our previous fact we’ll go ahead and guess
\[ Y_p(t) = Ae^{7t} + B \]
. Differentiating and plugging in yields
\[ 49Ae^{7t} - 28Ae^{7t} - 12Ae^{7t} - 12B = e^{7t} + 12 \]
\[ 9Ae^{7t} - 12B = e^{7t} + 12. \]
Setting coefficients equal gives $A = \frac{1}{9}$ and $B = -1$, so our particular solution is
\[ Y_p(t) = \frac{1}{9}e^{7t} - 1. \]

Let’s practice writing down some guesses.

**Example 17.3.** Write down the form of a particular solution to
\[ y'' - 4y' - 12y = g(t) \]
for each of the following $g(t)$’s:
(1) $g(t) = 2\cos(3t) - 9\sin(3t)$
Our guess for the cosine is
\[ A\cos(3t) + B\sin(3t). \]
Additionally, our guess for the sine is
\[ C\cos(3t) + D\sin(3t). \]
So if we add the two of them together, we obtain
\[ A\cos(3t) + B\sin(3t) + C\cos(3t) + D\sin(3t) = (A + C)\cos(3t) + (B + D)\sin(3t). \]
But $A + C$ and $B + D$ are just some constants, so we can replace them as such as we just yet a guess of

$$Y_p(t) = A \cos(3t) + B \sin(3t).$$

(2) $g(t) = \sin(t) - 2 \sin(14t) - 5 \cos(14t)$

Again, we start by writing down the guesses for the individual terms. The guess for $\sin(t)$ is

$$A \cos(t) + B \sin(t).$$

Since they have the same argument, the previous example showed us that we can combine the guesses for $2 \sin(14t)$ and $-5 \cos(14t)$ into

$$C \cos(14t) + D \sin(14t).$$

We won’t be able to further combine anything, since the trig functions here have different arguments, and so we end up with a guess of

$$Y_p(t) = A \cos(t) + B \sin(t) + C \cos(14t) + D \sin(14t).$$

(3) $g(t) = 7 \sin(10t) - 5t^2 + 4t$

Here we have the sum of a trig function ($7 \sin(10t)$) and a quadratic polynomial ($-5t^2 + 4t$).

The guess for the sine is

$$A \cos(10t) + B \sin(10t).$$

While the guess for the quadratic is

$$Ct^2 + Dt + E.$$

Since there’s nothing to be consolidated, our guess is

$$Y_p(t) = A \cos(10t) + B \sin(10t) + Ct^2 + Dt + E.$$

(4) $g(t) = 9e^t + 3te^{-5t} - 5e^{-5t}$

If we write down the sum of each of the individual guesses, we would obtain a guess of

$$Ae^t + (Bt + C)e^{-5t} + De^{-5t} = Ae^t + (Bt + (C + D))e^{-5t}.$$

Of course, $C + D$ is just another constant, so our final guess would be

$$Y_p(t) = Ae^t + (Bt + C)e^{-5t}.$$

We could also have noticed that $g(t)$ could be rewritten as

$$g(t) = 9e^t + (3t - 5)e^{-5t},$$

in which case writing down the sum of our guesses directly works out to what we just obtained through a slightly more indirect approach.

(5) $g(t) = t^2 \sin(t) + 4 \cos(t)$

Writing down the sum of the guesses gives

$$(At^2 + Bt + C) \cos(t) + (Dt^2 + Et + F) \sin(t) + G \cos(t) + H \sin(t),$$

which simplifies to

$$(At^2 + Bt + (C + G)) \cos(t) + (Dt^2 + Et + (F + H)) \sin(t).$$

Once again, since $C + G$ and $F + H$ are just constants, we end up with

$$Y_p(t) = (At^2 + Bt + C) \cos(t) + (Dt^2 + Et + F) \sin(t).$$

The last two examples have shown us that if we have two terms in our $g(t)$ whose guesses differ only by a polynomial factor, then we can just look for the term corresponding to the highest degree polynomial and the guess for that term will include the guess for the second one. Without making this observation, of course, we can just do what we did here and observe that certain coefficients combine together.
(6) $g(t) = 3e^{-3t} + e^{-3t} \sin(3t) + \cos(3t)$.

We’ll need to start here by writing down the sums of each of the guesses:

$$Ae^{-3t} + e^{-3t}(B \cos(3t) + C \sin(3t)) + D \cos(3t) + E \sin(3t).$$

Notice that we can’t combine anything, since the various terms that look similar differ by nonconstant factors, not just constant coefficients. So our guess is just

$$Y_p(t) = Ae^{-3t} + e^{-3t}(B \cos(3t) + C \sin(3t)) + D \cos(3t) + E \sin(3t).$$

As we’ve seen, sums are quite straightforward to deal with as long as we’re on the look out for redundant terms that can be combined.

So, that’s all for undetermined coefficients, right? Well...not quite. There’s one potential complication that we need to be able to deal with.

**Example 17.4.** Find a particular solution to

$$y'' - 4y' - 12y = e^{6t}.$$ 

This seems straightforward enough: our nonhomogeneous term is an exponential, so we just guess $Y_p(t) = Ae^{6t}$. If we differentiate and plug in, we get

$$36Ae^{6t} - 24Ae^{6t} - 12Ae^{6t} = e^{6t}$$

$$0 = e^{6t}.$$ 

Exponentials are never zero. So this is no good at all. We weren’t able to solve for our coefficient, which means we made a mistake in our original guess. But what?

Recall that the complimentary solution in this case is

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}.$$ 

So our guess is just the first component of our complimentary solution, which is the solution to

$$y'' - 4y' - 12y = 0.$$ 

As a result, we should have expected that this guess would give us 0 when we plugged it into the left hand side of our equation, since it’s a solution to the associated homogeneous equation. So how do we fix this?

Our guess should still involve the exponential $e^{6t}$, but it can’t just be a constant times this exponential. The next thing to try might be $Y_p(t) = At e^{6t}$, since all we’ve done to our original guess is to multiply by a linear factor. Let’s try it.

$$(36Ate^{6t} + 12Ae^{6t}) - 4(6Ate^{6t} + Ae^{6t}) - 12Ate^{6t} = e^{6t}$$

$$(36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} = e^{6t}$$

$$8Ae^{6t} = e^{6t}$$

Setting coefficients equal, we conclude that $A = \frac{1}{8}$, so

$$Y_p(t) = \frac{1}{8}te^{6t}.$$ 

Notice in Example 17.4 that when we plugged in our new guess, the terms with the $t$ all vanished. This is just a consequence of the product rule: those terms corresponded to differentiating only $e^{6t}$, and we already saw that those vanish when plugged into the DE. This is why multiplying by $t$ works: the $t$ goes away and the product rule gives us some new terms which don’t cancel.

What is the lesson of Example 17.4? We had a problem with our previous guessing method because it led us to guess a term in the complimentary solution. The solution, then, is to find the
complimentary solution first, write down guesses, and after comparing the two equations multiply any redundant terms by $t$.

We do have to be slightly more careful than just that, though: remember that in the repeated roots case, our complimentary solution has the form

$$y_c(t) = c_1 e^{rt} + c_2 te^{rt}.$$ 

If our $g(t) = e^{rt}$, multiplication by $t$ won’t be enough, since $te^{rt}$ is also a term in the complimentary solution. So we’ll actually need to multiply our guess by $t^2$, giving a final guess of

$$Y_p(t) = At^2 e^{rt}.$$
LECTURE 18

Undetermined Coefficients: The Revenge and Variation of Parameters

1. This Is It For Undetermined Coefficients, I Promise

We ended last class by noticing that if part of our guess for a particular solution $Y_p(t)$ coincided with a bit of the complimentary solution, we needed to multiply the offending part by $t$ to ensure that it doesn’t vanish when we plug it into the equation. Let’s look at some examples.

**Example 18.1.** Write down a guess for the form of a particular solution to the following differential equations.

1. $y'' - y' - 20y = 6t + e^{-4t} - 2$

   First, we find the complimentary solution. It is $y_c(t) = c_1 e^{5t} + c_2 e^{-4t}$.

   Our nonhomogeneous term is $g(t) = 6t + e^{-4t} - 2$, which we can rearrange to group the polynomial terms together as $g(t) = 6t - 2 + e^{-4t}$. Thus our initial guess would be $At + B + C e^{-4t}$.

   The first two terms aren’t a problem, but the $Ce^{-4t}$ term also appears in the complimentary solution. Since $Cte^{-4t}$ doesn’t show up in the complimentary solution, our final guess is $Y_p(t) = At + B + C e^{-4t}$.

2. $y'' - 64y = t^2 e^{8t} + \cos(t)$

   The complimentary solution is $y_c(t) = c_1 e^{8t} + c_2 e^{-8t}$.

   Our initial guess for a particular solution is $(At + Bt + C)e^{8t} + D \cos(t) + E \sin(t)$.

   If we distributed the exponential through the polynomial, we’d have a $Ce^{8t}$ term which also showed up in our complimentary solution. What we’ll need to do is to multiply the *entire* first term by $t$ (to see why, just differentiate and plug in...you’ll see that if we don’t, we’ll end up losing a coefficient which we’ll need later). So our final guess is $Y_p(t) = t(At^2 + Bt + C)e^{8t} + D \cos(t) + E \sin(t)$.

3. $y'' + 4y' = e^{-t} \cos(2t) + t \sin(2t)$

   The complimentary solution is $y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)$.

   Our first guess for a particular solution would be $e^{-t}(A \cos(2t) + B \sin(2t)) + (Ct + D) \cos(2t) + (Et + F) \sin(2t)$.

   First, we notice that despite having similar looking terms, we can’t actually combine anything, since the similar terms are multiplied by factors which don’t just differ by constant coefficients.

   Next, we notice that both the second and third terms involve components of the complimentary
solution: $D \cos(2t)$ and $F \sin(2t)$. Thus we’ll need to multiply those two terms by $t$. The first term is ok, though, since if we multiplied it out we would have a product of an exponential and a sine or a cosine, and those aren’t terms in the complimentary solution. So we end up with

$$Y_p(t) = e^{-t}(A \cos(2t) + B \sin(2t)) + t(Ct + D) \cos(2t) + t(Et + F) \sin(2t).$$

(4) $y'' + 2y' + 5y = e^{-t} \cos(2t) + t \sin(2t)$

Notice that the nonhomogeneous term in this example is the same as in the previous one; we’ve just changed the differential equation. Here, the complimentary solution is

$$y_c(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t).$$

Our initial guess for the particular solution is the same as in the last example:

$$e^{-t}(A \cos(2t) + B \sin(2t)) + (Ct + D) \cos(2t) + (Et + F) \sin(2t).$$

This time, the first term causes the problem, while the second and third are fine just as they are. So we’ll multiply the first by $t$:

$$Y_p(t) = te^{-t}(A \cos(2t) + B \sin(2t)) + (Ct + D) \cos(2t) + (Et + F) \sin(2t).$$

(5) $y'' + 4y' + 4y = t^2 e^{-2t} + 2 e^{-2t}$

Here the complimentary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Notice that we can factor out a $e^{-2t}$ from our nonhomogeneous term, which then becomes $g(t) = (t^2 + 2) e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2 + Bt + C)e^{-2t}.$$ But this $Ce^{-2t}$ term is the first term in our complimentary solution. Also, we have $Bt e^{-2t}$, which is the second term in the complimentary solution. So this is no good. Next, we try multiplying by $t$:

$$t(At^2 + Bt + C)e^{-2t}.$$

This still causes problems: the $Ct e^{-2t}$ term is still the second term in our complimentary solution. If we multiply by $t^2$, though, we have no problems, and so our final guess is

$$Y_p(t) = t^2(At^2 + Bt + C)e^{-2t}.$$ 

So that’s about it for Undetermined Coefficients. As long as you’re comfortable with the guesses for the basic types, are on the lookout for coefficients that should be combined, and check that you’re not replicating any part of the complimentary solution, you can’t go wrong.

### 2. Variation of Parameters

The other major method for determining a particular solution to the linear nonhomogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

is Variation of Parameters. Undetermined Coefficients works well enough, but it only works for a handful of equations.

Variation of Parameters is a much more general method, but it has its drawbacks. We could sometimes do Undetermined Coefficients without the complimentary solution, but this is never possible for Variation of Parameters. Also, while Undetermined Coefficients reduces the problem of finding a particular solution to an algebraic one, Variation of Parameters involves taking some integrals, which we may not always be able to do. So we’ll always be able to write down a formula
for the solution, but we may not be able to explicitly find it. Still, its generality makes it worth discussing.

How does it work? For this method, we start by dividing through by \( p(t) \): we’ll want to start with \( y'' \) having a coefficient of 1. So the equation we’re dealing with is really

\[
y'' + q(t)y' + r(t)y = g(t). \tag{18.1}
\]

Next, suppose we know the complimentary solution

\[
y_c(t) = c_1y_1(t) + c_2y_2(t). \tag{18.2}
\]

Recall that this is the general solution to the homogeneous equation

\[
y'' + q(t)y' + r(t)y = 0.
\]

We’re going to see if we can find two functions \( u_1(t) \) and \( u_2(t) \) such that

\[
Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{18.3}
\]

This is similar in spirit to doing reduction of order: we know that combinations of \( y_1 \) and \( y_2 \) with constant coefficients gives solutions to Equation 18.2, so the next thing to try is a combination of \( y_1 \) and \( y_2 \) involving nonconstant functions.

However, to do this, we need to make an assumption. If we differentiate Equation 18.3, we obtain

\[
Y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.
\]

The assumption we will make is that

\[
u_1'y_1 + u_2'y_2 = 0.
\]

Why do we make this assumption? Unfortunately, I don’t have a good answer for this. It turns out that it works: it simplifies the expression we’ll end up getting so that we can solve it, and it doesn’t interfere with obtaining a solution. With this assumption, the first derivative becomes

\[
Y_p' = u_1'y_1 + u_2y_2'.
\]

Then the second derivative is

\[
Y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2'' + u_2y_2'.'
\]

Let’s plug the derivatives into Equation 18.1.

\[
(u_1'y_1 + u_1'y_1 + u_2'y_2' + u_2'y_2') + q(t) (u_1'y_1 + u_2'y_2') + r(t) (u_1'y_1 + u_2'y_2) = g(t)
\]

Let’s rearrange this to take advantage of \( y_1 \) and \( y_2 \) solving Equation 18.2:

\[
(u_1'y_1' + u_2'y_2') + u_1 (y_1'' + q(t)y_1' + r(t)y_1) + u_2 (y_2'' + q(t)y_2' + r(t)y_2) = g(t)
\]

\[
\begin{align*}
u_1'y_1' + u_2'y_2' & = g(t) \\
u_1'y_1' + u_2'y_2' & = g(t)
\end{align*}
\]

This is why we wanted the coefficient of \( y'' \) to be 1. Otherwise, we would have a fraction on the right hand side of the previous equation, which would in general be nasty.

Remember that we started by supposing that the form of a particular solution was

\[
Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).
\]

So we’re trying to find these unknown functions \( u_1(t) \) and \( u_2(t) \). We have two conditions on these functions:

\[
\begin{align*}
u_1'y_1 + u_2'y_2 & = 0 \\
u_1'y_1' + u_2'y_2' & = g(t)
\end{align*}
\]
Let’s solve this system of equations.

\[
\begin{align*}
    u_1' &= -\frac{u_2'y_2}{y_1} \\
    \left( -\frac{u_2'y_2}{y_1} \right) y_1' + u_2'y_2' &= g(t) \\
    u_2'(y_2' - \frac{y_2'y_1}{y_1}) &= g(t) \\
    u_2' \left( \frac{y_1'y_2 - y_2'y_1}{y_1} \right) &= g(t) \\
    u_2' &= \frac{y_1g(t)}{y_1'y_2 - y_2'y_1} \\
    u_1' &= -\frac{y_2g(t)}{y_1'y_2 - y_2'y_1}
\end{align*}
\]

Notice that the quantity in these denominators is the Wronskian of \(y_1\) and \(y_2\), \(W(y_1, y_2)\). We know that the Wronskian is nonzero because we started by assuming that we knew \(y_1\) and \(y_2\) were a fundamental set of solutions. By integrating, we get

\[
\begin{align*}
    u_1(t) &= -\int \frac{y_2g(t)}{W(y_1, y_2)} dt \\
    u_2(t) &= \int \frac{y_1g(t)}{W(y_1, y_2)} dt
\end{align*}
\]

To sum, if we consider the differential equation

\[
y'' + q(t)y' + r(t)y = g(t)
\]

with complimentary solution

\[
y_c(t) = c_1y_1(t) + c_2y_2(t),
\]

then a particular solution is given by

\[
Y_p(t) = -y_1 \int \frac{y_2g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1g(t)}{W(y_1, y_2)} dt.
\]

Before we do some examples, let’s see what will happen to the constants of integration. If we explicitly put them in, we’ll get

\[
Y_p(t) = -y_1 \left( \int \frac{y_2g(t)}{W(y_1, y_2)} dt + c \right) + y_2 \left( \int \frac{y_1g(t)}{W(y_1, y_2)} dt + k \right)
\]

But we know that this final quantity in parentheses will contribute zero when plugged into Equation (18.1) for any \(c\) and \(k\), so it doesn’t end up mattering what the constants of integration are. Thus we will assume that they are zero.

Now, let’s do an example. In this example, we’ll still deal with constant coefficient equations, but you should be aware that this method will, in principle, work for all differential equations of the form (18.1).

**Example 18.2.** Find a general solution to the nonhomogeneous equation

\[
y'' - 2y' + y = \frac{e^t}{t^2 + 1}.
\]

First, we need the complimentary solution. In this case, it’s

\[
y_c(t) = c_1e^t + c_2te^t.
\]
Thus we have $y_1(t) = e^t$ and $y_2(t) = te^t$. We need the Wronskian of these two functions, which is
\[ W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^t(e^t + te^t) - e^t(te^t) = e^{2t}. \]

Then the particular solution is
\[
Y_p(t) = -e^t \int \frac{te^t}{e^{2t}(t^2 + 1)} dt + te^t \int \frac{e^t}{e^{2t}(t^2 + 1)} dt \\
= -e^t \int \frac{t}{t^2 + 1} dt + te^t \int \frac{1}{t^2 + 1} dt \\
= -\frac{1}{2} e^t \ln(t^2 + 1) + te^t \arctan(t).
\]

Thus the general solution is
\[ Y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} e^t \ln(t^2 + 1) + te^t \arctan(t). \]
LECTURE 19

Mechanical Vibrations

1. An Application!

We’re finally at a point where we can stop discussing solutions and properties of second order linear equations. Now, we’ll turn our attention to an application: mechanical vibrations. The particular application we’ll be considering is an object of given mass $m$ hanging from a spring of natural length $l$, but there are a number of applications in various branches of engineering that only differ from our setup by some specifics and notation.

Our convention is always that downward displacements and forces are positive, while upward displacements and forces are negative. It’s important to be consistent. We also measure all displacements of the object from its equilibrium position. Thus, if our displacement function is $u(y)$, $u = 0$ corresponds to the center of gravity as it hangs at rest from the spring. These conventions are depicted in Figure 19.1.

The first task is to develop a differential equation to model the displacement $u$ of the object. First, recall Newton’s Second Law

$$ma = F,$$

where $m$ is the mass of the object. We want our equation to be for displacement, so we’ll replace $a$ by $u''$, and Newton’s Second Law becomes

$$mu'' = F(t, u, u').$$

We now need to understand the various forces acting on the system. Four forces will be considered. Two (the gravitational force and the spring force) will act on the system in every scenario, while

![Figure 19.1](image_url)

**Figure 19.1.** Illustration of the lengths and sign conventions used spring-mass problems. $l$ is the natural spring length and $L$ the additional length induced by the mass. A displacement of 0, denoted by $u = 0$, is at the center of gravity of the mass at rest, and downward displacements from this resting position are taken to be positive ($u > 0$).
the other two (a damping force and an external force) will act on certain scenarios in various combinations.

1. **Gravity, \(F_g\)**
   The gravitational force always acts on the object. It’s given by
   \[ F_g = mg, \]
   where \(g\) is the acceleration due to gravity. For simpler computations, we’ll take \(g = 10\text{m/s}^2\). Notice that this force is positive, since it always acts downward.

2. **Spring, \(F_s\)**
   Since we’re attaching the object to a spring, the spring will itself always exert a force on the object. We’ll assume Hooke’s Law governs this force. Hooke’s Law says that the spring force is proportional to the displacement of the spring from its natural length. What is the displacement of the spring? When we attach an object to a spring, the spring gets stretched. Let’s denote the additional length of the stretched spring by \(L\). Then its displacement from its natural length is \(L + u\).
   So, the spring force is given by
   \[ F_s = -k(L + u), \]
   where \(k > 0\) is the **spring constant**. Why is that negative there? It ensures that the force has the correct direction. If \(u > -L\), i.e., the spring has been stretched beyond its natural length, then \(u + L > 0\) and so \(F_s < 0\), which is good because we would expect the spring to pull upward on the object in this situation. If \(u < -L\), so that the spring is compressed, the spring force would push the object back downwards, and so we expect to find \(F_s > 0\). Surely enough, that’s what we get.

3. **Damping, \(F_d\)**
   We will consider some situations where a damper is attached to the system. They will not always be present, but we’ll always note when there is a damper involved. Dampers work to counteract motion (a good example of one is the shocks on your car), so that they will always oppose the direction of the object’s velocity.
   In other words, if the object has downward velocity \(u' > 0\), we’d want the damping force to be acting in the upwards direction, so that \(F_d < 0\). Similarly, if \(u' < 0\), we want \(F_d > 0\). We’ll further assume that all of our dampers are linear. So we end up with
   \[ F_d = -\gamma u', \]
   where \(\gamma > 0\) is the **damping coefficient**.

4. **External Forces, \(F(t)\)**
   This is more of a catchall than a particular force. Sometimes we’ll want our spring-mass system to have some external forces acting on it (for example, we might hook our system up to a generator which will exert some additional force on it). We call \(F(t)\) the **forcing function**, and it’s just the sum of any of these external forces we have in a particular situation.
   It’s important to reiterate here that in a given problem, all of these forces may not necessarily act on the spring-mass system. So our force function will change depending on the particular situation. When actually writing down our differential equation for a particular situation, we’ll need to be aware of what forces are and are not present. But let’s go ahead and write down the most general form of our equation, and we’ll discuss how it changes in particular cases.
   We have \(F(t, u, u') = F_g + F_s + F_d + F(t)\), so that Newton’s Second Law becomes
   \[ mu'' = mg - k(L + u) - \gamma u' + F(t), \]
   or, upon reordering this a little,
   \[ mu'' + \gamma u' + ku = mg - kL + F(t). \]
Differential Equations Lecture 19: Mechanical Vibrations

Let’s now think about what happens when the object is at rest. When the object is at equilibrium $u = 0$, there are only two forces acting on the object: gravity and the spring force. Since the object is at rest, these two forces must be balancing each other out: $F_g + F_s = 0$. In other words, $mg = kL$.

So our equation simplifies to

$$mu'' + \gamma u' + ku = F(t),$$ \hspace{1cm} (19.1)

and this is the most general form of our equation, with all forces present. Along with this differential equation, we’ll have initial conditions of the form

- $u(0) = u_0$, Initial displacement from the equilibrium position
- $u'(0) = u'_0$, Initial velocity.

It’s important to keep our sign conventions in mind when writing these down.

Before we start talking about particular cases, we need to discuss how we might go about figuring out these constants $k$ and $\gamma$ if they’re not explicitly given to us in a problem. Let’s start with the spring constant $k$. We know that if the spring is attached to some object with mass $m$, the object stretches the spring by some length $L$ when it’s at rest. We observed above that at equilibrium, $mg = kL$. Thus, if we know how much some object with a known mass stretches the spring when it’s at rest, we can compute

$$k = \frac{mg}{L}.$$ 

This may not necessarily be the same object as the one in the spring-mass system, but that doesn’t matter.

How do we compute $\gamma$? If we don’t explicitly know the damping coefficient from the beginning, we’ll know how much force the damper exerts to oppose motion of a given speed. Then we can set $|F_d| = \gamma |u'|$, where $|F_d|$ is the magnitude of the damping force and $|u'|$ is the speed of motion. So we have $\gamma = \frac{F_d}{u'}$. We’ll see examples of this computation when we consider damped motion.

Let’s start looking at specific cases. These will be defined by which forces are acting on our spring-mass system.

2. Free, Undamped Motion

We’ll begin by assuming that there are no dampers or external forces acting on our system. This is the simplest situation, and as we have no damping, we can take $\gamma = 0$. Our differential equation is then just

$$mu'' + ku = 0,$$ \hspace{1cm} (19.2)

where $m, k > 0$. The characteristic equation is

$$mr^2 + k = 0,$$

which has roots

$$r_{1,2} = \pm i \sqrt{\frac{k}{m}}.$$ 

We’ll write

$$r_{1,2} = \pm i \omega_0,$$

where we’ve substituted

$$\omega_0 = \sqrt{\frac{k}{m}}.$$ 

$\omega_0$ is called the natural frequency of the system, for reasons that will be clear shortly.

Since the roots of our characteristic equation are imaginary, the form of our general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$ \hspace{1cm} (19.3)
This is why we called \( \omega_0 \) the natural frequency of the system: it’s the frequency of motion when the spring-mass system has no interference from dampers or external forces.

This is all well and good: we know it’s a general solution to our equation and it will easily let us solve for the constants \( c_1 \) and \( c_2 \) given initial conditions. It’s not an ideal form for the solution, though, since there’s important physical information that we can’t necessarily obtain just from looking at it. We can’t immediately recover the amplitude of the motion, for example. So what we will sometimes want to do, after we’ve solved Equation (19.3) for the constants, is to rewrite Equation 19.3 as

\[
 u(t) = R \cos(\omega_0 t - \delta), \tag{19.4}
\]

where \( R > 0 \) is the amplitude of displacement and \( \delta \) is the phase angle of displacement, sometimes referred to as the phase shift.

Before we talk about how to rewrite Equation 19.3 as Equation (19.4), let’s discuss briefly the pros and cons of both forms. When the displacement is written as Equation 19.3, it’s a lot easier to find the constants of integration. But Equation (19.4) is easier to work with: we can directly read off quantities like the amplitude, and it’s a lot easier to graph Equation (19.4) than it is to graph Equation (19.3). So the ideal situation would be for us to solve Equation (19.2) in the form of Equation (19.3), find the constants \( c_1 \) and \( c_2 \) using initial conditions, and then convert to Equation (19.4).

Let’s assume that, for a given problem, we’ve found \( c_1 \) and \( c_2 \). How do we compute \( R \) and \( \delta \)? Consider Equation (19.4). Using a trig identity, we can write it as

\[
 u(t) = R \cos(\delta) \cos(\omega_0 t) + R \sin(\delta) \sin(\omega_0 t). \tag{19.5}
\]

Comparing Equation 19.5 to Equation 19.3, we see that

\[
 c_1 = R \cos(\delta) \quad c_2 = R \sin(\delta). \nonumber
\]

Notice

\[
 c_1^2 + c_2^2 = R^2 (\cos^2(\delta) + \sin^2(\delta)) = R^2, \nonumber
\]

so that, assuming \( R > 0 \),

\[
 R = \sqrt{c_1^2 + c_2^2}. \nonumber
\]

Now, let’s consider the quantity

\[
 \frac{c_2}{c_1} = \frac{\sin(\delta)}{\cos(\delta)} = \tan(\delta). \nonumber
\]

Some care will be needed here, though, due to the fact that the range of \( \arctan \) is \((-\frac{\pi}{2}, \frac{\pi}{2})\). So we can’t just take the \( \arctan \) of both sides, since that would always result in our angle \( \delta \) being in Quadrants I or IV. What we’ll need to do, then, is determine what quadrant \( \delta \) should be in by considering its sine, \( c_1 \), and its cosine, \( c_2 \). At that point, we can solve

\[
 \tan(\delta) = \frac{c_2}{c_1} \nonumber
\]

for the correct angle \( \delta \). We’ll see examples of this shortly.

Let’s do an example.

**Example 19.1.** *A 2kg object is attached to a spring, which stretches by \( \frac{5}{8} \) m. The object is given an initial displacement of 1 m upwards and given an initial downwards velocity of 4 m/sec. Assuming there are no other forces acting on the spring-mass system, find the displacement of the object at any time \( t \) and express it as a single cosine.*

The first step is to write down the initial value problem for this setup. We’ll need to find \( m \) and \( k \). \( m \) is easy: we know the mass of the object is 2kg. How about \( k \)? From our earlier discussion.
we know that
\[ k = \frac{mg}{L} = \frac{(2)(10)}{\frac{5}{8}} = 32. \]

So our differential equation is
\[ 2u'' + 32u = 0. \]

The initial conditions are given by
\[ u(0) = -1 \quad u'(0) = 4. \]

The characteristic equation here is
\[ 2r^2 + 32 = 0, \]
and this has roots \( r_{1,2} = \pm 4i. \) Hence \( \omega_0 = 4, \) and our general solution is
\[ u(t) = c_1 \cos(4t) + c_2 \sin(4t). \]

Using our initial conditions, we see
\[ -1 = u(0) = c_1 \]
\[ 4 = u'(0) = 4c_2 \Rightarrow c_2 = 1. \]

So the particular solution in this case is
\[ u(t) = \cos(4t) + \sin(4t). \]

We want to write this as a single cosine, since that will make it easier to visualize the motion in question. We start by computing \( R: \)
\[ R = \sqrt{c_1^2 + c_2^2} = \sqrt{2}. \]

Next, let’s consider \( \delta. \) We know
\[ \tan(\delta) = \frac{c_2}{c_1} = -1. \]

This means \( \delta \) is either in Quadrants II or IV. To decide which, we need to look at the values of \( \sin(\delta) \) and \( \cos(\delta). \) We have
\[ \sin(\delta) = c_2 > 0 \]
\[ \cos(\delta) = c_1 < 0. \]

Our conclusion is that \( \delta \) is in Quadrant II. If we take \( \arctan(-1) = -\frac{\pi}{4}, \) this has a value in Quadrant IV. Since \( \tan \) is \( \pi \)–periodic, however, \( -\frac{\pi}{4} + \pi = \frac{3\pi}{4} \) is in Quadrant II and also has a tangent of \(-1.\) Thus our desired phase angle is
\[ \delta = \arctan \left( \frac{c_2}{c_1} \right) + \pi = \arctan(-1) + \pi = \frac{3\pi}{4}, \]
and our solution in the form of Equation (19.4) is
\[ u(t) = \sqrt{2} \cos \left( 4t - \frac{3\pi}{4} \right). \]

The displacement of this system is graphed in Figure 19.2. \( \square \)
Figure 19.2. Graph of the solution $u(t)$ to Example 19.1. As this is free, undamped motion, the oscillation is periodic. Notice the amplitude of displacement of $R = \sqrt{2}$ and the phase shift of $\delta = \frac{3\pi}{4}$. Obtaining these quantities is a good reason to know how to calculate them from the coefficients in the particular solution.
Good (Mechanical) Vibrations

1. Recap

Last class, we saw how to derive a differential equation for a spring-mass system from Newton’s Second Law. The equation we derived was

\[ mu'' + \gamma u' + ku = F(t) \]  

(20.1)

where \( u(t) \) is the displacement of the object at time \( t \) (with downward displacement being positive), \( m \) is the object’s mass, \( \gamma \leq 0 \) is the damping coefficient (0 if there is no damping, \( > 0 \) if there is), \( k > 0 \) is the spring coefficient, and \( F(t) \) is the sum of any external forces we may have present in our system.

It’s critical to remember the sign conventions we’re using for writing down initial conditions: downward displacements and forces are positive, upward ones are negative.

We also considered the case of free, undamped motion, where there are no external forces nor any damping. In this case, the equation becomes

\[ mu'' + ku = 0. \]

We saw that this has solution

\[ u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \]

where \( \omega_0 = \sqrt{k/m} \) is referred to as the natural frequency of the system, since it is the frequency the system will oscillate at in the absence of any damping or external factors. We also saw that we can rewrite this equation as a single cosine, in the form

\[ u(t) = R \cos(\omega_0 t - \delta), \]

where \( R = \sqrt{c_1^2 + c_2^2} \) is the amplitude of displacement and \( \delta \), given by \( \tan(\delta) = \frac{c_2}{c_1} \), is the phase angle of displacement. This form is preferable in many situations, because it lets us know, at a glance, what the amplitude and any phase shifting are, whereas the previous form only readily gives us the frequency. The general strategy here is to calculate the constants \( c_1 \) and \( c_2 \) using the initial conditions in the first form, then convert to the second, more physically revealing, form.

2. Free, Damped Motion

Now, let’s consider what happens if we add a damper into the system with damping coefficient \( \gamma \). We’re still considering free motion, so \( F(t) = 0 \), and our differential equation becomes

\[ mu'' + \gamma u' + ku = 0. \]

The characteristic equation is

\[ mr^2 + \gamma r + k = 0, \]

and this has solution

\[ r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}. \]

There are three different cases we need to consider, corresponding to the discriminant being positive, zero, or negative.
(1) $\gamma^2 - 4mk = 0$

This case gives us a double root of $r = -\frac{\gamma}{2m}$, and so the general solution to our equation is

$$u(t) = c_1 e^{-\frac{\gamma}{2m}} + c_2 te^{-\frac{\gamma}{2m}}.$$  

Notice that $\lim_{t \to \infty} u(t) = 0$, which is good, since that’s the whole point of damping.

This case is called critical damping, and it occurs when

$$\gamma^2 - 4mk = 0$$

$$\gamma = \sqrt{4mk} = 2\sqrt{mk}$$

This value of $\gamma = 2\sqrt{mk}$ is denoted by $\gamma_{CR}$ and is called the critical damping coefficient. Since this case separates the other two (which have very different behaviors, as we will see), it’s generally useful to be able to calculate this coefficient for a given spring-mass system, which we can do using this formula.

Critically damped systems may cross $u = 0$ once (depending on the coefficients $c_1$ and $c_2$), but they will never cross it more than that. Their general behavior is for the motion to settle back to equilibrium.

(2) $\gamma^2 - 4mk > 0$

In this case, the discriminant is positive and so we will get two distinct, real roots $r_1$ and $r_2$.

Hence our general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$  

But what is the behavior of this solution? The solution should die out, since we have damping, but we’ll need to be sure of this.

We want to be sure that in this case, we still have $\lim_{t \to \infty} u(t) = 0$. To do this, we’ll need to rewrite our roots a little.

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

$$= \frac{-\gamma \pm \gamma \sqrt{1 - \frac{4mk}{\gamma^2}}}{2m}$$

$$= -\frac{\gamma}{2m} \left( 1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}} \right)$$

By assumption, we have that $\gamma^2 > 4mk$. Hence

$$1 - \frac{4mk}{\gamma^2} < 1$$

and so

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1.$$  

So, the quantity in parentheses above is guaranteed to be positive, which means both of our roots will be negative.

Thus the damping in this case has the desired effect, and the vibration will die out in the limit.

This case, which occurs when $\gamma > \gamma_{CR}$, is called overdamping.

The general behavior here is similar to that in the critically damped case. The solution won’t oscillate around equilibrium, but rather, it will just settle back into place.

(3) $\gamma^2 < 4mk$
The final case occurs when $\gamma < \gamma_{CR}$. In this case, the characteristic equation has complex roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \alpha + i\beta.$$ 

The displacement is

$$u(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) = e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t)).$$

In analogy to the free, undamped case, we can write this as

$$u(t) = Re^{\alpha t} \cos(\beta t - \delta).$$

We know $\alpha < 0$. Hence the displacement in this case will also settle back to equilibrium. There is a difference, though: solutions in this case will oscillate even as the oscillations have smaller and smaller amplitude. This case is called underdamped.

Notice that the solution $u(t)$ isn’t quite periodic. It has the form of a cosine, but the amplitude isn’t constant. A function $u(t)$ is called quasi-periodic, since it oscillates with a constant “frequency” but has varying amplitude. $\beta$ is called the quasi-frequency of the oscillation.

So when we have free, damped vibrations, there are three cases. There’s underdamping, where the damping is there, so the vibrations die out, but the system still oscillates around the equilibrium. Mathematically, this corresponds to the characteristic equation having complex roots. Then, there’s overdamping, where there’s enough damping to suppress oscillation. Here, we have distinct real roots of the characteristic equation. The separating case, critical damping, occurs for a particular damping coefficient, $\gamma_{CR} = 2\sqrt{mk}$, that makes the discriminant of the characteristic equation zero. In this case, like in the overdamped case, there is no oscillation.

A good example to keep in mind when considering the types of damping are a car’s shocks. If your shocks are new, your car is likely overdamped: when you hit a bump in the road, the car settles back into place. As the car’s shocks wear, there’s more of an initial jostle, but the car still doesn’t bounce around. Eventually, though, your car behaves more like a beat up Volkswagon: when you hit a bump, your car bounces up and down for a few minutes, though it eventually settles.

This last case corresponds to the car’s shocks only underdamping the car’s motion. The previous case (generally) corresponds to the overdamped case, and the critical point where the car goes from settling to bouncing is the critically damped case.

Another example to keep in mind is that of a washing machine: new washing machines don’t vibrate significantly due to the presence of good dampers. Old (or cheap) washing machines, on the other hand, vibrate quite a lot.

In practice, we want to avoid underdamping. We don’t want our cars to bounce around on the road and we don’t want the top of tall buildings to sway in the wind. With critical damping, we have the right sort of behavior, but it’s too precise a case: if our dampers wear even a little we’re in the underdamped case. Thus, in practice, we want to be overdamped, so there’s some margin for the dampers to wear without going into the overdamped case. Oscillations are generally good things in the real world.

Now, let’s do some examples of damped vibrations.

**Example 20.1.** A 2kg object stretches a spring by $\frac{5}{8}$ m. A damper is attached that exerts a resistive force of 48 N when the speed is 3 m/sec. If the initial displacement is 1 m upwards and the initial velocity is 2 m downwards, find the displacement $u(t)$ at any time $t$.

This is actually the example from last class with a damper attached and slightly different initial conditions. We’ve already calculated the spring coefficient $k = 32$. What is the damping coefficient
Figure 20.1. Graph of the solution $u(t) = -e^{-4t} - 2te^{-4t}$ to Example 20.1. Observe the limiting behavior to zero and the lack of oscillation. While this example involves critical damping, other critically damped problems may cross the $t$-axis before settling down.

We know that $|F_d| = 48$ when the speed $|u'| = 3$. So the damping coefficient is given by

$$\gamma = \frac{|F_d|}{|u'|} = \frac{48}{3} = 16.$$ 

Thus the initial value problem is

$$2u'' + 16u' + 32u = 0 \quad u(0) = -1 \quad u'(0) = 2.$$ 

Before we solve it, let’s see which case we’re in. To do so, let’s calculate the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{mk} = 2\sqrt{64} = 16.$$ 

So we’re critically damped. This means we should get a double root. Surely enough, the characteristic equation has the double root $r = -4$, and so the general solution is

$$u(t) = c_1 e^{-4t} + c_2 te^{-4t}.$$ 

The initial conditions give coefficients $c_1 = -1$ and $c_2 = -2$. So the solution is

$$u(t) = -e^{-4t} - 2te^{-4t}.$$ 

The graph of this solution can be seen in Figure 20.1. Notice the lack of oscillation in this case. \hfill \square

Example 20.2. For the same spring-mass system as in the previous example, attach a damper that exerts a force of 40 N when the speed is 2 m/s. Find the displacement at any time $t$.

The only difference between this example and the previous one is the damping force. Let’s compute the damping coefficient.

$$\gamma = \frac{|F_d|}{|u'|} = \frac{40}{2} = 20.$$
Figure 20.2. Graph of the solution $u(t) = -e^{-2t}$ to Example 20.2. Observe the limiting behavior to zero and the lack of oscillation. This is typical of overdamped spring-mass systems.

Since we computed $\gamma_{CR} = 16$, this means we are overdamped and the characteristic equation should give us distinct real roots. The initial value problem is

$$2u'' + 20u' + 32u = 0 \quad u(0) = -1 \quad u(0) = 2.$$  

The characteristic equation has roots $r_1 = -8$ and $r_2 = -2$. So the general solution is

$$u(t) = c_1 e^{-8t} + c_2 e^{-2t}.$$  

The initial conditions give $c_1 = 0$ and $c_2 = -1$, so the displacement is

$$u(t) = -e^{-2t}.$$  

Notice here again, we don’t actually have a “vibration” as we normally think of them. This can be seen in Figure 20.2. The damper is strong enough to force the vibrations to die out so quickly that we don’t notice much, if any, of them.

Example 20.3. For the same spring-mass system as in the previous two examples, add a damper that exerts a force of 16 N when the speed is 2 m/s.

In this case, the damping coefficient is

$$\gamma = \frac{16}{2} = 8,$$

which tells us that this case is underdamped as $\gamma < \gamma_{CR} = 16$. We should expect to get complex roots of the characteristic equation. The initial value problem is

$$2u'' + 8u' + 32u = 0 \quad u(0) = -1 \quad u(0) = 3.$$  

The characteristic equation has roots

$$r_{1,2} = \frac{-8 \pm \sqrt{192}}{4} = -2 \pm i \sqrt{12}.$$  

Thus our general solution is

$$u(t) = c_1 e^{-2t} \cos \left( \sqrt{12}t \right) + c_2 e^{-2t} \sin \left( \sqrt{12}t \right).$$
Figure 20.3. Graph of the solution $u(t)$ to Example 20.3. Observe the oscillation with amplitude tending to zero. This is characteristic of underdamped systems — the damping results in the displacement dying, but the relatively low magnitude of the damping force allows for oscillation.

The initial conditions give constants $c_1 = -1$ and $c_2 = \frac{1}{\sqrt{12}}$, so we have

$$u(t) = -e^{-2t} \cos\left(\sqrt{\frac{12}{13}} t\right) + \frac{1}{\sqrt{12}} e^{-2t} \left(\sqrt{\frac{12}{13}} t\right).$$

Let’s write this as a single cosine.

$$R = \sqrt{(-1)^2 + \left(\frac{1}{\sqrt{12}}\right)^2} = \sqrt{\frac{13}{12}}$$

$$\tan(\delta) = -\frac{1}{\sqrt{12}}$$

As in the undamped case, we look at the signs of $c_1$ and $c_2$ to figure out which quadrant $\delta$ is in. By doing so, we see that $\delta$ has [negative cosine and positive sine, so it’s in Quadrant II. Hence we need to take the arctangent and add $\pi$ to it.

$$\delta = \arctan\left(-\frac{1}{\sqrt{12}}\right) + \pi.$$

Thus our displacement is

$$u(t) = \sqrt{\frac{13}{12}} e^{-2t} \cos\left(\sqrt{\frac{12}{13}} t - \arctan\left(-\frac{1}{\sqrt{12}}\right) - \pi\right).$$

In this case, we actually get a vibration, even though its amplitude steadily decreases until it becomes negligible. This can be seen in Figure 20.3. The vibration has quasifrequency $\sqrt{\frac{12}{13}}$. □
LECTURE 21

Even Better (Mechanical) Vibrations

We’ve finished discussing what happens when we have free vibrations. In this situation, the differential equation modeling the displacement of an object attached to a spring is

\[ mu'' + \gamma u' + ku = 0, \]

where \( m \) is the mass of the object, \( \gamma \) is the damping coefficient of any damper that might be present, and \( k \) is the spring coefficient.

When there is no damping (\( \gamma = 0 \)), our solution is

\[ u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = R \cos(\omega_0 t - \delta). \]

In this case, the solution oscillates with amplitude \( R \) and frequency \( \omega_0 \), which is called the natural frequency of the system.

If we hook up a damper with damping coefficient \( \gamma \), the solution depends on what \( \gamma \) is. If \( \gamma = \gamma_{CR} = 2\sqrt{mk} \), we have a critically damped system, and, as the only root for the characteristic equation is \( r = -\frac{\gamma}{2m} \), the solution is

\[ u(t) = c_1 e^{-\frac{\gamma}{2m} t} + c_2 te^{-\frac{\gamma}{2m} t}. \]

If \( \gamma > \gamma_{CR} \), we have an overdamped system, and the characteristic equation has real and distinct roots \( r_1 \neq r_2 \), where both roots are negative. This gives a solution of

\[ u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \]

In both the critically damped and overdamped cases, the damping is strong enough to cancel out almost all oscillation (there may be a single crossing of the equilibrium position, but beyond that, the entire system settles back to equilibrium). If \( \gamma < \gamma_{CR} \), though, we have an underdamped situation, where the characteristic equation has complex roots \( r_{1,2} = \alpha \pm i\beta \), where \( \alpha < 0 \). This means the solution has the form

\[ u(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) = Re^{\alpha t} \cos(\beta t - \delta). \]

Here the solution still goes to the equilibrium \( u = 0 \) in the limit, but it does so after oscillating (this is the only damped case where we actually get true “vibrations.”) We can’t call the solution periodic, however, since the amplitude is changing. So we call it quasi-periodic, with quasi-frequency \( \beta \).

1. Forced, Undamped Motion

It’s now time to discuss what happens if we allow some external force \( F(t) \) to act on our system. This function \( F(t) \) is called, appropriately enough, the forcing function. We start by considering the undamped case. Our differential equation is

\[ mu'' + ku = F(t). \]
Differential Equations  Lecture 21: Even Better (Mechanical) Vibrations

This is a nonhomogeneous equation, which means that the general solution has the form

\[ u(t) = u_c(t) + U_p(t), \]

where \( u_c(t) \) is the general solution to the associated homogeneous equation, which is just the analogous free, undamped case, and \( U_p(t) \) is a particular solution to the nonhomogeneous equation, which we can find using undetermined coefficients (if \( F(t) \) is of an appropriate form) or variation of parameters.

We will restrict our attention to the interesting case where \( F(t) = F_0 \cos(\omega t) \) or \( F(t) = F_0 \sin(\omega t) \).

In other words, the force that we’re applying to our spring-mass system is a simple periodic function with frequency \( \omega \). In the subsequent discussion, I’ll assume that \( F(t) = F_0 \cos(\omega t) \), but everything is completely analogous if \( F(t) \) is a sine function. So, the equation that we’ll consider is just

\[ mu'' + ku' = F_0 \cos(\omega t). \]

Since the complimentary solution is the solution to the analogous free, undamped equation, as noted earlier it is just

\[ u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \]

where \( \omega_0 = \sqrt{\frac{k}{m}} \) is the natural frequency.

We can use the method of undetermined coefficients for this nonhomogeneous term \( F(t) \). The initial guess for the form of the particular solution is

\[ U_p(t) = A \cos(\omega t) + B \sin(\omega t). \]

This guess is fine if \( \omega \neq \omega_0 \), but if the frequency of the forcing function is the same as the natural frequency, this guess is precisely the complimentary solution \( u_c \). Thus, if \( \omega = \omega_0 \), we’ll need to add a factor of \( t \) to our guess.

As a result, we have two cases we need to consider.

(1) \( \omega \neq \omega_0 \)

In this case, our initial guess is distinct from the complimentary solution, and so the form of our particular solution will be

\[ U_p(t) = A \cos(\omega t) + B \sin(\omega t). \]

Differentiating this guess and plugging it in to our differential equation, we get

\[
m\omega^2 (-A \cos(\omega t) - B \sin(\omega t)) + k(A \cos(\omega t) + B \sin(\omega t)) = F_0 \cos(\omega t) \\
(-m\omega^2 A + kA) \cos(\omega t) + (-m\omega^2 B + kB) \sin(\omega t) = F_0 \cos(\omega t).
\]

Setting coefficients equal, we get

\[
\begin{align*}
\cos(\omega t) : & \quad (-m\omega^2 + k)A = F_0 \\
\sin(\omega t) : & \quad (-m\omega^2 + k)B = 0
\end{align*}
\]

\[ \Rightarrow \quad A = \frac{F_0}{k - m\omega^2} \quad \text{and} \quad B = 0. \]

So our particular solution is

\[ U_p(t) = \frac{F_0}{k - m\omega^2} \cos(\omega t) \\
= \frac{F_0}{m \left( \frac{k}{m} - \omega^2 \right)} \cos(\omega t) \\
= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t). \]
Notice that the amplitude of the particular solution is dependent on two things: the amplitude of the forcing function \(F_0\) and the difference between the natural frequency and the forcing frequency.

We can write our displacement function in two forms, depending on which form we use for the complimentary solution.

\[
u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)
\]

\[
u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)
\]

Again, we’d get an analogous solution if our forcing function had been \(F(t) = F_0 \sin(\omega t)\).

The key feature of this case can be seen from the second form of our displacement function; namely, we have two cosine functions with different frequencies. These will interfere with each other, causing the net oscillation to vary between bouts of greater and lesser amplitude. This phenomenon is called “beats,” in analogy to the musical phenomenon. Think of this case like hitting a tuning fork when another one is already ringing; the increasing/decreasing volume that one hears in that situation as the waves interfere with each other is the same as what happens to the size of the oscillations for this spring-mass system.

(2) \(\omega = \omega_0\)

If the frequency of the forcing function is the same as the natural frequency, our guess for the particular solution has to become

\[
U_p(t) = At \cos(\omega_0 t) + Bt \sin(\omega_0 t).
\]

Differentiating, plugging in, and simplifying gives

\[
(-m\omega_0^2 + k)At \cos(\omega_0 t) + (-m\omega_0^2 + k)Bt \sin(\omega_0 t) + 2m\omega_0 B \cos(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t) = F_0 \cos(\omega_0 t).
\]

To begin further simplification, let’s recall that \(\omega_0^2 = \frac{k}{m}\), so \(m\omega_0^2 = k\). This means the first two terms will actually vanish (as we would expect, since there are no analogous terms on the right hand side), and we get

\[2m\omega_0 B \cos(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t) = F_0 \cos(\omega_0 t).\]

Now, let’s set coefficients equal.

\[
\begin{align*}
\cos(\omega_0 t) & : \\
& 2m\omega_0 B = F_0 \quad \quad B = \frac{F_0}{2m\omega_0} \\
\sin(\omega_0 t) & : \\
& -2m\omega_0 A = 0 \quad \quad A = 0
\end{align*}
\]

Thus the particular solution in this case is

\[U_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)\]

and the displacement is

\[u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)\]

or

\[u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).\]

What stands out the most about this equation? Notice that as \(t \to \infty\), \(u(t) \to \infty\) due to the form of the particular solution. Thus, in the case where the forcing frequency is the same as the natural frequency, the oscillation will have an amplitude that continues to increase for all
time since the external force adds energy to the system in a way that reinforces the natural motion of the system.

This phenomenon is called resonance. Resonance is the phenomenon behind microwave ovens: the microwave radiation strikes the water molecules in what’s being heated at their natural frequency, causing them to vibrate faster and faster, which generates heat. A similar phenomenon occurs in the Bay of Fundy, where tidal forces cause the ocean to resonate, yielding larger and larger tides. Resonance in the ear causes us to be able to distinguish between different tones in sound.

A common example that’s cited of resonance is the collapse of the Old Tacoma Narrows Bridge. This is incorrect, however: the oscillation that led to the collapse of the bridge was from a far more complicated phenomenon than the simple resonance we’re considering now.

In general, for engineering purposes, resonance is something that we’d like to avoid unless we thoroughly understand the situation and what the effect of the resonance will be.

There’s no reason to suppose that the forcing function was sinusoidal as we did. It provides us with enough interesting behavior to be worthwhile, though. If you’re faced with a forcing function that is of a different sort, you can still approach it using these techniques, though the phenomena that result will be different.

So, to sum up, when we drive our system at a different frequency than the natural frequency, the two frequencies interfere and we observe beats in the motion. When the system is driven at its natural frequency, the natural motion of the system is reinforced, causing the amplitude of the motion to increase to infinity.

Let’s do an example.

**Example 21.1.** A 4 kg object is attached to a spring, which it stretches by 40 cm. There is no damping, but the system is forced with forcing function

\[ F(t) = 8 \cos(\omega t) \]

such that the system will experience resonance. If the object is initially displaced 10 cm downward and given an initial upward velocity of 25 cm/sec, find the displacement at any time \( t \).

We need to be aware of units: all lengths need to be in meters. The first thing to do is to find \( k \).

\[ k = \frac{mg \cdot L}{4} = 100 \]

Next, we’re told that the system experiences resonance. Thus, the forcing frequency \( \omega \) must be the same as the natural frequency \( \omega_0 \).

\[ \omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5 \]

Thus our initial value problem is

\[ 4u'' + 100u = 8 \cos(5t) \quad u(0) = \frac{1}{10} \quad u'(0) = -\frac{1}{4}. \]

The complimentary solution is the general solution to the associated free, undamped case. As we’ve computed the natural frequency already, the complimentary solution will just be

\[ u_c(t) = c_1 \cos(5t) + c_2 \sin(5t). \]

The particular solution (using the formula derived above) is

\[ U_p(t) = \frac{1}{5} t \sin(5t), \]

and so the general solution is

\[ u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{5} t \sin(5t). \]
Figure 21.1. Graph of the solution $u(t)$ to Example 21.1. Here $\omega = \omega_0$, and as the external force and the natural oscillation of the system are in-phase, the resulting resonance causes the amplitude to increase as the external energy is added to the mechanical energy of the vibrating system rather than countering it.

The initial conditions give that $c_1 = \frac{1}{10}$ and $c_2 = -\frac{1}{20}$, so the displacement can be given as

$$u(t) = \frac{1}{10} \cos(5t) - \frac{1}{20} \sin(5t) + \frac{1}{5} t \sin(5t).$$

Let’s convert the first two terms to a single cosine.

$$R = \sqrt{ \left( \frac{1}{4} \right)^2 + \left( -\frac{1}{20} \right)^2 } = \sqrt{ \frac{26}{400} } = \frac{\sqrt{26}}{20}$$

$$\tan(\delta) = -\frac{\frac{1}{20}}{\frac{1}{4}} = -\frac{1}{5}$$

Looking at the signs of $c_1$ and $c_2$, we see that $\cos(\delta) > 0$ and $\sin(\delta) < 0$. Thus $\delta$ is in Quadrant IV, and so we can just take the arctangent.

$$\delta = \arctan\left( -\frac{1}{5} \right)$$

The displacement is then

$$u(t) = \frac{\sqrt{26}}{20} \cos \left( 5t - \arctan \left( -\frac{1}{5} \right) \right) + \frac{1}{5} t \sin(5t).$$

This is plotted in Figure 21.1.

Example 21.2. A 4 kg object is attached to a spring, which it stretches by 40 cm. There is no damping, but the system is forced with forcing function

$$F(t) = 11 \cos(6t)$$

If the object is initially displaced 10 cm downward and given an initial upward velocity of 25 cm/sec, find the displacement at any time $t$. 

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Observe that this is the same example as above, other than the forcing function, which only affects the particular solution. In Example 21.2, we calculated the complimentary solution to be

$$u(t) = \frac{1}{10} \cos(5t) - \frac{1}{20} \sin(5t).$$

We have that the natural frequency $\omega_0 = 5$ while the forcing frequency $\omega = 6$, so the particular solution will be given by

$$U_p(t) = -\frac{1}{4} \cos(6t),$$

and the displacement is

$$u(t) = \frac{1}{10} \cos(5t) - \frac{1}{20} \sin(5t) - \frac{1}{4} \cos(6t).$$

With the general solution written as a single cosine, the displacement is

$$u(t) = \frac{\sqrt{26}}{20} \cos\left(5t - \arctan\left(-\frac{1}{5}\right)\right) - \frac{1}{4} \cos(6t).$$

This is plotted in Figure 21.2.

**Example 21.3.** A 4 kg object is attached to a spring, which it stretches by 40 cm. There is no damping, but the system is forced with forcing function

$$F(t) = 9 \cos(4t)$$

If the object is initially displaced 10 cm downward and given an initial upward velocity of 25 cm/sec, find the displacement at any time $t$. 

---

**Figure 21.2.** Graph of the solution $u(t)$ to Example 21.2. Here $\omega > \omega_0$, so the external force acts at a higher frequency than the natural oscillation of the spring-mass system. Observe how this results in an oscillation of the amplitude of the overall displacement depending on if the external force is in-phase or out-of-phase with the natural oscillation.
Once again, the complimentary solution is identical to the previous examples. The particular solution will be given by

\[ U_p(t) = \frac{1}{4} \cos(4t), \]

and the displacement is

\[ u(t) = \frac{1}{10} \cos(5t) - \frac{1}{20} \sin(5t) + \frac{1}{4} \cos(4t). \]

With the general solution written as a single cosine, the displacement is

\[ u(t) = \frac{\sqrt{26}}{20} \cos \left( 5t - \arctan \left( -\frac{1}{5} \right) \right) + \frac{1}{4} \cos(4t). \]

This is plotted in Figure 21.3.

\[ \square \]

2. Forced, Damped Motion

The final case to consider occurs when we have both forcing and damping. Our system has all the forces we listed in the first vibrations lecture acting on it, so the equation becomes

\[ mu'' + \gamma u' + ku = F(t). \]

Again, we’ll assume for the purposes of this discussion that \( F(t) \) is sinusoidal, but this isn’t necessarily going to be the case in practice. We won’t go into too much detail here, but notice that our general solution, once again, has the form

\[ u(t) = u_c(t) + U_p(t). \]
The complimentary solution \( u_c(t) \) is just the solution to the associated free, damped case, and it will have different forms depending on whether the system is overdamped, underdamped, or critically damped. Regardless, we know from our discussion of free, damped motion that there will be an exponential involved, and as a result, \( \lim_{t \to \infty} u(t) = 0 \).

Since every term of \( u_c(t) \) has an exponential, and our forcing function is of the form
\[
F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t),
\]
our initial guess for the particular solution of
\[
U_p(t) = A \cos(\omega t) + B \sin(\omega t)
\]
will be correct. As a result, the general solution will look like
\[
u(t) = u_c(t) + A \cos(\omega t) + B \sin(\omega t) = u_c(t) + R \cos(\omega t - \delta).
\]

Since we know \( u_c(t) \to 0 \) we call \( u_c(t) \) the transient solution. \( U_p(t) \) will persist (and continue to oscillate with constant amplitude \( R \)), and in the limit, \( u(t) \to U_p(t) \). We call \( U_p(t) \) the steady-state solution. Thus, our solution will initially look like the sum of these two functions, but eventually \( u_c \) dies out and the displacement looks just like \( U_p \), which comes from the forcing.

Observe that with this choice of \( F(t) \), we won’t get any resonance as we could have in the undamped case, even if \( \omega = \omega_0 \). If our system is underdamped, we might get a beating phenomenon, but it will die out as the transient solution vanishes and the steady-state solution takes over.

**Exercise.** Assume our system is underdamped. If \( F(t) = F_0 e^{-\alpha t} \cos(\beta t) \), where \( \alpha \pm i\beta \) are the roots of the characteristic equation of
\[
mu'' + \gamma u' + ku,
\]
our guess at the form of \( U_p(t) \) will need to be adjusted. What will the displacement function \( u(t) \) look like in this case, and what happens to it as \( t \to \infty \)? Do we have a similar decomposition into a transient solution and a steady-state solution, or will it vanish? Is there an analogous phenomenon to resonance in this case?
Part 4

Laplace Transforms
LECTURE 22

Laplace Transforms

One approach to solving differential equations is to tackle them directly, which is what we’ve been doing. Sometimes, however, it’s convenient to transform the problem into a different one, which we can then solve more easily and transform back. One example of such a transformation is the Laplace transform, which we’ll discuss now. In some cases, it lets us more easily solve problems we could solve with other methods, while in others, it lets us deal with problems we have no other method to solve.

For the basic examples, such as the sorts of linear homogeneous equations with constant coefficients that we looked at earlier, we’ll find that the Laplace transform is more messy than we would otherwise require. When we start having some basic nonhomogeneous equations, we could still use other methods, but the amount of work is generally a wash. When our forcing functions start to get more complicated, however, Laplace transforms are a great method to have at hand.

1. The Definition

There is one basic notion we have to get out of the way before we can define the Laplace transform.

**Definition 22.1.** A function \( f \) is called piecewise continuous on an interval \([a, b]\) if \([a, b]\) can be broken into a finite number of subintervals \([a_n, b_n]\) such that \( f \) is continuous on each open subinterval \((a_n, b_n)\) and has a finite limit at every endpoint \(a_n, b_n\).

In other words, a piecewise continuous function has only a finite number of “jumps” and doesn’t have any asymptotes where it blows up to plus or minus infinity. There are many examples of these; we’ll see several down the road.

Now, we can define the Laplace transform of a function.

**Definition 22.2.** Suppose that \( f(t) \) is a piecewise continuous function. The Laplace transform of \( f(t) \), denoted by \( \mathcal{L} \{ f(t) \} \), is given by

\[
\mathcal{L} \{ f(t) \} = \int_{0}^{\infty} e^{-st} f(t) \, dt. \tag{22.1}
\]

**Remark.** There is an alternate notation for the Laplace transform that we will commonly use. Notice that the definition of \( \mathcal{L} \{ f(t) \} \) introduces a new variable, \( s \), then is a definite integral with respect to \( t \). As a result, computing this transform yields a function which depends on \( s \). Thus, we will use the notation

\[
\mathcal{L} \{ f(t) \} = F(s).
\]

It should also be noted that the integral in the definition of \( \mathcal{L} \{ f(t) \} \) is an improper integral. In our first examples of computing Laplace transforms, we’ll review how these work.

**Example 22.3.** Compute \( \mathcal{L} \{ 1 \} \). Plugging \( f(t) = 1 \) into the Definition 22.1, we have

\[
\mathcal{L} \{ 1 \} = \int_{0}^{\infty} e^{-st} \, dt.
\]
Recall that to calculate this improper integral, we need to convert it to a limit as follows.

\[
= \lim_{N \to \infty} \int_0^N e^{-st} \, dt
\]

\[
= \lim_{N \to \infty} \left[ -\frac{1}{s} e^{-st} \right]_0^N
\]

\[
= \lim_{N \to \infty} \left( -\frac{1}{s} e^{-Ns} + \frac{1}{s} \right)
\]

At this point we need to be careful: the value of \( s \) will affect our answer. If \( s < 0 \), the exponent of our exponential is positive, so the limit in question will diverge as the exponential goes to infinity. On the other hand, if \( s > 0 \), the exponential will go to 0 and the limit will converge.

Thus, we restrict our attention to the case where \( s > 0 \) and conclude that

\[
\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for } s > 0.
\]

\[
\square
\]

Notice that we had to put a restriction on the domain of our Laplace transform. This will always be the case: these integrals will not always converge for any \( s \). At the moment, we can brush this to the side as a technical detail, but it’s important to keep in mind that Laplace transforms are not defined for all \( s \).

**Example 22.4.** Compute \( \mathcal{L}\{e^{at}\} \) for \( a \neq 0 \).

By definition,

\[
\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt
\]

\[
= \int_0^\infty e^{(a-s)t} \, dt
\]

\[
= \lim_{N \to \infty} \left[ \frac{1}{a-s} e^{(a-s)t} \right]_0^N
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{a-s} e^{(a-s)N} - 1 \right)
\]

\[
= \frac{1}{s-a} \quad \text{for } s > a.
\]

\[
\square
\]

**Example 22.5.** Compute \( \mathcal{L}\{\sin(at)\} \).

\[
\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st} \sin(at) \, dt
\]

\[
= \lim_{N \to \infty} \int_0^N e^{-st} \sin(at) \, dt.
\]

Integration by parts yields

\[
= \lim_{N \to \infty} \left( \frac{1}{a} (1 - e^{-sn} \cos(an)) - \frac{s}{a} \left( \frac{1}{a} e^{-sn} \sin(an) + \frac{s}{a} \int_0^n e^{-st} \sin(at) \, dt \right) \right).
\]
Differential Equations Lecture 22: Laplace Transforms

Doing a bit more jiggling, we get
\[
F(s) = \frac{1}{a} - \frac{s}{a} F(s)
\]
\[
\mathcal{L}\{\sin(at)\} = F(s) = \frac{a}{s^2 + a^2} \quad \text{provided } s > 0.
\]

\[\square\]

**Example 22.6.** If \(f(t)\) is a piecewise continuous function with piecewise continuous derivative \(f'(t)\), express \(\mathcal{L}\{f'(t)\}\) in terms of \(\mathcal{L}\{f(t)\}\).

We plug \(f'\) into the definition (22.1).
\[
\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t) \, dt
\]
\[
= \lim_{N \to \infty} \int_0^N e^{-st} f'(t) \, dt
\]
The next step is to integrate by parts.
\[
= \lim_{N \to \infty} \left( e^{-st} f\big|_0^N + s \int_0^N e^{-st} f(t) \, dt \right)
\]
\[
= \lim_{N \to \infty} e^{-SN} f(N) - f(0) + s \int_0^\infty e^{-st} f(t) \, dt
\]
\[
= s \mathcal{L}\{f(t)\} - f(0) \quad \text{provided } s > 0
\]

\[\square\]

Doing this repeatedly, one can find
\[
\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).
\]

**Example 22.7.** If \(f(t)\) is a piecewise continuous function, express \(\mathcal{L}\{e^{at} f(t)\}\) in terms of \(\mathcal{L}\{f(t)\}\).

We begin by plugging into Equation (22.1).
\[
\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) \, dt
\]
\[
= \int_0^\infty e^{(a-s)t} f(t) \, dt
\]
This looks rather like the definition of \(\mathcal{L}\{f(t)\} = F(s)\), but it’s not quite, since the exponent is \(a - s\). However, if we substitute \(u = s - a\), we get the following.
\[
= \int_0^\infty e^{-ut} f(t) \, dt
\]
\[
= F(u)
\]
\[
= F(s - a).
\]

Thus if we take the Laplace transform of a function multiplied by \(e^{at}\), we’ll get the Laplace transform of the original function shifted by \(a\). This will be useful to keep in mind. \[\square\]
2. Laplace Transforms

In general, we won’t be computing our Laplace transforms from scratch; we’ll be using a table. The table doesn’t include every Laplace transform we might encounter, but it does have the ones we will commonly see. My recommendation is to know the transforms you see coming up all the time, and use the table as insurance or if you encounter one you’re not familiar with. From now on, the examples we work will be done with a thought process that supposes we have a table in front of us, even if we don’t.

There is one important fact we need to get out of the way: the Laplace transform is linear.

**Theorem 22.1.** Given piecewise continuous functions \( f(t) \) and \( g(t) \),

\[
L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}
\]

for any constants \( a, b \).

This follows from the linearity of integration. From a practical perspective, this is great. It means that we don’t have to worry about constants or sums; we can just decompose our function into individual pieces, transform them, and then put everything back together. Let’s do a few examples.

**Example 22.8.** Find the Laplace transforms of the following functions.

(i) \( f(t) = 2e^{3t} - e^{-t} + 6t^2 - 10 \)

\[
F(s) = L\{f(t)\} = 2L\{e^{3t}\} - L\{e^{-t}\} + 6L\{t^2\} - 10L\{1\}
= 2 \frac{1}{s - 3} - \frac{1}{s - (-1)} + 6 \frac{2!}{s^{2+1}} - 10 \frac{1}{s}
= 2 \frac{1}{s - 3} - \frac{1}{s + 1} + \frac{12}{s^3} - \frac{10}{s}
\]

(ii) \( g(t) = \cos(6t) - 2 \sin(6t) - 3 \cos(8t) \)

\[
G(s) = L\{g(t)\} = L\{\cos(6t)\} - 2L\{\sin(6t)\} - 3L\{\cos(8t)\}
= \frac{s}{s^2 + 6^2} - 2 \frac{6}{s^2 + 6^2} - 3 \frac{s}{s^2 + 8^2}
= \frac{s - 12}{s^2 + 36} - \frac{3s}{s^2 + 64}
\]

(iii) \( h(t) = e^{2t} + \cos(3t) - e^{2t} \cos(3t) \)

\[
H(t) = L\{h(t)\} = L\{e^{2t}\} + L\{\cos(3t)\} - L\{e^{2t} \cos(3t)\}
= \frac{1}{s - 2} + \frac{s}{s^2 + 3^2} - \frac{s - 2}{(s - 2)^2 + 3^2}
= \frac{1}{s - 2} + \frac{2}{s^2 + 9} - \frac{s - 2}{(s - 2)^2 + 9}
\]

□

3. Application to Initial Value Problems

As interesting as that was, we’re not interested in playing with Laplace transforms for their own sake. After all, this is a differential equations class. The reason we’re discussing Laplace transforms is that they will help us solve certain initial value problems. Now that we know what the Laplace
transform is, let’s look at an example to see how this works and figure out what we have left to learn about Laplace transforms before we can apply them.

**Example 22.9.** Solve the following initial value problem using Laplace transforms.

\[ y'' - 6y' + 5y = 7t \quad y(0) = 1 \quad y'(0) = 2. \]

The first step in using Laplace transforms to solve an initial value problem is to transform both sides of the equation.

\[
\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 7\mathcal{L}\{t\}
\]

\[
s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 5Y(s) = \frac{7}{s^2}
\]

\[
s^2Y(s) + s - 2 - 6(sY(s) + 1) + 5Y(s) = \frac{7}{s^2}
\]

Next, we solve for \(Y(s)\).

\[
(s^2 - 6s + 5)Y(s) + s - 8 = \frac{7}{s^2}
\]

\[
Y(s) = \frac{7}{s^2(s^2 - 6s + 5)} + \frac{8 - s}{s^2 - 6s + 5}
\]

Now what? We want to solve for \(y(t)\), but we have an expression for \(Y(s) = \mathcal{L}\{y(t)\}\). Thus, to able to actually finish solving this problem, we’ll have to discuss how to go backwards; we’ll need to learn how to take inverse Laplace transforms.
LECTURE 23

Inverse Laplace Transforms

At the end of last class, we saw that upon using Laplace transforms on an initial value problem, we end up with an expression for \( Y(s) \), the Laplace transform of the desired solution \( y(t) \). Thus, we need to know how to go backwards: given a transformed function \( Y(s) \), how do we find the original function \( y(t) \)? This is a slightly more complicated process than taking transforms, which was quite straightforward. We refer to \( f(t) \) as the inverse Laplace transform of \( F(s) \) and use the notation

\[
f(t) = \mathcal{L}^{-1}\{F(s)\}.
\]

Our starting point is that the inverse Laplace transform is linear, just like the original transform was.

**Theorem 23.1.** Given two Laplace transforms \( F(s) \) and \( G(s) \),

\[
\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}
\]

for any constants \( a, b \).

So, we’ll decompose our original transformed function into pieces, inverse transform, and then put everything back together.

This is where familiarity with the basic Laplace transforms and the table becomes handy. The key is to try to identify the desired inverse transform by looking at the denominator. For the most part, this will tell us what the original function will have to be, but occasionally, we will have to look at the numerator to distinguish between two potential inverses (e.g., the denominators for the transforms of \( \sin(at) \) and \( \cos(at) \) are the same, but the numerators differ). Then, we know precisely how we have to write our function \( F(s) \) so that it is the inverse transform of the function we’ve identified as the inverse. Sometimes this requires a little bit of algebra or arithmetic.

Let’s look at some examples.

**Example 23.1.** Find the inverse transforms of the following.

(i) \( F(s) = \frac{4}{s} - \frac{1}{s - 2} + \frac{2}{s - 3} \)

This one is quite straightforward. The denominator of the first term, \( \frac{4}{s} \), indicates that this will be the Laplace transform of 1. Since \( \mathcal{L}\{1\} = \frac{1}{s} \), we’ll factor out the 4 before taking the inverse transform. For the second term, this is just the Laplace transform of \( e^{2t} \), and there’s nothing else to do with it. The third term is also an exponential, \( e^{3t} \), and we’ll need to factor out the 2 in the numerator before we inverse transform.

So we have

\[
\mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} + 2\mathcal{L}^{-1}\{1\} s - 3
\]

\[
f(t) = 4(1) - e^{2t} + 2(e^{3t})
\]

\[
= 4 - e^{2t} + 2e^{3t}.
\]

We usually won’t put quite so much detail into these.
Differential Equations  

Lecture 23: Inverse Laplace Transforms

(ii) \( G(s) = \frac{12}{s + 3} - \frac{1}{2s - 4} + \frac{2}{s^4}. \)

The first term is just the transform of \( e^{-3t} \) multiplied by 12, which we’ll factor out before inverse transforming.

The second term looks like it ought to be an exponential, but it’s got a 2s instead of an s in the denominator, and transforms of exponentials should just have s. We can fix this by factoring a 2 out of the denominator and then taking the inverse transform.

The third term has \( s^4 \) as its denominator. This indicates that it will be related to the transform of \( t^3 \). The numerator isn’t quite correct, though, since \( \mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} \). So we would need the numerator to be 6, and right now it’s 2. How do we fix this? We’ll multiply by \( \frac{3}{2} \), absorb the top 3 into the transform, and keep the \( \frac{1}{3} \) out front.

Let’s start by rewriting the transform, with these fixes incorporated.

\[
G(s) = 12 \frac{1}{s - (-3)} - \frac{1}{2(s - 2)} + \frac{3}{3 s^4} = 12 \frac{1}{s - (-3)} - \frac{1}{2} \frac{1}{s - 2} + \frac{1}{3} \frac{6}{s^4}
\]

Now we can take the inverse transform.

\[
g(t) = 12e^{-3t} - \frac{1}{2}e^{2t} + \frac{1}{3}t^3
\]

(iii) \( H(s) = \frac{4s}{s^2 + 25} + \frac{3}{s^2 + 16} \)

The denominator of the first term, \( s^2 + 25 \), indicates that this should be the transform of either \( \sin(5t) \) or \( \cos(5t) \). The numerator is 4s, though, which tells us that once we factor out the 4, it will be the transform of \( \cos(5t) \).

The second term’s denominator is \( s^2 + 16 \), so it will be the transform of either \( \sin(4t) \) or \( \cos(4t) \). The numerator is a constant, 3, so it will be the transform of \( \sin(4t) \). The only problem is that the numerator of \( \mathcal{L}\{\sin(4t)\} \) should be 4, while here it is 3. We’ll fix this, as in the previous example, by multiplying by \( \frac{4}{4} \).

We rewrite the transform.

\[
H(s) = 4 \frac{1}{s^2 + (5)^2} + \frac{3}{4} \frac{1}{s^2 + (4)^2}
\]

\[
= 4 \frac{1}{s^2 + 25} + \frac{3}{4} \frac{4}{s^2 + 16}
\]

Then we take the inverse.

\[
h(t) = 4 \cos(5t) + \frac{3}{4} \sin(4t).
\]

Let’s do some now that require a little more work.

Example 23.2. Find the inverse Laplace transforms for each of the following.

(i) \( F(s) = \frac{4s + 3}{s^2 + 16} \)

Looking at the denominator, we recognize that this will involve a sine or a cosine, as it has the form \( s^2 + a^2 \). It’s not quite either, though, since it has both an s and a constant in the numerator.

As a result, we need to split up this fraction into the difference of two fractions. Once we manipulate these individual fractions as we did in the previous example, we’ll be able to take
our inverse transform.

\[ F(s) = \frac{4s + 3}{s^2 + 16} \]

\[ = \frac{4s}{s^2 + 16} + \frac{3}{4} \frac{s^2 + 16}{s^2 + 16} \]

\[ = \frac{4s}{s^2 + 16} + \frac{3}{4} \frac{4}{s^2 + 16} \]

Now each term is in the correct form, and we can take the inverse transform.

\[ f(t) = 4 \cos(4t) + \frac{3}{4} \sin(4t) \]

(ii) \[ G(s) = \frac{3 - 5s}{s^2 + 4s + 9} \]

If we look at our table of Laplace transforms, we might see that there are no denominators that look like a quadratic polynomial. Also, this polynomial doesn’t factor nicely. However, there are terms in the table that have denominators of the form \((s - a)^2 + b^2\): those for \(e^{at} \cos(bt)\) and \(e^{at} \sin(bt)\). We can put this denominator in that form if we complete the square. Then we can fix the numerators to figure out the inverse transform.

\[ s^2 + 4s + 9 = s^2 + 4s + 4 - 4 + 9 \]

\[ = s^2 + 4s + 4 + 5 \]

\[ = (s + 2)^2 + 5 \]

Thus, our transformed function can be written as

\[ G(s) = \frac{3 - 5s}{(s + 2)^2 + 5}. \]

We won’t split this up into two pieces yet. First, we’ll fix the \(s\) in the numerator to be \(s + 2\), which we’ll need for the numerator of \(e^{-2t} \cos(\sqrt{5}t)\). We do this by adding and subtracting 2 to the \(s\). This will produce some other constant term, which we’ll combine with the already present constant, and then we can worry about fixing that numerator to correspond to the numerator of the transform of \(e^{-2t} \sin(\sqrt{5}t)\).

\[ G(s) = \frac{3 - 5(s + 2 - 2)}{(s + 2)^2 + 5} \]

\[ = \frac{3 - 5(s + 2) + 10}{(s + 2)^2 + 5} \]

\[ = -5(s + 2) + 13 \]

\[ = (s + 2)^2 + 5 \]

Now we can break our transform up into two pieces, one of which will correspond to the cosine and the other to the sine. At that point, fixing the numerators is the same as in the last couple of examples.

\[ G(s) = -5 \frac{s + 2}{(s + 2)^2 + 5} + \frac{13}{\sqrt{5}} \frac{\sqrt{5}}{(s + 2)^2 + 5} \]

\[ g(t) = -5e^{-2t} \cos(\sqrt{5}t) + \frac{13}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t) \]
This should appear similar at first glance to the previous example, but there’s a difference: this time, we can factor the denominator. This requires us to deal with the inverse transform differently. Factoring, we see

\[ H(s) = \frac{s + 2}{(s-3)(s-7)}. \]

We know that if we have a linear denominator, that will correspond to an exponential. In this case, we have the product of two linear factors. This by itself isn’t the denominator of any particular Laplace transform, but we know a method for turning certain rational functions with factored denominators into a sum of more simple rational functions with those factors in each denominator: partial fractions. We’ll need to do that here.

We start by writing

\[ H(s) = \frac{s + 2}{(s-3)(s-7)} = \frac{A}{s-3} + \frac{B}{s-7}. \]

There are two methods for finding these constants \(A\) and \(B\): the first is much simpler, but doesn’t always work, while the second can result in some algebra, but will always work. In doing this example, we’ll review the first, and we’ll see the second in a later example where it will be required.

The first method requires us to put our partial fraction decomposition over a single denominator:

\[ \frac{s + 2}{(s-3)(s-7)} = \frac{A(s-7) + B(s-3)}{(s-3)(s-7)}. \]

This needs to be true for any value of \(s\); in particular, the numerators must match for every value of \(s\):

\[ s + 2 = A(s-7) + B(s-3). \]

As a result, we can choose values of \(s\) to plug in that will isolate an individual constant. Let’s do this for each.

\[
\begin{align*}
s = 3 & : \quad 5 = -4A \quad \Rightarrow \quad A = -\frac{5}{4} \\
s = 7 & : \quad 9 = 4B \quad \Rightarrow \quad B = \frac{9}{4}
\end{align*}
\]

Thus, our transform can be written as

\[ H(s) = \frac{-\frac{5}{4}}{s-3} + \frac{\frac{9}{4}}{s-7} \]

and taking the inverse transforms, we get

\[ h(t) = -\frac{5}{4}e^{3t} + \frac{9}{4}e^{7t}. \]

Remark. We could have done the last part of the previous example as we had the previous parts by completing the square. However, this would have left us with expressions involving the hyperbolic sine \(\sinh\) and the hyperbolic cosine \(\cosh\). These are interesting functions which can be written in terms of exponentials (as we got in the form of our answer to the previous example), but it will be much easier for us to work with the exponentials, so we’re better off just doing partial fractions even though it’s slightly more work.
Table 23.1. Translation from factored denominator to partial fractions.

<table>
<thead>
<tr>
<th>Factor in Denominator</th>
<th>Partial Fractions Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ax + b$</td>
<td>$\frac{A}{ax + b}$</td>
</tr>
<tr>
<td>$(ax + b)^k$</td>
<td>$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$</td>
</tr>
<tr>
<td>$ax^2 + bx + c$</td>
<td>$\frac{Ax + B}{ax^2 + bx + c}$</td>
</tr>
<tr>
<td>$(ax^2 + bx + c)^k$</td>
<td>$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$</td>
</tr>
</tbody>
</table>

Partial fractions and completing the square are a part of life when it comes to Laplace transforms. Being comfortable with these techniques is especially important when we’re working with initial value problems, since most of our answers will involve some combinations of exponentials, sines, and cosines.

Let’s quickly review partial fractions. The first step is to factor the denominator as much as you can. Then, using Table 23.1, we can find each of the terms for our partial fractions decomposition. This table isn’t exhaustive, but we’ll only worry about having linear or quadratic factors.

Let’s do some more examples that require partial fractions.

**Example 23.3.** Find the inverse transform of each of the following.

1. \( F(s) = \frac{2s + 3}{(s - 2)(s + 3)(2s - 1)} \)

   The form of the decomposition will be

   \[ G(s) = \frac{A}{s - 2} + \frac{B}{s + 3} + \frac{C}{2s - 1} \]

   since all of the factors in our denominator are linear. Putting the right hand side over a common denominator and setting numerators equal, we have

   \[ 2s + 3 = A(s + 3)(2s - 1) + B(s - 2)(2s - 1) + C(s - 2)(s + 3). \]

   We can once again use the method from the previous example where we choose key values of \( s \) that will isolate the coefficients.

   \[
   \begin{align*}
   s = 2 : & \quad 7 = A(5)(9) \quad \Rightarrow \quad A = \frac{7}{45} \\
   s = -3 : & \quad -3 = B(-5)(-7) \quad \Rightarrow \quad B = \frac{3}{35} \\
   s = \frac{1}{2} : & \quad 4 = C \left( -3 \right) \left( \frac{7}{2} \right) \quad \Rightarrow \quad C = \frac{-16}{21} 
   \end{align*}
   \]

   Thus, the partial fraction decomposition for this transform is

   \[ F(s) = \frac{7}{45} \frac{1}{s - 2} - \frac{3}{35} \frac{1}{s + 3} - \frac{16}{21} \frac{1}{2s - 1}. \]
To take the inverse transform, we’ll need to factor a 2 out of the denominator in the last term. The corrected transform and inverse transform then become

\[ G(s) = \frac{7}{45} \frac{1}{s - 2} - \frac{3}{35} \frac{1}{s + 3} - \frac{8}{21} \frac{1}{s - \frac{1}{2}} \]

\[ g(t) = \frac{7}{45} e^{2t} - \frac{3}{35} e^{-3t} - \frac{8}{21} e^{\frac{1}{2}t}. \]

(ii) \( G(s) = \frac{-1 - 3s}{(s - 2)(s^2 + 3)} \)

Now we have a quadratic in the denominator. Looking at Table 23.1, we see that the form of the partial fractions decomposition will be

\[ G(s) = \frac{A}{s - 2} + \frac{Bs + C}{s^2 + 3}. \]

If we put everything over a common denominator and setting the numerators equal, we will get

\[-1 - 3s = A(s^2 + 3) + (Bs + C)(s - 2).\]

Notice that we can’t use the method from the previous examples: there are only two “key values” for \( s \), but there are 3 constants, so we would be stuck at some point. Thus we need to use the “long” method, which requires us to multiply out the right side and compare coefficients for like terms.

\[-1 - 3s = A(s^2 + 3) + (Bs + C)(s - 2) = As^2 + 3A + Bs^2 - 2Bs + Cs - 2C \]

\[= (A + B)s^2 + (-2B + C)s + (3A - 2C)\]

We have the following system of equations to solve.

\[(s^2) : \quad A + B = 0\]

\[(s^1) : \quad -2B + C = -3 \quad \Rightarrow \quad A = -1 \quad B = 1 \quad C = -1\]

\[(s^0) : \quad 3A - 2C = -1\]

Thus the partial fraction decomposition is

\[ G(s) = -\frac{1}{s - 2} + \frac{s}{s^2 + 3} - \frac{1}{s^2 + 3} \]

\[= -\frac{1}{s - 2} + \frac{s}{s^2 + 3} - \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{s^2 + 3} \]

and the inverse transform is

\[ g(t) = -e^{2t} + \cos \left( \sqrt{3}t \right) - \frac{1}{\sqrt{3}} \sin \left( \sqrt{3}t \right). \]

(iii) \( H(s) = \frac{16}{s^3(s - 2)} \)

The partial fraction decomposition in this case is

\[ H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s - 2}. \]

Setting numerators equal and multiplying out gives

\[16 = As^2(s - 2) + Bs(s - 2) + C(s - 2) + Ds^3\]

\[= As^3 - 2As^2 + Bs^2 - 2Bs + Cs - 2C + Ds^3\]
and we have to solve the system of equations

\[(s^3) : \quad A + D = 0\]
\[(s^2) : \quad -2A + B = 0\]
\[(s^1) : \quad -2B + C = 0\]
\[(s^0) : \quad -2C = 16.\]

Thus our partial fractions decomposition becomes

\[
H(s) = -\frac{2}{s} - \frac{4}{s^2} - \frac{8}{s^3} + \frac{2}{s - 2}
\]

\[
= -\frac{2}{s} - \frac{4}{s^2} - \frac{8}{2! s^3} + \frac{2}{s - 2}
\]

and the inverse transform is

\[
h(t) = -2 + 4t - 4t^2 + 2e^{2t}.
\]
Step Functions

When we defined the Laplace transform, we made mention of the fact that we didn’t have to transform only continuous functions; we could transform a function that was piecewise continuous instead. So far, however, all of the examples of transforms and inverse transforms we’ve considered have involved only continuous functions.

It would be nice to be able to solve differential equations with forcing functions that weren’t continuous, but had isolated points of discontinuity where the forcing function jumped from one value to another abruptly. We can easily imagine applications where these would arise: a mechanical vibration where we add some extra force later on or an electrical circuit where a voltage is switched on or off at some specified time. This is where Laplace transforms come into their own: for the piecewise continuous forcing functions that arise in these sorts of situations, we can use these transforms to solve the initial value problem much more efficiently than we could otherwise.

Without transforms, we would have to split these initial value problems into several different problems, with each subsequent initial condition derived from the earlier solutions. But with Laplace transforms, we can deal with them in a single stroke. To do so, we’ll need to introduce some new notation.

1. Step Functions

Consider the following function.

\[ u_c(t) = \begin{cases} 
0 & \text{if } t < c \\
1 & \text{if } t \geq c 
\end{cases} \]

This function is called the step or Heaviside function at \( c \). It represents a jump at \( t = c \) from zero (which means anything multiplied by it doesn’t contribute anything prior to time \( c \)) to one at \( t = c \) (where any coefficient it has will begin to contribute). An example graph of \( u_c(t) \) is plotted in Figure 24.1. You can think of a step function as a switch that turns on its coefficient at a specified time. You may also sometimes see a slightly more symmetrical notation,

\[ u(t - c) = u_c(t). \]

The step function itself takes only values of 0 or 1, but it’s easy enough to make it “turn on” any value we desire. For example, \( 4u_c(t) \) will contribute 4 once \( t = c \) and beyond while contributing nothing earlier, while \( -7u_c(t) \) will contribute \(-7\) once \( t \) hits \( c \) and nothing earlier.

We can also play with the step function to get a switch that turns off at \( t = c \). If we consider the function

\[ 1 - u_c(t) = \begin{cases} 
1 - 0 = 1 & \text{if } t < c \\
1 - 1 = 0 & \text{if } t \geq c 
\end{cases} \]

it will exhibit precisely this sort of behavior. We use this, again, to get any contribute we might want only prior to \( t = c \): e.g., \( 3 - 3u_c(t) \) will give us a value of 3 when \( t < c \) and 0 when \( t \geq c \).

Step functions are the key to neatly writing down piecewise continuous functions as a single expression, rather than as a system of cases.
Figure 24.1. Plot of the step (or Heaviside) function $u_c(t)$. The function is zero at for $t < c$ and one for $t \geq c$. This function is useful for representing non-continuous changes in forcing functions, which occur often in engineering applications.

Figure 24.2. Plot of the function $f(t)$ from Example 24.1. This function, which illustrates several jump discontinuities, can be written in terms of step functions, as the Example shows.

Example 24.1. Write the following function in terms of step functions.

$$f(t) = \begin{cases} 
-6 & \text{if } t < 2 \\
14 & \text{if } 2 \leq t < 6 \\
20 & \text{if } 6 \leq t < 9 \\
9 & \text{if } 9 \leq t 
\end{cases}$$

There are three jumps in this function: at $t = 2$, $t = 6$, and $t = 9$, as seen in Figure 24.2. So we will need a total of three step functions, each of which will correspond to one of these jumps.
Differential Equations

Lecture 24: Step Functions

In terms of step functions,

\[ f(t) = -6 + 20u_2(t) + 6u_6(t) - 11u_9(t). \]

How did we find that?

When \( t < 2 \), all of the step functions have a value of 0. So the only contributing term in our expression for \( f(t) \) is \(-6\), and on this region \( f(t) = -6 \).

On the next interval, \( 2 \leq t < 6 \), we want \( f(t) = 14 \). The first step function \( u_2(t) \) is on, while the others are off. Notice that the \(-6\) term is still contributing, as nothing gets turned off. As a result, the coefficient of \( u_2(t) \) will need to cause the sum of it and \(-6\) to equal 14. Thus it must be 20.

On the third interval, \( 6 \leq t < 9 \), we now have two “on” step functions while the last one is off. The first two terms, the \(-6\) and the \( 20u_2(t) \), continue to contribute, as again they never turn “off.” Thus the coefficient of the newly contributing step function, \( u_6(t) \), will need to 14 to give us our desired value \( f(t) = 20 \). Thus it must be 6.

Lastly, we have the interval \( 9 < t \). Now all of the terms contribute, and the coefficient of \( u_9(t) \), the step function corresponding to the final jump, should move us from our previous value of 20 to our new value, \( f(t) = 9 \). As a result, it must be \(-11\).

So the function written above has the correct value on all intervals. \( \square \)

We’re not usually interested just in situations where our forcing function takes constant values on intervals, however. In the case of mechanical vibrations of the sort we considered earlier, we might want to add in a new external force which is sinusoidal. Or a nonconstant voltage might be switched on at some time after we’ve started considering an electrical circuit.

So we will consider the following piecewise continuous function: \( g(t) = u_c(t)f(t - c) \), where \( f(t) \) is some function. We shift it by \( c \), the starting point of the step function, to indicate that we want it to start working at \( t = c \) instead of \( t = 0 \), which it would normally. Think of this graphically (as illustrated in Figure 24.3): to get the graph of \( g(t) \), what we want to do is take the graph of \( f(t) \), starting at \( t = 0 \), and “push” it to start at \( t = c \) with a value of 0 prior to this time. This requires us to shift the argument of \( f \) by \( c \).
2. Laplace Transform

What is the Laplace transform \( L\{g(t)\} \)?

\[
L\{g(t)\} = L\{u_c(t)f(t-c)\}
\]
\[
= \int_0^\infty u_c(t)e^{-st}f(t-c)\,dt
\]
\[
= \int_c^\infty e^{-st}f(t-c) \quad \text{using the definition of the step function.}
\]

Now, this looks sort of like a Laplace transform, except that the integral starts at \( t = c \) instead of \( t = 0 \). So we’ll introduce a new variable \( u = t - c \) to shift the integral to start at 0.

\[
G(s) = \int_0^\infty e^{-s(u+c)}f(u)\,du
\]
\[
= e^{-sc}\int_0^\infty e^{-su}f(u)\,du
\]
\[
= e^{-sc}F(s) \quad \text{using the notation } F(s) = L\{f(u)\}.
\]

Notice that the Laplace transform we end up with is the Laplace transform of the original function \( f(t) \) multiplied by an exponential related to the step function’s “on time”, even though we had shifted the function by \( c \) to begin with. Summarizing, we have the formula

\[
L\{u_c(t)f(t-c)\} = e^{-sc}F(s) = e^{-sc}L\{f(t)\}.
\]  \hspace{1cm} (24.1)

It is critical that we write the function to be transformed in the correct form, as a different function shifted by \( c \), before we transform by using Equation (24.1). Again, it cannot be noted strongly enough that when using this formula, we end up computing the transform of \( f(t) \), not the shifted function \( f(t-c) \). This is something that can be a sticking point initially, but with practice it usually becomes clear.

We can, of course, use Equation (24.1) to get a formula for a step function by itself. To do so, we consider a step function multiplied by the constant function \( f(t) = 1 \). In this case, \( f(t-c) = 1 \) as well, since it doesn’t matter what value of \( t \) we input into \( f \), we still get the same output 1.

Doing this gives

\[
L\{u_c(t)\cdot 1\} = e^{-cs}L\{1\} = \frac{1}{s}e^{-cs}.
\]  \hspace{1cm} (24.2)

**Example 24.2.** Find the Laplace transforms of each of the following.

(i) \( f(t) = 4u_6(t) + 3(t-4)^2u_4(t) - (1 + e^{10-2t})u_5(t) \)

Recall that we must write each piece in the form \( u_c(t)h(t-c) \) before we take the transform. If it isn’t in that form, we have to put it in that form.

There are three terms in \( f(t) \). We will use the linearity of the Laplace transform to treat them separately, then add them all together in the end. Let’s write

\[
f(t) = f_1(t) + f_2(t) + f_3(t)
\]

to keep ourselves organized.

\( f_1(t) = 4u_6(t) \), so it is just a constant times a step function. We can thus use Equation (24.2) to determine its Laplace transform.

\[
L\{f_1(t)\} = L\{4u_6(t)\} = \frac{4e^{-6s}}{s}
\]

\( f_2(t) = 3(t-4)^2u_4(t) \), so we have to do two things: write it as a function shifted by 4 (if it isn’t in that form already) and isolate the function that was shifted and transform it. In this
case, we’re good: we can write $f_2(t) = h(t - 4)u_4(t)$, with $h(t) = 3t^2$. Thus

$$\mathcal{L} \{f_2(t)\} = e^{-4s} \mathcal{L} \{3t^2\} = 3e^{-4s} \frac{2}{s^3} = \frac{6e^{-4s}}{s^3}.$$  

Finally, we have $f_3(t) = -(1 + e^{10-2t})u_5(t)$. Again, we have to express it as a function shifted by 5 and then identify the unshifted function so that we may transform it. This can be accomplished by rewriting

$$f_3(t) = -(1 + e^{-2(t-5)})u_5(t),$$

so, writing $g_3(t) = h(t - 5)u_5(t)$, we have $h(t) = -(1 + e^{-2t})$ as the unshifted coefficient function. Thus

$$\mathcal{L} \{f_3(t)\} = e^{-5s} \mathcal{L} \{- (1 + e^{-2t})\} = -e^{-5s} \left( \frac{1}{s} + \frac{1}{s + 2} \right).$$

Putting it all together,

$$F(s) = \frac{4e^{-6s}}{s} + \frac{6e^{-4s}}{s^3} - e^{-5s} \left( \frac{1}{s} + \frac{1}{s + 2} \right).$$

(ii) $g(t) = t^2u_2(t) + \cos(t)u_5(t)$. 

In the last example, it turned out that all of the coefficient functions were pre-shifted (the most we had to do was pull out a constant to see that). In this example, that is definitely not the case. So what we want to do is to write each of our coefficient functions as the shift (by whichever constant is appropriate for that step function) of a different function. The idea is that we add and subtract the desired quantity, then simplify, keeping the correct shifted term. So, let’s write $g(t) = g_1(t) + g_2(t) + g_3(t)$. $g_1(t) = t^2u_2(t)$.

$$g_1(t) = (t - 2 + 2)^2 u_2(t)$$

This isn’t quite right, but we’ll use the associativity property of addition:

$$g_1(t) = ((t - 2) + 2)^2 u_2(t)$$

$$= (t - 2)^2 + 4(t - 2) + 4) u_2(t).$$

Now we can see that $g_1(t) = h(t - 2)u_2(t)$, where $h(t) = t^2 + 4t + 4$.

$$\mathcal{L} \{g_1(t)\} = e^{-2s} \mathcal{L} \{t^2 + 4t + 4\} = e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right)$$

The second term is similar. We start with

$$g_2(t) = \cos(t)u_5(t) = \cos ((t - 5) + 5) u_5(t).$$

Here we need to use the trig identity

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b).$$

This yields

$$g_2(t) = (\cos(t - 5) \cos(5) - \sin(t - 5) \sin(5)) u_5(t).$$

Since $\cos(5)$ and $\sin(5)$ are just constants, we get (after using the linearity of the Laplace transform)

$$\mathcal{L} \{g_2(t)\} = e^{-5s} \left( \cos(5) \mathcal{L} \{\cos(t)\} - \sin(5) \mathcal{L} \{\sin(t)\} \right)$$

$$= e^{-5s} \left( \frac{s \cos(5)}{s^2 + 1} - \frac{\sin(5)}{s^2 + 1} \right).$$
Piecing everything back together, we get

\[ G(s) = e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) + e^{-5s} \left( \frac{s \cos(5) - \sin(5)}{s^2 + 1} \right). \]

(iii) \( f(t) = \begin{cases} 
  t^3 & \text{if } t < 4 \\
  t^3 + 2 \sin \left( \frac{t}{12} - \frac{1}{3} \right) & 4 \leq t
\end{cases} \)

The first step here is to write \( f(t) \) as a single expression using step functions.

Next, we want to write the coefficients of \( u_4(t) \) as another function shifted by 4.

Since everything is appropriately shifted, we have

\[ F(s) = \mathcal{L} \left\{ t^3 \right\} + 2e^{-4s} \mathcal{L} \left\{ \sin \left( \frac{t}{12} \right) \right\} \]

\[ = \frac{3!}{s^4} + 2e^{-4s} \frac{1}{s^2} + \left( \frac{1}{144} \right)^2 \]

\[ = \frac{6}{s^4} + \frac{e^{-4s}}{12 \left( s^2 + \frac{1}{144} \right)} \]

\[ = \frac{6}{s^4} + \frac{e^{-4s}}{12s^2 + \frac{1}{12}}. \]

(iv) \( g(t) = \begin{cases} 
  t & \text{if } t < 2 \\
  3 + (t - 2)^2 & \text{if } 2 \leq t
\end{cases} \)

First, we need to write \( g(t) \) using step functions.

\[ g(t) = t + \left( -t + 3 + (t - 2)^2 \right) u_2(t). \]

Notice that we had to subtract \( t \) from the coefficient of \( u_2(t) \) in order to make \( g(t) \) have the correct value when \( t \geq 2 \). However, this means that the coefficient function of \( u_2(t) \) is no longer properly shifted. As a result, we need to add and subtract 2 from that \( t \) to make it have the proper form.

\[ g(t) = t + \left( -\left( t - 2 \right) + 3 + (t - 2)^2 \right) u_2(t) \]

\[ = t + \left( -\left( t - 2 \right) - 2 + 3 + (t - 2)^2 \right) u_2(t) \]

\[ = t + \left( -\left( t - 2 \right) + 1 + (t - 2)^2 \right) u_2(t) \]

So, we have

\[ G(s) = \mathcal{L} \left\{ t \right\} + e^{-2s} \left( \mathcal{L} \left\{ -t \right\} + \mathcal{L} \left\{ 1 \right\} + \mathcal{L} \left\{ t^2 \right\} \right) \]

\[ = \frac{1}{s^2} + e^{-2s} \left( -\frac{1}{s^2} + \frac{1}{s} + \frac{2}{s^3} \right). \]

□

As you can see, taking Laplace transforms of functions involving step functions can be a bit more complicated than taking Laplace transforms of other functions. It’s still not too bad, we just have to be careful to make sure that our coefficient functions are appropriately shifted.
LECTURE 25

Inverse Transforms of Step Functions

Last class, we started to consider piecewise continuous functions. We saw that we could write them in terms of step functions $u_c(t)$ and that the Laplace transform

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\},$$

where $f(t-c)$ is the coefficient function of $u_c(t)$ written as a function shifted by $c$.

Now, let’s look at some inverse transforms. The previous formula’s associated inverse transform is

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c), \quad (25.1)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$. So here, we need to be careful about shifting: this time, though, we do it at the end, after finding the inverse transform of the coefficient of the exponential.

**Example 25.1.** Find the inverse Laplace transform of the following.

(i) $F(s) = \frac{se^{-4s}}{(2s-4)(s+3)}$

Whenever we do these, it’s a good idea to ignore the exponential and determine the inverse transform of whatever’s left over first. When it comes to splitting things up into terms, we should use our discretion: sometimes there’s no way around it, but sometimes we can save ourselves work by not splitting things up. If you want, you can err on the side of treating everything separately. Then, by Equation (25.1), the inverse transform of that term will be the appropriate step function multiplied by the shifted inverse transform we calculated.

In this case, we can’t split anything up, as there’s only one exponential and no terms without an exponential. So we’ll pull out the exponential and ignore it for the time being:

$$F(s) = e^{-4s} \frac{s}{(2s-4)(s+3)} = e^{-4s}H(s).$$

We want to determine $h(t)$; once we have that, Equation (25.1) tells us that the inverse transform will be

$$f(t) = h(t-4)u_4(t).$$

Now we need to partial fraction $H(s)$ so that we can take its inverse transform. The form of the decomposition is

$$H(s) = \frac{A}{2s-4} + \frac{B}{s+3} = \frac{A(s+3) + B(2s-4)}{(2s-4)(s+3)}.$$

Setting numerators equal, we have

$$s = A(s+3) + B(2s-4).$$

We can use the quick method of finding “key” values of $s$ here.

$$s = 2: \quad 2 = 5A \quad \Rightarrow \quad A = \frac{2}{5}$$

$$s = -3: \quad -3 = -10B \quad \Rightarrow \quad B = \frac{3}{10}$$

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So the partial fraction decomposition is
\[ H(s) = \frac{2}{5} \frac{1}{2s - 4} + \frac{3}{10} \frac{1}{s + 3} \]
\[ = \frac{2}{5} \frac{1}{2(s - 2)} + \frac{3}{10} \frac{1}{s + 3}. \]
Notice here that we factored a 2 out of the denominator. Taking the inverse transform, we have
\[ h(t) = \frac{1}{5} e^{2t} + \frac{3}{10} e^{-3t}. \]
Let’s return to the original problem. We wanted to find the inverse transform of
\[ F(s) = e^{-4s} H(s). \]
This will be, by Equation (25.1),
\[ f(t) = h(t - 4) u_4(t), \]
where \( h(t) \) is what we just found above. So
\[ f(t) = u_4(t) \left( \frac{e^{2(t-4)}}{5} + \frac{3e^{-3(t-4)}}{10} \right) \]
\[ = u_4(t) \left( \frac{e^{2t-8}}{5} + \frac{3e^{-3t+12}}{10} \right). \]
(ii) \( G(s) = \frac{2e^{-3s} + e^{-7s}}{(s - 3)(s^2 + 4)} \)
As in the previous example, we want to begin by ignoring the exponentials. We could begin by writing
\[ G(s) = e^{-3s} \frac{2}{(s - 3)(s^2 + 4)} + e^{-7s} \frac{1}{(s - 3)(s^2 + 4)}, \]
then call the first fraction \( F(s) \), the second \( G(s) \), and begin doing partial fractions. But notice that if we pull out the constant from the first term, we have
\[ G(s) = 2e^{-3s} \frac{1}{(s - 3)(s^2 + 4)} + e^{-7s} \frac{1}{(s - 3)(s^2 + 4)}, \]
and the two fractions are in fact the same function. So we can save ourselves some effort by writing
\[ G(s) = (2e^{-3s} + e^{-7s}) \frac{1}{(s - 3)(s^2 + 4)} = (2e^{-3s} + e^{-7s}) H(s). \]
If you wanted to go the first route, that would be fine. It’s best to use your discretion and go with whichever you’re comfortable with.
Now we want to find the inverse transform of
\[ H(s) = \frac{1}{(s - 3)(s^2 + 4)}. \]
The partial fraction decomposition is
\[ H(s) = \frac{A}{s - 3} + \frac{Bs + C}{s^2 + 4} = \frac{A(s^2 + 4) + (Bs + C)(s - 3)}{(s - 3)(s^2 + 4)}. \]
Setting numerators equal and simplifying gives
\[ 1 = A(s^2 + 4) + (Bs + C)(s - 3) \]
\[ = (A + B)s^2 + (-3B + C)s + (4A - 3C). \]
Setting coefficients equal and solving gives
\[
\begin{align*}
    s^2 : & \quad A + B = 0 \\
    s^1 : & \quad -3B + C = 0 \quad \Rightarrow \quad A = \frac{1}{13} \quad B = -\frac{1}{13} \quad C = \frac{-3}{13} \\
    s^0 : & \quad 4A - 3C = 1
\end{align*}
\]
Substituting back into the transform (and pulling out the denominator of 13), we get
\[
H(s) = \frac{1}{13} \left( \frac{1}{s - 3} + \frac{-s - 3}{s^2 + 4} \right) = \frac{1}{13} \left( \frac{1}{s - 3} - \frac{s}{s^2 + 4} - \frac{3}{2} \frac{2}{s^2 + 4} \right).
\]
Notice that, as in the previous lecture, we had to multiply the last term by \( \frac{2}{2} \) so that we could get the correct denominator for inverse transforming. Now, if we take the inverse transform, we get
\[
h(t) = \frac{1}{13} \left( e^{3t} - \cos(2t) - \frac{3}{2} \sin(2t) \right).
\]
Returning to the original problem, we had
\[
G(s) = (2e^{-3s} + e^{-7s}) H(s) = 2e^{-3s} H(s) + e^{-7s} H(s).
\]
We had to distribute \( H(s) \) through the parentheses to use (25.1), since we must end up with each term containing one step function and one coefficient function. By (25.1), we have
\[
g(t) = 2h(t - 3)u_3(t) + h(t - 7)u_7(t)
\]
\[
\quad = \frac{1}{13} \left( 2e^{3t-9} - 2 \cos(2t - 6) - 3 \sin(2t - 6) + e^{3t-21} - \cos(2t - 14) - \frac{3}{2} \sin(2t - 14) \right).
\]
(iii) \( F(s) = \frac{4s + 2e^{-4s}}{s^2(s - 2)} \)

Here we will first have to break up our transform into two pieces, since one term has a constant and the other has an \( s \). The exponential is, for these considerations, irrelevant. We write
\[
F(s) = \frac{4s}{s^2(s - 2)} + e^{-4s} \frac{2}{s^2(s - 2)} = F_1(s) + e^{-4s} F_2(s).
\]
We’ll need to partial fraction the functions \( F_1(s) \) and \( F_2(s) \) separately. Let’s consider \( F_1(s) \) first.
\[
F_1(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 2}
\]
\[
4s = As(s - 2) + B(s - 2) + Cs^2 = (A + C)s^2 + (-2A + B)s - 2B
\]
Now we find the constants
\[
A + C = 0 \\
-2A + B = 4 \quad \Rightarrow \quad A = -2 \quad B = 0 \quad C = 2 \\
-2B = 0
\]
So \( F_1(s) \) and its inverse transform are
\[
F_1(s) = -\frac{2}{s} + \frac{2}{s - 2} \\
\]
\[ f_1(t) = -2 + 2e^{2t} \]

Now we’ll repeat the process for \( F_2(s) \).
\[
F_2(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 2} \\
2 = As(s - 2) + B(s - 2) + Cs^2 \\
= (A + C)s^2 + (-2A + B)s - 2B
\]
So we have
\[
A + C = 0 \\
-2A + B = 0 \quad \Rightarrow \quad A = -\frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}.
\]
Thus \( F_2(s) \) and its inverse transform are
\[
F_2(s) = -\frac{1}{2} s - \frac{1}{s^2} + \frac{1}{2} \frac{1}{s - 2} \\
f_2(t) = -\frac{1}{2} - t + \frac{1}{2} e^{2t}.
\]

Our original transformed function was
\[
F(s) = F_1(s) + e^{-4s}F_2(s).
\]
Now, by (25.1), our inverse transform will be
\[
f(t) = f_1(t) + f_2(t - 4)u_4(t) \\
= -2 + 2e^{2t} + \left(-\frac{1}{2} - (t - 4) + \frac{1}{2} e^{2(t-4)}\right) u_4(t) \\
= -2 + 2e^{2t} + \left(\frac{7}{2} - t + \frac{1}{2} e^{2t-8}\right) u_4(t)
\]

(iv) \( G(s) = \frac{2 - se^{-2s}}{s^2 - 2s + 10} \)

In this case, we won’t have to do partial fractions, since the denominator doesn’t factor. Instead, we’ll have to complete the square, and we get
\[
G(s) = \frac{2 - se^{-2s}}{(s - 1)^2 + 9} = \frac{2}{(s - 1)^2 + 9} - e^{-2s} \frac{s}{(s - 1)^2 + 9} = G_1(s) - e^{-2s}G_2(s).
\]
As in the last example, we need to treat \( G_1(s) \) and \( G_2(s) \) separately. For \( G_1(s) \), we need the numerator to involve 3, so we’ll multiply top and bottom by \( \frac{3}{3} \).
\[
G_1(s) = \frac{2}{(s - 1)^2 + 9} \\
= \frac{2}{3} \frac{3}{(s - 1)^2 + 9} \\
g_1(t) = \frac{2}{3} e^t \sin(3t)
\]
For $G_2(s)$, however, we need to use a trick from last class. We need the numerator to involve $s - 1$, but there is only an $s$. So we have to add and subtract 1.

$$G_2(s) = \frac{s}{(s - 1)^2 + 9}$$

$$= \frac{s - 1 + 1}{(s - 1)^2 + 9}$$

$$= \frac{s - 1}{(s - 1)^2 + 9} + \frac{1}{3(s - 1)^2 + 9}$$

$$g_2(t) = e^t \cos(3t) + \frac{1}{3} e^t \sin(3t)$$

Our original transform was

$$G(s) = G_1(s) + e^{-2s} G_2(s).$$

By Equation (25.1),

$$g(t) = g_1(t) + g_2(t - 2) u_2(t)$$

$$= \frac{2}{3} e^t \sin(3t) + \left( e^{t-2} \cos(3(t - 2)) + \frac{1}{3} e^{t-2} \sin(3(t - 2)) \right) u_2(t)$$

$$= \frac{2}{3} e^t \sin(3t) + \left( e^{t-2} \cos(3t - 6) + \frac{1}{3} e^{t-2} \sin(3t - 6) \right) u_2(t).$$
Differential Equations

LECTURE 26

IVPs with Laplace Transforms

Now that we have a good grasp of how to take Laplace transforms and inverse transforms, let’s return to differential equations. First, we should recall the following formula, with \( f^{(n)} \) denoting the \( n \)th derivative of \( f \):

\[
\mathcal{L}\left\{ f^{(n)} \right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).
\]

We’ll be dealing exclusively in this lecture with second order differential equations, so in particular, we’ll need

\[
\mathcal{L}\{y'\} = sY(s) - y(0)
\]

and

\[
\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0).
\]

You should be familiar with the general formula, however.

Remark. Notice that we must have our initial conditions at \( t = 0 \) to use Laplace transforms.

Example 26.1. Solve the following IVP using Laplace transforms.

\[
y'' - 5y' - 6y = 5t \quad y(0) = -1 \quad y'(0) = 2
\]

We begin by transforming both sides of the equation:

\[
\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 5\mathcal{L}\{t\}
\]

\[
s^2 Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) = \frac{5}{s^2}
\]

\[
(s^2 - 5s - 6) Y(s) + s - 2 - 5 = \frac{5}{s^2}.
\]

As we’ve already begun doing, now we solve for \( Y(s) \).

\[
Y(s) = \frac{5}{s^2(s^2 - 5s - 6)} + \frac{7 - s}{s^2 - 5s - 6}
\]

\[
= \frac{5}{s^2(s - 6)(s + 1)} + \frac{7 - s}{(s - 6)(s + 1)}
\]

\[
= \frac{5 + 7s^2 - s^3}{s^2(s - 6)(s + 1)}
\]

We now have an expression for \( Y(s) \), which is the Laplace transform of the solution \( y(t) \) to the initial value problem. We’ve simplified it as much as we can; now it’s time to take the inverse transform. The partial fraction decomposition is

\[
Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 6} + \frac{D}{s + 1}.
\]

Setting numerators equal gives us

\[
6 + 7s^2 - s^3 = As(s - 6)(s + 1) + B(s - 6)(s + 1) + Cs^2(s + 1) + Ds^2(s - 6).
\]
We can find constants by choosing key values of $s$.

$s = 0$ : $6 = -6B \Rightarrow B = -1$

$s = 6$ : $42 = 252C \Rightarrow C = \frac{1}{6}$

$s = -1$ : $14 = -7D \Rightarrow D = -\frac{1}{2}$

$s = 1$ : $12 = -10A + \frac{77}{6} \Rightarrow A = \frac{1}{12}$

So

$$Y(s) = \frac{1}{12} s - \frac{1}{2} \frac{1}{s^2} + \frac{1}{6} \frac{1}{s - 6} - \frac{1}{2} \frac{1}{s + 1}$$

$$y(t) = \frac{1}{12} - t + \frac{1}{6} e^{6t} - \frac{1}{2} e^{-t}.$$  

\[\square\]

**Exercise.** Solve the initial value problem in the previous example using Undetermined Coefficients. Do you get the same thing? Which method took less work?

**Example 26.2.** Solve the following initial value problem.

$$y'' + 2y' + 5y = \cos(t) - 10t \quad y(0) = 0 \quad y'(0) = 1$$

We begin by transforming the entire equation and solving for $Y(s)$.

\[
\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{\cos(t)\} - 10\mathcal{L}\{t\}
\]

\[
s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{s}{s^2 + 1} - \frac{10}{s^2}
\]

\[
(s^2 + 2s + 5)Y(s) - 1 = \frac{s}{s^2 + 1} - \frac{10}{s^2}
\]

So we have

$$Y(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 5)} - \frac{10}{s^2(s^2 + 2s + 5)} + \frac{1}{s^2 + 2s + 5}$$

$$= Y_1(s) + Y_2(s) + Y_3(s).$$

Now we'll have to take inverse transforms. This will require doing partial fractions on the first two pieces.

Let's start with the first one.

$$Y_1(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 5)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 5}$$

After putting everything over a common denominator, we get numerators equal.

\[
s = (A + C)s^3 + (2A + B + D)s^2 + (5A + 2B + C)s + (5B + D)
\]

This gives us the following system of equations, which we solve.

\[
A + C = 0 \\
2A + B + D = 0 \quad \Rightarrow \quad A = \frac{1}{5} \quad B = \frac{1}{10} \quad C = -\frac{1}{5} \quad D = -\frac{1}{2}
\]

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Thus our first term becomes

\[ Y_1(s) = \frac{1}{5} \frac{s}{s^2 + 1} + \frac{1}{10} \frac{1}{s^2 + 1} - \frac{1}{5} \frac{s}{s^2 + 2s + 5} - \frac{1}{2} \frac{1}{s^2 + 2s + 5}. \]

We’ll hold off on taking the inverse transform for the time being.

Now, let’s deal with \( Y_2(s) \).

\[ Y_2(s) = -\frac{10}{s^2(s^2 + 2s + 5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 5} \]

We put everything over a common denominator and set numerators equal.

\[-10 = As(s^2 + 2s + 5) + B(s^2 + 2s + 5) + Cs^3 + Ds^2 \]

\[= (A + C)s^3 + (2A + B + D)s^2 + (5A + 2B)s + 5B \]

This gives the following system of equations.

\[ A + C = 0 \]
\[ 2A + B + D = 0 \]
\[ 5A + 2B = 0 \]
\[ 5B = -10 \]

This gives us

\[ Y_2(s) = \frac{4}{5} \frac{1}{s} - \frac{2}{s^2} - \frac{4}{5} \frac{s}{s^2 + 2s + 5} + \frac{2}{5} \frac{1}{s^2 + 2s + 5} \]

Let’s return to our original function.

\[ Y(s) = Y_1(s) + Y_2(s) + Y_3(s) \]

\[= \frac{1}{5} \frac{s}{s^2 + 1} + \frac{1}{10} \frac{1}{s^2 + 1} + \frac{4}{5} \frac{1}{s} - \frac{2}{s^2} + \left( -\frac{1}{5} - \frac{4}{5} \right) \frac{s}{s^2 + 2s + 5} + \left( -\frac{1}{2} + \frac{2}{5} + 1 \right) \frac{1}{s^2 + 2s + 5} \]

\[= \frac{1}{5} \frac{s}{s^2 + 1} + \frac{1}{10} \frac{1}{s^2 + 1} + \frac{4}{5} \frac{1}{s} - \frac{2}{s^2} - \frac{s}{(s + 1)^2 + 4} + \frac{9}{10} \frac{1}{(s + 1)^2 + 4} \]

Now we have to adjust the last two terms to make them suitable for the inverse transform. Namely, we need to have \( s + 1 \) in the numerator of the second to last, and 2 in the numerator of the last.

\[= \frac{1}{5} \frac{s}{s^2 + 1} + \frac{1}{10} \frac{1}{s^2 + 1} + \frac{4}{5} \frac{1}{s} - \frac{2}{s^2} - \frac{s + 1}{(s + 1)^2 + 4} + \frac{9}{10} \frac{1}{(s + 1)^2 + 4} \]

\[= \frac{1}{5} \frac{s}{s^2 + 1} + \frac{1}{10} \frac{1}{s^2 + 1} + \frac{4}{5} \frac{1}{s} - \frac{2}{s^2} - \frac{s + 1}{(s + 1)^2 + 4} + \frac{19}{20} \frac{2}{(s + 1)^2 + 4} \]

So our solution is

\[ y(t) = \frac{1}{5} \cos(t) + \frac{1}{10} \sin(t) + \frac{4}{5} - 2t - e^{-t} \cos(2t) + \frac{19}{20} e^{-t} \sin(2t). \]

We could have done both of the preceding examples using Undetermined Coefficients. In fact, it would have been a lot less work. Let’s do some involving step functions; recall that none of our previous solution methods could cope with non-continuous forcing functions.
**Example 26.3.** Solve the following initial value problem.
\[ y'' - 5y' + 6y = 2 - u_2(t)e^{2(t-4)} \quad y(0) = 0 \quad y'(0) = 0 \]

As before, we begin by transforming everything. Before we do that, however, we need to write the coefficient function of \( u_2(t) \) as a function evaluated at \( t-2 \).
\[ y'' - 5y' + 6y = 2 - u_2(t)e^{2(t-2)} \]

Now we can transform.
\[
\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = 2\mathcal{L}\{1\} - \mathcal{L}\{u_2(t)e^{2(t-2)}\}
\]
\[
s^2Y(s) - sy(0) - y'(0) - 5sY(s) + 5y(0) - 6Y(s) = \frac{2}{s} - e^{-2s}\mathcal{L}\{e^{2t}\}
\]
\[
(s^2 - 5s + 6)Y(s) = \frac{2}{s} - e^{-2s}\frac{1}{s - 2}
\]

So we end up with
\[
Y(s) = \frac{2}{s(s-3)(s-2)} - e^{-2s}\frac{1}{(s-3)(s-2)^2}
\]
\[
= Y_1(s) + e^{-2s}Y_2(s).
\]

Since one of these terms has an exponential, we’ll need to deal with them separately. I’ll leave it to you to check all of the partial fractions.
\[
Y_1(s) = \frac{11}{3s} + \frac{2}{3s-3} - \frac{1}{s-2}
\]
\[
Y_2(s) = -\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2}
\]

Thus we have
\[
Y(s) = \frac{11}{2s} + \frac{2}{3s-3} - \frac{1}{s-2} + e^{-2s}\left(-\frac{1}{s-3} + \frac{1}{s-2} + \frac{1}{(s-2)^2}\right)
\]

and
\[
y(t) = \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3(t-2)} + e^{2(t-2)} + (t-2)e^{2(t-2)}\right)
\]
\[
= \frac{1}{2} + \frac{2}{3}e^{3t} - e^{2t} + u_2(t)\left(-e^{3t-6} - e^{2t-4} + te^{2t-4}\right)
\]

once we observe that \( \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}. \]

**Example 26.4.** Solve the following initial value problem.
\[ y'' + 4y = 8 + tu_4(t) \quad y(0) = 0 \quad y'(0) = 0 \]

We need to first write the coefficient function of \( u_4(t) \) in the form \( h(t-4) \) for some function \( h(t) \). So we write \( h(t-4) = t = t - 4 + 4 \) and conclude \( h(t) = t + 4 \). So our equation is
\[ y'' + 4y = 8 + ((t-4) + 4)u_4(t) \]

Now, we want to Laplace transform everything.
\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 8\mathcal{L}\{1\} + \mathcal{L}\{((t-4) + 4)u_4(t)\}
\]
\[
s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{8}{s} + e^{-4s}\mathcal{L}\{t + 4\}
\]
\[
(s^2 + 4)Y(s) = \frac{8}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)
\]
So we have
\[ Y(s) = \frac{8}{s(s^2 + 4)} + e^{-4s} \left( \frac{1}{s^2(s^2 + 4)} + \frac{4}{s(s^2 + 4)} \right) \]
\[ = \frac{8}{s(s^2 + 4)} + e^{-4s} \frac{1 + 4s}{s^2(s^2 + 4)} = Y_1(s) + e^{-4s}Y_2(s), \]
where we’ve consolidated the two fractions being multiplied by the exponential to reduce the number of partial fraction decompositions we need to compute. After doing partial fractions (leaving the details for you to check), we have
\[ Y_1(s) = \frac{2}{s} - \frac{2s}{s^2 + 4} \]
and
\[ Y_2(s) = \frac{1}{s} + \frac{1}{4s^2} - \frac{s}{s^2 + 4} - \frac{1}{4} \frac{1}{s^2 + 4}, \]
so
\[ Y(s) = \frac{2}{s} - \frac{2s}{s^2 + 4} + e^{-4s} \left( \frac{1}{s} + \frac{1}{4s^2} - \frac{s}{s^2 + 4} - \frac{1}{4} \frac{1}{s^2 + 4} \right) \]
\[ = \frac{2}{s} - \frac{2s}{s^2 + 4} + e^{-4s} \left( \frac{1}{s} + \frac{1}{4s^2} - \frac{s}{s^2 + 4} - \frac{1}{8} \frac{1}{s^2 + 4} \right) \]
and the solution is
\[ y(t) = 2 - 2 \cos(2t) + u_4(t) \left( 1 + \frac{1}{4}(t - 4) - \cos(2(t - 4)) - \frac{1}{8} \sin(2(t - 4)) \right) \]
\[ = 2 - 2 \cos(2t) + u_4(t) \left( \frac{1}{4}t - \cos(2t - 8) - \frac{1}{8} \sin(2t - 8) \right). \]
\[ \square \]
A few lectures ago, we discussed step (or Heaviside) functions, which we could think of as a "switch," changing the nonhomogeneous term at specified times. From an applications viewpoint, we could interpret the presence of a step function as a new external force which would be applied to the system starting at a certain point.

But what if, instead of having a new force which would persist for the rest of the system’s use, all we wanted to do was apply a large force over a very small, close to instantaneous time interval? Instead of turning on a new voltage in an electrical circuit, we might want to model a short. A mechanical interpretation might be the use of a hammer to strike an object or the striking of a baseball with a bat. It would be nice if we had a mathematical way of representing these types of forces.

To this end, we will introduce a new "function", the Dirac delta function.

1. The Dirac Delta

There are several ways to define the Dirac delta, but we will do so by requiring it to satisfy several properties.

**Definition 27.1.** The Dirac delta at \( t = c \), denoted \( \delta(t - c) \), satisfies the following properties:

1. \( \delta(t - c) = 0 \) \( t \neq c \);
2. \( \int_{c-\epsilon}^{c+\epsilon} \delta(t - c) \, dt = 1 \) for any \( \epsilon > 0 \);
3. \( \int_{c-\epsilon}^{c+\epsilon} f(t) \delta(t - c) \, dt = f(c) \) for any \( \epsilon > 0 \).

Heuristically, we can think of \( \delta(t - c) \) as having an "infinite" value at \( t = c \), so that its total energy is 1, all concentrated at that point. So the Dirac delta can be thought of as an instantaneous impulse at time \( t = c \). Notice that the second and third properties work when the limits are the endpoints of any interval including \( t = c \).

This should sound like a very odd function to you. It’s zero everywhere but at one point, and yet its integral is 1; this does not seem consistent with the geometric understanding of integration that we have from calculus. This is why I put “function” in quotation marks earlier: the Dirac delta is not really a function, but is instead an example of something called a *distribution*. However, it’s perfect for our purposes: it does a fantastic job of modeling a sudden impulse to a system.

---

1Paul Dirac (1902-1984) was a British physicist who helped found quantum mechanics. Dirac was well regarded for his mathematical proficiency, which led him to make connections between certain physical principles and mathematical formalisms. Dirac proposed the "Dirac equation" as a relativistic equation of motion for an electron’s wavefunction. This led to him predicting the existence of the positron, the antiparticle of the electron. Dirac’s work also led to the way for Feynman’s introduction of the path integral formulation of quantum mechanics. Dirac’s books on general relativity and quantum mechanics are excellent introductory textbooks to those subjects.

On a lighter note, when Dirac was asked about what he thought of poetry, he answered "In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it’s the exact opposite.”
2. Laplace Transform of the Dirac Delta

We will need to know the Laplace transform of $\delta(t - c)$. By definition,

$$\mathcal{L}\{\delta(t - c)\} = \int_0^\infty e^{-st}\delta(t - c)\,dt = e^{-cs}$$

by the third property of the Dirac delta. Notice that this requires $c > 0$, since otherwise the integral in question will just vanish.

Now, let’s try solving some initial value problems involving the Dirac delta.

**Example 27.2.** Solve the following initial value problem.

$$y'' + 3y' - 10y = 4\delta(t - 2) \quad y(0) = 2 \quad y'(0) = -3$$

We begin, as usual, by taking the Laplace transform of the entire equation.

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) - 10Y(s) = 4e^{-2s}$$

$$(s^2 + 3s - 10)Y(s) - 2s - 3 = 4e^{-2s}$$

$$Y(s) = \frac{4e^{-2t}}{(s + 5)(s - 2)} + \frac{2s + 3}{(s + 5)(s - 2)}$$

$$= Y_1(s)e^{-2t} + Y_2(s)$$

We’ll leave it to you to verify that the partial fractions of each of the pieces are

$$Y_1(s) = \frac{4}{(s + 5)(s - 2)} = \frac{4}{7} - \frac{4}{7s + 5}$$

$$Y_2(s) = \frac{2s + 3}{(s + 5)(s - 2)} = \frac{1}{s - 2} + \frac{1}{s + 5}.$$

Thus they have the inverse transforms

$$y_1(t) = \frac{4}{7}e^{2t} - \frac{4}{7}e^{-5t}$$

$$y_2(t) = e^{2t} + e^{-5t}$$

and the solution is then

$$y(t) = y_1(t - 2)u_2(t) + y_2(t)$$

$$= u_2(t)\left(\frac{4}{7}e^{2(t-2)} - \frac{4}{7}e^{-5(t-2)}\right) + e^{2t} + e^{-5t}$$

$$= u_2(t)\left(\frac{4}{7}e^{2t-4} - \frac{4}{7}e^{-5t+10}\right) + e^{2t} + e^{-5t}.$$

□

Notice that, even though the exponential in the transform $Y(s)$ came originally from the delta function, once we inverse transformed the corresponding term it became a step function. This will be the case in general; we shouldn’t be surprised, because it turns out that there’s a relationship between the step function $u_c(t)$ and the delta $\delta(t - c)$.

We begin with the integral

$$\int_{-\infty}^{t} \delta(u - c)\,du = \begin{cases} 0 & t < c \\ 1 & t > c \end{cases}$$

$$= u_c(t)$$
Then the Fundamental Theorem of Calculus says

$$u'_c(t) = \frac{d}{dt} \left( \int_{-\infty}^{t} \delta(u - c) \, du \right) = \delta(t - c).$$

Thus the Dirac delta at $t = c$ is actually the derivative of the step function at $t = c$, which we can think of geometrically by remembering that the graph of $u_c(t)$ is horizontal at every $t \neq c$, hence at those points $t = 0$, and it has a jump of one at $t = c$.

Let’s do another couple of examples of initial value problems involving the Dirac delta.

**Example 27.3.** Solve the following initial value problem.

$$y'' + 4y' + 9y = 2\delta(t - 1) + e^t \quad y(0) = 0 \quad y'(0) = -1$$

First, we Laplace transform both sides and solve for $Y(s)$.

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 9Y(s) = 2e^{-s} + \frac{1}{s - 1}$$

$$(s^2 + 4s + 9)Y(s) + 1 = 2e^{-s} + \frac{1}{s - 1}$$

$$Y(s) = \frac{2e^{-s}}{s^2 + 4s + 9} + \frac{1}{(s - 1)(s^2 + 4s + 9)} - \frac{1}{s^2 + 4s + 9}$$

$$= Y_1(s)e^{-s} + Y_2(s) - Y_3(s)$$

Next, we have to prepare $Y(s)$ for the inverse transform. This will require completing the square for $Y_1(s)$ and $Y_3(s)$, while we’ll need to first partial fraction $Y_2(s)$. We’ll leave the details to you to verify, but, to get everything in the correct form for inverse transforming, we obtain

$$Y_1(s) = \frac{2}{s^2 + 4s + 9} = \frac{2}{(s + 2)^2 + 5}$$

$$= \frac{2}{\sqrt{5}(s + 2)^2 + 5}$$

$$Y_2(s) = \frac{1}{(s - 1)(s^2 + 4s + 9)} = \frac{1}{14} \left( \frac{1}{s - 1} - \frac{s + 5}{(s + 2)^2 + 5} \right)$$

$$= \frac{1}{14} \left( \frac{1}{s - 1} - \frac{s + 2}{(s + 2)^2 + 5} + \frac{3}{(s + 2)^2 + 5} \right)$$

$$Y_3(s) = \frac{1}{(s^2 + 4s + 9)} = \frac{1}{(s + 2)^2 + 5}$$

$$= \frac{1}{\sqrt{5}(s + 2)^2 + 5}.$$
So their inverse transforms are
\[ y_1(t) = \frac{2}{\sqrt{5}} e^{-2t} \sin \left( \sqrt{5} t \right) \]
\[ y_2(t) = \frac{1}{14} \left( e^t - e^{-2t} \cos \left( \sqrt{5} t \right) - \frac{3}{\sqrt{5}} e^{-2t} \sin \left( \sqrt{5} t \right) \right) \]
\[ y_3(t) = \frac{1}{\sqrt{5}} e^{-2t} \sin \left( \sqrt{5} t \right) . \]

Thus, since our original transformed function was
\[ Y(s) = Y_1(s)e^{-s} + Y_2(s) - Y_3(s), \]
we obtain
\[ y(t) = u_1(t)y_1(t - 1) + y_2(t) - y_3(t) \]
\[ = u_1(t) \left( \frac{2}{\sqrt{5}} e^{-2(t+1)} \sin \left( \sqrt{5} (t - 1) \right) \right) \]
\[ + \frac{1}{14} \left( e^t - e^{-2t} \cos \left( \sqrt{5} t \right) - \frac{3}{\sqrt{5}} e^{-2t} \sin \left( \sqrt{5} t \right) \right) \]
\[ - \frac{1}{\sqrt{5}} e^{-2t} \sin \left( \sqrt{5} t \right) . \]

**Example 27.4.** Solve the following initial value problem.
\[ y'' + 16y = 2u_3(t) + 5\delta(t - 1) \quad y(0) = 1 \quad y'(0) = 2 \]

Again, we begin by taking the Laplace transform of the entire equation and applying our initial conditions, then solving for \( Y(s) \).
\[ s^2 Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{2e^{-3s}}{s} + 5e^{-s} \]
\[ (s^2 + 16)Y(s) - s - 2 = \frac{2e^{-3s}}{s} + 5e^{-s} \]
\[ Y(s) = \frac{2e^{-3s}}{s(s^2 + 16)} + \frac{5e^{-s}}{s^2 + 16} + \frac{s + 2}{s^2 + 16} \]
\[ = Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s) \]

The only one of these three functions that needs partial fractioning\(^2\) is the first one. The rest can be dealt with directly; all they need is a little modification. We end up with
\[ Y_1(s) = \frac{2}{s(s^2 + 16)} = \frac{11}{8} \frac{1}{s} - \frac{1}{8} \frac{s}{s^2 + 16} \]
\[ Y_2(s) = \frac{5}{s^2 + 16} = \frac{5}{4} \frac{4}{s^2 + 16} \]
\[ Y_3(s) = \frac{s + 2}{s^2 + 16} = \frac{s}{s^2 + 16} + \frac{1}{2} \frac{4}{s^2 + 16} \]

\(^2\)Verbification of nouns is fun, easy, and useful!
and so the associated inverse transforms are
\[ y_1(t) = \frac{1}{8} - \frac{1}{8} \cos(4t) \]
\[ y_2(t) = \frac{5}{4} \sin(4t) \]
\[ y_3(t) = \cos(4t) + \frac{1}{2} \sin(4t). \]

Our solution is the inverse transform of
\[ Y(s) = Y_1(s)e^{-3s} + Y_2(s)e^{-s} + Y_3(s), \]
and this will be
\[ y(t) = u_3(t)y_1(t-3) + u_1(t)y_2(t-1) + y_3(t) \]
\[ = u_3(t)\left( \frac{1}{8} - \frac{1}{8} \cos(4(t-3)) \right) + \frac{5}{4} u_1(t) \sin(4(t-1)) + \cos(4t) + \frac{1}{2} \sin(4t) \]
\[ = u_3(t)\left( \frac{1}{8} - \frac{1}{8} \cos(4t - 12) \right) + \frac{5}{4} u_1(t) \sin(4t - 4) + \cos(4t) + \frac{1}{2} \sin(4t). \]
Part 5

Series Solutions
Differential Equations

LECTURE 28

Series Solutions to Differential Equations

In Lecture 1, I remarked that the differential equations that we would be dealing with in this course are not representative of the world of differential equations at large. This meant two things: first, that most differential equations do not necessarily have unique solutions, while existence and uniqueness is guaranteed for any of the classes of differential equations that we are working with; and second, even when a differential equation has a solution, often numerical methods are required to approximate that solution, whereas in this course we are focusing on obtaining closed-form solutions to particularly nice equations.

One of the methods for finding approximate solutions to differential equations (without requiring the use of a computer) is to find a power series representation of the solution. This is about as useful as approximation methods can be: if the series converges, we can approximate the solution as closely as we want, and often in practice having a power series approximation is more useful than having the solution itself. This is similar (though not exactly the same) as finding the Taylor series of a function.

1. Power Series Review

We should first review some facts about power series. Recall that a power series has the form

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \]

for some \( x_0 \) and some coefficients \( a_n \). We can pull out as many terms as we would like from the infinite sum; for example, the following are all equations for \( f(x) \):

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \]
\[ = a_0 + \sum_{n=1}^{\infty} a_n(x - x_0)^n \]
\[ = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \sum_{n=3}^{\infty} a_n(x - x_0)^n. \]

Further, a power series can be re-indexed to start at any initial index value by shifting the index in the sum. Suppose we start with the power series

\[ \sum_{n=2}^{\infty} (n - 1)a_n x^{n+3}, \]
but we want it to begin at \( n = 0 \) instead of \( n = 2 \). To do this, define \( m = n - 2 \). Then \( n = m + 2 \), and we rewrite the series in terms of \( m \):

\[
\sum_{n=2}^{\infty} (n - 1)a_n x^{n+3} = \sum_{m=0}^{\infty} (m + 2 - 1)a_{m+2} x^{m+2+3} = \sum_{m=0}^{\infty} (m + 1)a_{m+2} x^{m+5}.
\]

Then, since the index is really nothing more than a dummy variable, we can replace \( m \) with \( n \):

\[
\sum_{n=0}^{\infty} (n + 1)a_{n+2} x^{n+5}.
\]

Once you get comfortable with this process, you can see that all that was needed was to replace every instance of \( n \) with \( n + 2 \), causing the starting index to be \( n = 0 \) (the starting index value goes in the opposite direction of the variable shift, so if \( n \) is increased by two, the starting index decreases by two), and the intermediate work can be skipped.

Index shifts are important for adding or subtracting two power series together. If we have two power series, adding them together is quite simple (conceptually) as a power series is just a special type of infinite sum. However, for the formulas to be consolidated into a single sum, we would need the two series to start at the same index value, and for each index value to correspond to the same power of \((x - x_0)\).

**Example 28.1.** Write the following as a single power series:

\[
\sum_{n=0}^{\infty} na_n (x - 2)^{n+1} + \sum_{n=2}^{\infty} n^2 a_n (x - 2)^n.
\]

There are two things that need to be done here: first, we need the exponents of \( x - 2 \) to match up; and second, we need the two series need to start at the same index value of \( n \). This will require us to shift the index of the first sum down by 1, so that the exponent is \( n \) rather than \( n + 1 \). This will cause the sum to start at \( n = 1 \) rather than the current \( n = 1 \). Doing so, we get

\[
\sum_{n=1}^{\infty} (n - 1)a_{n-1}(x - 2)^n + \sum_{n=2}^{\infty} n^2 a_n (x - 2)^n.
\]

Now, we need the two series to start at the same index value. We can’t do any more index shifts, as this would cause the exponents to no longer line up, but we can pull the first term out of the first sum (the term corresponding to \( n = 1 \)). Then both sums will start at \( n = 2 \). In this case, that first term is equal to 0, but in general we will be left with a term in front of the sum. Working through this process, the sums combine into the single power series

\[
(1 - 1)a_{1-1}(x - 2) + \sum_{n=2}^{\infty} [(n - 1)a_{n-1} + n^2 b_n] (x - 2)^n = \sum_{n=2}^{\infty} [(n - 1)a_{n-1} + n^2 b_n] (x - 2)^n.
\]

Notice the change in notation for the coefficients of the second sum from \( a_n \) to \( b_n \); this was only done to avoid confusion between the two coefficient series.\(\square\)

Differentiating a power series can be done term-by-term, as it is a giant summation. That is, if

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots,
\]
we have
\[ f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \ldots = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}. \]

Notice as well that we could start the power series representation of \( f'(x) \) at \( n = 0 \) without having to shift indices, as the coefficient of \( n \) would make that term equal to zero.

1.1. Convergence. Whenever we deal with an infinite series, we have to be worried about whether the infinite sum actually adds up to anything or if it will grow without bound. In the case of a power series, we also need to be concerned about whether the sum will converge or diverge. A power series converges at \( x = c \) if the limit of partial sums
\[ \lim_{N \to \infty} \sum_{n=0}^{N} a_n(c - x_0)^n \]
exists. Any power series will always converge at \( x = x_0 \), so any given power series will always converge at least one point. How many other points the series converges at is captured by its radius of convergence, which is the (possibly infinite) value \( 0 \leq \rho \leq \infty \) such that the series converges for all \( x \) such that \( |x - x_0| < \rho \) and the series will diverge for all \( x \) such that \( |x - x_0| > \rho \). Whether or not the series converges for a given \( x \) with \( |x - x_0| = \rho \) has to be determined on a case-by-case basis.

The most common way to determine the radius of convergence for a series is to use the ratio test. To use this test, compute
\[ L = |x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|. \]
If \( L < 1 \), the series will converge, if \( L > 1 \), it will diverge, and if \( L = 1 \) the test fails and we will need to use another method.

If at this point, you need more details on power series, a calculus reference would be a good resource.

2. Series Solutions

Now suppose we have the homogeneous linear second order differential equation
\[ p(x)y'' + q(x)y' + r(x) = 0. \quad (28.1) \]

Definition 28.2. We say that a point \( x = x_0 \) is an ordinary point if the functions \( q(x)/p(x) \) and \( r(x)/p(x) \) are analytic at \( x_0 \); that is, if they have Taylor series expansions at \( x_0 \) with a positive radius of convergence and which converge appropriately to the original functions.

A point that is not an ordinary point is a singular point.

It is generally the case that singular points occur where \( p(x) = 0 \), but as the next example demonstrates, this is not always the case.

Example 28.3. The point \( x = 0 \) is a ordinary point of the equation
\[ xy'' + \sin(x)y' + x^2y = 0, \]
since \( \sin(x)/x \) and \( x \) are analytic at \( x = 0 \). On the other hand, \( x = 0 \) is a singular point for
\[ xy'' + y' + x^2y = 0. \]
It will turn out that we can only find series solutions to differential equations near ordinary points, but in practice, this is not as big of an obstacle as it might seem. For any equation that we will look at, there aren’t many singular points (though I won’t go into the technical definition of “many” in this class). The procedure for finding a series solution to Equation (28.1) is to suppose that a series representation of the solution exists,

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0), \]

and then we can determine what the coefficients \( a_n \) need to be by differentiating Equation (2) and plugging these derivatives into Equation (28.1). Once we have the appropriate coefficients, we say that Equation (2) is the series solution to Equation (28.1) near \( x = x_0 \). This series solution will be valid within the radius of convergence of Equation (2).

Let’s demonstrate this method on a basic example with constant coefficients, so we can make sure that the resulting series solutions match our intuition.

**Example 28.4.** Determine a series solution to

\[ y'' - y = 0 \]

near \( x = 0 \).

First, we observe that \( p(x) = 1 \), and so the function that we need to be analytic near \( x = 0 \) is \( q(x) = 1 \), which is obvious. We suppose that the series solution has the form

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

Differentiating,

\[ y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \]

and

\[ y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \]

Plugging these into the differential equation yields

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0. \]

We want to write the left-hand side as a single series; this will involve re-indexing the first sum so that \( x \) has the same exponent in each sum.

\[ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0. \]

In general, to combine these sums, we would need to pull out any extra terms so that the sums started at the same index value of \( n \). In this example, that is unnecessary, so we can proceed, and we get

\[ \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0. \]

Now, recall that if a power series is identically 0 for any \( x \), each coefficient must be zero. Thus,

\[ (n+2)(n+1)a_{n+2} - a_n = 0. \quad (28.2) \]

This is an example of a recurrence relation, and we need to use it to determine the values of each \( a_n \). We can’t do that directly, but we can use Equation (28.2) to solve for \( a_{n+2} \) in terms of \( a_n \).
This will allow us to write down an expression for each $a_n$ in terms of $a_0$ and $a_1$, recursively. We get

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}, \quad (28.3)$$

To proceed, we need to figure out what Equation (28.3) tells us about specific values of $a_n$. The most convenient way to do that is to plug in values of $n$ and see what emerges.

\[
\begin{align*}
(n = 0) & \quad a_2 = \frac{a_0}{(2)(1)} & (n = 1) & \quad a_3 = \frac{a_1}{(3)(2)} \\
(n = 2) & \quad a_4 = \frac{a_2}{(4)(3)} & (n = 3) & \quad a_5 = \frac{a_3}{(5)(4)} \\
& \quad = \frac{a_0}{(4)(3)(2)(1)} & & \quad = \frac{a_1}{(5)(4)(3)(2)} \\
(n = 4) & \quad a_6 = \frac{a_4}{(6)(5)} & (n = 5) & \quad a_7 = \frac{a_5}{(7)(6)} \\
& \quad = \frac{a_0}{(6)(5)(4)(3)(2)(1)} & & \quad = \frac{a_1}{(7)(6)(5)(4)(3)(2)} \\
& \quad \vdots & & \vdots \\
& \quad a_{2k} = \frac{a_0}{(2k)!} & a_{2k+1} = \frac{a_1}{(2k+1)!}
\end{align*}
\]

Observe that the both coefficient formulas work for all values of $k$, including $k = 0$. This may not always be the case. Now, we can plug these coefficients into the series and collect the $a_0$ and $a_1$ terms:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \ldots$$

$$= a_0 + a_1 x + \frac{a_0}{2!} x^2 + \frac{a_1}{3!} x^3 + \ldots + \frac{a_0}{(2k)!} x^{2k} + \frac{a_1}{(2k+1)!} x^{2k+1} + \ldots$$

$$= a_0 \left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2k}}{(2k)!} \right) + a_1 \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} \right)$$

$$= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad (28.5)$$

We don’t know the values of $a_0$ and $a_1$, but this is ok; what we’ve calculated is a series representation of the general solution, and if we had initial conditions we would be able to find $a_0$ and $a_1$, as we will see shortly. \[\square\]

We aren’t quite finished with Example 28.4. We know from earlier that the general solution to the differential equation $y'' - y = 0$ is

$$y(x) = c_1 e^x + c_2 e^{-x}. \quad (28.6)$$
The series solution that we obtained should be equivalent to this on its radius of convergence. If we take the Taylor expansion of Equation (28.6), we get:

\[
y(x) = c_1 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\
= c_1 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + c_2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) \\
= (c_1 + c_2)1 + (c_1 - c_2)x + \ldots + (c_1 + c_2)\frac{x^{2k}}{(2k)!} + (c_1 - c_2)\frac{x^{2k+1}}{(2k+1)!} + \ldots \\
= (c_1 + c_2) \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + (c_1 - c_2) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.
\]

This is the same as the series solution (28.5) with \(a_0 = c_1 + c_2\) and \(a_1 = c_1 - c_2\). This also illustrates that initial conditions can be used to find \(a_0\) and \(a_1\), as we would have used them to find \(c_1\) and \(c_2\).

**Exercise.** Calculate the radius of convergence for Equation (28.5). Is this the same as the radius of convergence for the Taylor series representation of \(e^x\)?

Next, let’s do an example involving non-constant coefficients. Our only method so far that can deal with these types of equations is Variation of Parameters, which does not always yield integrals that can be evaluated in a closed form. We will also include initial conditions.

**Example 28.5.** Compute a series solution around \(x = 0\) for the initial value problem

\[
y'' - xy = 0 \quad y(0) = -1 \quad y'(0) = 2.
\]

Once again, every point is an ordinary point, so \(x = 0\) in particular will have a series solution nearby. We have:

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n \\
y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \\
y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.
\]

Plugging into the equation,

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0.
\]

We need to move the coefficient in front of the second sum into the series before they can be combined. In this case, we can just distribute it through:

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.
\]

Next, the exponents of \(x\) need to match up; we will do this by shifting the first series down by 2 and the second up by 1:

\[
\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.
\]
The last step before we can combine them is to get them to start at the same index value of \( n \). The clearest way to do this here is to pull out the \( n = 0 \) term in the first series, so that the sum starts at \( n = 1 \):

\[
(2)(1)a_2x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0
\]

\[
2a_2x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0.
\]

We know that each coefficient in this sum has to be equal to zero, including the coefficient of the \( n = 0 \) term \((2a_2)\).

\[
(n = 0) : \quad 2a_2 = 0 \quad \Rightarrow \quad a_2 = 0
\]

\[
(n > 0) : \quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0 \quad \Rightarrow \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}
\]

Plugging in values of \( n \):

\[
a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_5 = \frac{a_2}{(5)(4)} = 0
\]

\[
a_6 = \frac{a_3}{(6)(5)}, \quad a_7 = \frac{a_4}{(7)(6)}, \quad a_8 = \frac{a_5}{(8)(7)} = 0
\]

\[\vdots \quad \vdots\]

\[
a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)}, \quad a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \quad a_{3k+2} = 0,
\]

for \( k = 1,2,\ldots \), as the formulas do not hold for \( k = 0 \). Putting this together,

\[
y(x) = a_0 + a_1x + a_3x^3 + a_4x^4 + a_6x^6 + \ldots
\]

\[
= a_0 + a_1x + \ldots + a_0x^{3k} + \frac{a_1x^{3k+1}}{2(3)(5)(6)\cdots(3k-1)(3k)} + \frac{a_1x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} + \ldots
\]

\[
= a_0 \left( 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right) + a_1 \left( x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right).
\]

Finally, we will use the initial conditions to solve for \( a_0 \) and \( a_1 \). While we could get the same result from Equation (28.8), it will be easier to use Equation (28.7), from which we obtain

\[
-1 = y(0) = a_0
\]

\[
2 = y'(0) = a_1.
\]

Thus the solution to the initial value problem is

\[
y(x) = -\left( 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right) + 2 \left( x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right).
\]

As Example 28.5 illustrates, the process of finding series solutions can be tedious and it isn’t always obvious how to get the general form of the coefficients from the first few values. In general, you want to try to get these when it’s possible (and your skill at finding the patterns will develop.
over time), but it isn’t always possible, as the next example will show. In that case, just find however many terms are required to approximate the solution to the desired order.

Also, it is desirable to have initial conditions at the value of $x$ where the series solution is centered. Otherwise, we will not be able to directly calculate the constants for the particular solution unless the series are Taylor series for a nice function.

Let’s finish our discussion of series solutions by looking at the same problem as Example 28.5, but around a different point. We won’t bother trying to get the general form for the series coefficients, so we’ll just calculate several of them.

**Example 28.6.** Compute the first five terms in the series solution around $x = 1$ for the initial value problem

$$y'' - xy = 0 \quad y(1) = 1 \quad y'(1) = 2.$$ 

In this case, we assume that our solution has the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - 1)^n,$$

with derivatives

$$y'(x) = \sum_{n=1}^{\infty} na_n(x - 1)^{n-1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x - 1)^{n-2}.$$ 

So the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n(x - 1)^{n-2} - x \sum_{n=0}^{\infty} a_n(x - 1)^n = 0.$$

Unlike in the previous example, we can’t just distribute the $x$ in front of the second series into it. To give it the form $x - 1$, we need to add and subtract 1.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x - 1)^{n-2} - (x - 1 + 1) \sum_{n=0}^{\infty} a_n(x - 1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x - 1)^{n-2} - (x - 1) \sum_{n=0}^{\infty} a_n(x - 1)^n - \sum_{n=0}^{\infty} a_n(x - 1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x - 1)^{n-2} - \sum_{n=0}^{\infty} a_n(x - 1)^{n+1} - \sum_{n=0}^{\infty} a_n(x - 1)^n = 0$$

Next, we need to shift the first series up by 2 and the second down by 1 to get the same exponent.

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}(x - 1)^n - \sum_{n=1}^{\infty} a_{n-1}(x - 1)^n - \sum_{n=0}^{\infty} a_n(x - 1)^n = 0$$

Finally, we pull out the $n = 0$ terms so that all of the sums begin at $n = 1$ and combine.

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n + 2)(n + 1)a_{n+2} - a_{n+1} - a_n] (x - 1)^n = 0.$$
Setting the coefficients equal to 0,

\[(n = 0) : \quad 2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2}\]
\[(n > 0) : \quad (n+2)(n+1)a_{n+2} - a_{n-1} - a_n = 0 \quad \Rightarrow \quad a_{n+2} = \frac{a_{n-1} + a_n}{(n+2)(n+1)}.

As we only want the first five terms, we just need to plug in values of \(n\) until we have them (as opposed to needing to wait until we can see the general pattern).

\[(n = 0) : \quad a_2 = \frac{a_0}{2}\]
\[(n = 1) : \quad a_3 = \frac{a_0 + a_1}{(3)(2)} = \frac{a_0}{6} + \frac{a_1}{6}\]
\[(n = 2) : \quad a_4 = \frac{a_1 + a_2}{(4)(3)} = \frac{a_0}{24} + \frac{a_1}{12}\]
\[(n = 3) : \quad a_5 = \frac{a_2 + a_3}{(5)(4)} = \frac{a_0}{40} + \frac{a_0}{120} + \frac{a_1}{120}\]

Putting this together, the general series solution (up to fifth order) will be:

\[y(x) = \sum_{n=0}^{\infty} a_n(x - 1)^n\]

\[= a_0 + a_1(x - 1) + a_2(x - 1)^2 + a_3(x - 1)^3 + a_4(x - 1)^4 + a_5(x - 1)^5 + \ldots\]

\[= a_1 + a_1(x - 1) + \frac{a_0}{2}(x - 1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x - 1)^3\]
\[+ \left(\frac{a_0}{24} + \frac{a_1}{12}\right)(x - 1)^4 + \left(\frac{a_0}{30} + \frac{a_1}{120}\right)(x - 1)^5 + \ldots\]

\[= a_0 \left(1 + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{24} + \frac{(x - 1)^5}{30} + \ldots\right)\]
\[+ a_1 \left(x + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{24} + \frac{(x - 1)^5}{120} + \ldots\right).

Using the initial conditions, \(a_0 = 1\) and \(a_1 = 2\), so the particular series solution up to fifth order is

\[y(x) = \left(1 + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{24} + \frac{(x - 1)^5}{30} + \ldots\right)\]
\[+ 2 \left(x + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{12} + \frac{(x - 1)^5}{120} + \ldots\right).
\]

\[\square\]
Part 6

Systems of Differential Equations
LECTURE 29

Systems of Differential Equations

So far, we’ve only discussed solution methods for a single differential equation. However, it’s rare that a real world application is modeled using only a single function: quite often, there are several interplaying factors at work in the evolution of any real system.

A good example is population dynamics. It would be possible to model the size of a single population using a single differential equation, making certain assumptions about death and birth rates (namely, that they are constant). But in general, this won’t be the case: the death rate of a prey species is dependant on the size of a predator population and the size of the predator population will depend on the number of prey, or two species compete for the same resources, so an increase in one population affects the ability of the other to increase. To be able to write down a model for the size of any of these populations, we need to know the size of the other population. This would then give us a system of two interlocked differential equations.

An example of a system of first order linear equations is

\[
\begin{align*}
    x_1' &= x_1 + 4x_2 \\
    x_2' &= 2x_1 - 7x_2.
\end{align*}
\]

We call a system like this coupled because the values of \(x_1\) and \(x_2\) affect the rates of change of the other.

It’s important to note that there will be a lot of similarities between our discussion here and our earlier discussion of second and higher order linear equations. There’s a very good reason for this: any higher order linear equation can be written as a system of first order differential equations. Let’s see how this is done.

**Example 29.1.** Write the following second order differential equation as a system of first order linear differential equations.

\[3y'' + 2y' - 6y = 0 \quad y(2) = 0 \quad y(2) = -2\]

All that’s required to rewrite this equation as a first order system is a very simple change of variables. In fact, this is always the change of variables to use for a problem like this. We set

\[
\begin{align*}
    x_1(t) &= y(t) \\
    x_2(t) &= y'(t).
\end{align*}
\]

Then we have

\[
\begin{align*}
    x_1' &= y' = x_2 \\
    x_2' &= y'' = 2y - \frac{2}{3}y' = 2x_1 - \frac{2}{3}x_2.
\end{align*}
\]

Notice how we used the original differential equation to obtain the second equation. The first equation, \(x_1' = x_2\), is always something you should expect to see when doing this, just by virtue of the change of variables we use.
All we have left to do is to convert the initial conditions.
\[ x_1(2) = y(2) = 0 \]
\[ x_2(2) = y'(2) = -2 \]

Thus our original initial value problem has been transformed into the system
\[ x'_1 = x_2 \quad x_1(2) = 0 \]
\[ x'_2 = 2x_1 - \frac{2}{3}x_2 \quad x_2(2) = -2 \]

Let’s do an example to see how this works for higher order linear equations.

**Example 29.2.** Write
\[ y^{(4)} + 2y''' - y'' + 6y' - 3y = 0 \]
as a system of first order differential equations.

We want to start by making an analogous change of variables as in Example 29.1. The only difference is that, since our equation in this example is fourth order, we will need four new variables instead of just two.

\[
\begin{align*}
  x_1 & = y \\
  x_2 & = y' \\
  x_3 & = y'' \\
  x_4 & = y''' 
\end{align*}
\]

Then we have
\[
\begin{align*}
  x'_1 & = y' = x_2 \\
  x'_2 & = y'' = x_3 \\
  x'_3 & = y''' = x_4 \\
  x'_4 & = y^{(4)} = 3y - 6y' + y'' - 2y''' = 3x_1 - 6x_2 + x_3 - 2x_4 
\end{align*}
\]
as our system of equations.

To be able to solve these, we need to review some facts about systems of equations and linear algebra.

**1. Systems of Equations**

In this section, we will restrict our attention only to the linear algebra that might come up when studying systems of differential equations. This is far from a complete treatment, so if you’re curious, taking a linear algebra course would be a good idea.

Suppose we start with a system of \( n \) equations with \( n \) unknowns, \( x_1, x_2, \ldots, x_n \).

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\
  & \quad \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & = b_n 
\end{align*}
\]  
(29.1)

Here’s the basic fact about systems of equations with the same number of unknowns as equations, such as (29.1).
Theorem 29.1. Given a system of \( n \) equations with \( n \) unknowns, there are three possibilities for the number of solutions:

1. no solutions;
2. exactly one solution;
3. infinitely many solutions.

We have one more definition to give: a system of equations such as (29.1) is called nonhomogeneous if at least one \( b_i \neq 0 \). If every \( b_i = 0 \), the system is called homogeneous. A homogeneous system has the following form.

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{align*}
\]

(29.2)

Notice that there is always at least one solution, given by

\[ x_1 = x_2 = \ldots = x_n = 0. \]

This solution is called the trivial solution. This means that it is impossible for a homogeneous system to have zero solutions, and Theorem 29.1 can be modified as follows.

Theorem 29.2. Given a homogeneous system of \( n \) equations with \( n \) unknowns, there are two possibilities for the number of solutions:

1. exactly one solution, the trivial solution;
2. infinitely many non-zero solutions in addition to the trivial solution.

2. Linear Algebra

While we could, in principle, solve the systems of equations (29.1) and (29.2) directly, we have some very powerful tools available to us. This is why linear algebra was invented. The main "objects" of study in linear algebra are matrices and vectors.

An \( n \times n \) matrix (sometimes referred to as an \( n \)-dimensional matrix) is an array of numbers with \( n \) rows and \( n \) columns. It’s possible to consider matrices with different numbers of rows and columns, but this is more general than we will need. An \( n \times n \) matrix has the form

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}.
\]

There’s one special matrix we will need to be familiar with; this is the \( n \)-dimensional identity matrix

\[
I_n = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]

We will be focusing on \( 2 \times 2 \) matrices in this class; the principles of everything we will discuss extend to higher dimensional matrices, but the computations are much simpler in 2 dimensions.

Matrix addition and subtraction are fairly straightforward: everything is done componentwise. The same goes for multiplying a matrix by a constant, called scalar multiplication: we just multiply every component of the matrix by that constant. This will be illustrated in the following example.
Example 29.3. Given the matrices

\[
A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix},
\]

compute \( A - 2B \).

The first thing to do is to compute \( 2B \).

\[
2B = 2 \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix}
\]

Then we have

\[
A - 2B = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -4 & -3 \end{pmatrix}.
\]

\[
\square
\]

Notice that these operations require the dimensions of the matrices to be equal.

A vector is a one-dimensional array of numbers. For example,

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

is a vector of \( n \) unknowns.

We can multiply two matrices \( A \) and \( B \) together by ”multiplying” each row in \( A \) by each column in \( B \).

Example 29.4. Compute \( AB \), where

\[
A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}.
\]

\[
AB = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} (1)(0) + (2)(2) & (1)(1) + (2)(-3) \\ (-1)(0) + (3)(2) & (-1)(1) + (3)(-3) \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 6 & -10 \end{pmatrix}.
\]

\[
\square
\]

Notice that \( AB \neq BA \). Matrix multiplication is not commutative. Also, the dimensions of the matrices being multiplied are important: if the number of columns of \( A \) do not match the number of rows of \( B \), we cannot compute \( AB \). Also, the identity matrix \( I_n \) is the identity for matrix multiplication; \( I_n A = AI_n = A \) for any matrix \( A \).

In particular, we can multiply an \( n \)-dimensional matrix and a vector with \( n \) components together as in the following example.

Example 29.5. Compute

\[
\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix}.
\]
We proceed by “multiplying” each row in the matrix by the vector.

\[
\begin{pmatrix}
2 & -1 \\
3 & 2 \\
\end{pmatrix}
\begin{pmatrix}
-1 \\
4 \\
\end{pmatrix}
= \begin{pmatrix}
2(-1) + (-1)(4) \\
3(-1) + (2)(4) \\
\end{pmatrix}
= \begin{pmatrix}
-6 \\
5 \\
\end{pmatrix}
\]

Remark. Multiplication of a matrix with a vector yields another vector. We then have an interpretation of a matrix $A$ as a linear function on vectors (it’s not hard to see that matrix multiplication breaks up over sums). This point of view is not essential, but is useful for what will follow over the next few lectures. It gets fleshed out more in a linear algebra course.

2.1. Determinants. Every square $(n \times n)$ matrix has a number associated to it, called the determinant. We won’t learn how to compute determinants for $n > 2$, as the process gets more and more complicated as $n$ increases. The standard notation for the determinant of a matrix is

\[
\text{det}(A) = |A|.
\]

For a $2 \times 2$ matrix, the determinant is computed using the following formula.

\[
\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix} = ad - bc; 
\tag{29.3}
\]

that is, the determinant is the product of the main diagonal minus the product of the off diagonal.

Example 29.6. Compute the determinants of

\[
A = \begin{pmatrix}
2 & 3 \\
1 & 2 \\
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
1 & 2 \\
2 & 4 \\
\end{pmatrix}.
\]

There’s not much to do here but use Equation (29.3).

\[
\text{det}(A) = (2)(2) - (3)(1) = 4 - 3 = 1
\]

\[
\text{det}(B) = (1)(4) - (2)(2) = 4 - 4 = 0
\]

We call a matrix $A$ singular if $\text{det}(A) = 0$ and nonsingular otherwise. In the previous example, the first matrix was nonsingular while the second was singular.

Determinants give us important information about the existence of an inverse for a given matrix. The inverse of a matrix $A$, denoted $A^{-1}$, satisfies

\[
AA^{-1} = A^{-1}A = I_n
\]

Inverses do not necessarily exist for a given matrix.

Theorem 29.3. Given a matrix $A$,

1. if $A$ is nonsingular, an inverse $A^{-1}$ will exist;
2. if $A$ is singular, no inverse $A^{-1}$ will exist.
LECTURE 30

More Linear Algebra

1. Back to Systems

We return our attention now to the system of equations

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{align*}
\]

(30.1)

To express this system of equations in matrix form, we start by writing both sides as vectors.

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{pmatrix}
\]

Notice that the left hand side of the equation can be rewritten as a matrix-vector product.

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{pmatrix}
\]

We can simplify this notation by writing

\[
A x = b
\]

(30.2)

where \( x \) is the vector whose entries are the unknowns in the system, \( A \) is the matrix of coefficients of the system (called the coefficient matrix), and \( b \) is the vector whose entries are the right-hand sides of the equations. We call Equation (30.2) the matrix form of the system of equations (30.1).

We know that the system of equations (30.1), and hence Equation (30.2), have zero, one, or infinitely many solutions. We can actually go a little further. Suppose \( \det(A) \neq 0 \), i.e., \( A \) is nonsingular. Then Equation (30.2) has only one solution, namely,

\[
x = A^{-1}b.
\]

So we can rewrite our earlier Theorem 29.1 in the following way.

**Theorem 30.1.** Given the system of equations (30.2),

1. if \( \det(A) \neq 0 \), there is exactly one solution;
2. if \( \det(A) = 0 \), there are either zero or infinitely many solutions.

Recall that if our system (30.1) were homogeneous, i.e., if each \( b_i = 0 \), we always have the trivial solution \( x_i = 0 \). Denoting the vector with entries all 0 by \( \mathbf{0} \), the matrix form of a homogeneous system is

\[
A x = \mathbf{0}
\]

(30.3)

Thus we can express the earlier Theorem 29.2 as follows.
Theorem 30.2. Given the homogeneous system of equations (30.3),
(1) if \( \det(A) \neq 0 \), there is exactly one solution, \( x = 0 \);
(2) if \( \det(A) = 0 \), there will be infinitely many nonzero solutions.

2. Eigenvalues and Eigenvectors

We’ll now need a slight digression to one of the most important aspects of linear algebra. We’ve already observed that if we multiply a vector by a matrix, we get another vector, i.e.,

\[ A\eta = y. \]

A natural question to ask is when \( y \) is just a scalar multiple of \( \eta \); in other words, for what vectors \( \eta \) is multiplication by \( A \) equivalent to a "stretching" of \( \eta \), or, more formally, when do we have

\[ A\eta = \lambda \eta \quad (30.4) \]

If Equation (30.4) is satisfied for some constant \( \lambda \) and some vector \( \eta \), we call \( \eta \) an eigenvector of \( A \) with eigenvalue \( \lambda \). We can first notice that if \( \eta = 0 \), (30.4) will be satisfied for any \( \lambda \). We’re not generally interested in a trivial solution like this, though, so we will always assume \( \eta \neq 0 \).

So how can we find solutions to Equation (30.4)? Let’s start by rewriting it, recalling that \( I \) is the \( 2 \times 2 \) identity matrix.

\[ A\eta = \lambda \eta \]
\[ A\eta - \lambda I \eta = 0 \]
\[ (A - \lambda I)\eta = 0 \]

Notice that we had to multiply \( \lambda \) by the identity \( I \) before we could factor it out. This is because we can’t subtract a constant from a matrix, so we had to ”convert,” in a way, the constant \( \lambda \) into a matrix by multiplying it by \( I \).

So, we’ve turned Equation (30.4) into the equation

\[ (A - \lambda I)\eta = 0, \quad (30.5) \]

which is the matrix form for a homogeneous system of equations. By Theorem 30.3 from last class, if \( A - \lambda I \) is nonsingular, the only solution is the trivial solution \( \eta = 0 \), which we’ve already said we’re not interested in. On the other hand, if \( A - \lambda I \) is singular, we’ll have infinitely many nonzero solutions. Thus the condition that we’ll need to find any eigenvalues and eigenvectors that may exist for \( A \) is for

\[ \det(A - \lambda I) = 0. \]

It’s a basic fact that this equation, \( \det(A - \lambda I) = 0 \), is an \( n \)th degree polynomial if \( A \) is an \( n \times n \) matrix. It’s called the characteristic equation of the matrix \( A \).

The Fundamental Theorem of Algebra applied to the characteristic equation tells us that an \( n \times n \) matrix \( A \) has \( n \) eigenvalues, counting multiplicities. To find them, all we have to do is to find the roots of an \( n \)th degree polynomial, which is no problem for small \( n \). Suppose we’ve found these eigenvalues. What can we conclude about their associated eigenvectors?

We call \( k \) vectors \( x_1, x_2, \ldots, x_k \) linearly independent if the only constants \( c_1, c_2, \ldots, c_k \) satisfying

\[ c_1 x_1 + c_2 x_2 + \ldots + c_k x_k = 0 \]

are \( c_1 = c_2 = \ldots = c_k = 0 \). This definition should look similar; it’s entirely analogous to our earlier definition of linear independence for functions. Now, we have the following fact.

Theorem 30.3. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) is the complete list of eigenvalues of \( A \), including multiplicities, then
(1) if \( \lambda \) occurs only once in the list, it is called simple;
(2) if \( \lambda \) occurs \( k > 1 \) times, it has multiplicity \( k \);
(3) if \( \lambda_1, \lambda_2, \ldots, \lambda_k \) \((k \leq n)\) are the simple eigenvalues of \( A \) with corresponding eigenvectors \( \eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(k)} \), then these eigenvectors \( \eta^{(i)} \) are linearly independent;

(4) if \( \lambda \) is an eigenvalue with multiplicity \( k \), then \( \lambda \) will have anywhere from 1 to \( k \) linearly independent eigenvectors.

This fact should look familiar from our discussion of second and higher order equations; we had a similar result about roots of the characteristic equation in that case. This result tells us when we have linearly independent eigenvectors, which is handy when trying to solve systems of differential equations.

Now, once we have the eigenvalues, how do we find their associated eigenvectors? Let’s do a couple of examples to see.

**Example 30.1.** Find the eigenvalues and eigenvectors of

\[
A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}.
\]

The first thing we need to do is to find the roots of the characteristic equation of the matrix

\[
A - \lambda I = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 - \lambda & 4 \\ 2 & 1 - \lambda \end{pmatrix}.
\]

This is

\[
0 = \det(A - \lambda I) = (3 - \lambda)(1 - \lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).
\]

Thus the two eigenvalues of \( A \) are \( \lambda_1 = 5 \) and \( \lambda_2 = -1 \).

Now, to find the eigenvectors we need to plug each eigenvalue into Equation (30.5) and solve for \( \eta \).

(1) \( \lambda_1 = 5 \):

In this case, Equation (30.5) becomes the following system.

\[
\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \eta = 0
\]

Next, we’ll write out the components of the two vectors and multiply through.

\[
\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} -2\eta_1 + 4\eta_2 \\ 2\eta_1 - 4\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

For this vector equation to hold, the components must match up. So we’ve got to find a solution to the system

\[
-2\eta_1 + 4\eta_2 = 0
\]

\[
2\eta_1 - 4\eta_2 = 0.
\]

Notice that these are the same equation: the second line only differs from the first by multiplication by a constant, in this case \(-1\). This will always be the case if we’ve found our eigenvalues correctly, since we know that \( A - \lambda I \) is singular and so our system (30.5) should have infinitely many solutions. In fact, going back to our matrix \( A - \lambda I \), we could notice there that the rows only differed by a constant factor. This is a good place to check that our earlier algebra was correct: if the rows differ by more than just a constant factor, something’s gone wrong.

Since these equations are identical, we can just choose one (whichever is convenient works fine) and obtain a relation between the eigenvector components \( \eta_1 \) and \( \eta_2 \). Let’s choose the first. This gives

\[
2\eta_1 = 4\eta_2,
\]

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and so we have \( \eta_1 = 2 \eta_2 \). As a result, any eigenvector corresponding to \( \lambda_1 = 5 \) has the form
\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \eta_2 \\ \eta_2 \end{pmatrix}.
\]

There are infinitely many vectors of this form, of course; we only need one. We can select one by choosing a value for \( \eta_2 \). The only restriction is that we shouldn’t take \( \eta_2 = 0 \), since then \( \eta = 0 \), which we don’t want. We may choose, for example, \( \eta_2 = 1 \), and then we have
\[
\eta^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

(2) \( \lambda_2 = -1 \):
In the previous case, we went into more detail than we will in future examples. The process is exactly the same, however.
Plugging in \( \lambda_2 \) into Equation (30.5) gives the system
\[
\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
\[
\begin{pmatrix} 4 \eta_1 + 4 \eta_2 \\ 2 \eta_1 + 2 \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The two equations corresponding to this vector equation are
\[
4 \eta_1 + 4 \eta_2 = 0 \\
2 \eta_1 + 2 \eta_2 = 0.
\]
Once again, these only differ by a constant factor: the first equation is twice the second. Let’s choose the second equation to work with, though in this case it doesn’t matter at all; both are equally easy. We have
\[
\eta_1 = -\eta_2,
\]
and so any eigenvector has the form
\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}.
\]
We can once again choose \( \eta_2 = 1 \), giving us a second eigenvector of
\[
\eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]
Summarizing, the eigenvalue/eigenvector pairs of \( A \) are
\[
\lambda_1 = 5 \quad \eta^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
\lambda_2 = -1 \quad \eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Remark. We could have ended up with any number of different vectors for our eigenvectors \( \eta^{(1)} \) and \( \eta^{(2)} \), depending on the choices we made at the end. However, they would have only differed by a multiplicative constant, and this is ok.
Example 30.2. Find the eigenvalues and eigenvectors of

\[ A = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}. \]

The characteristic equation for this matrix is

\[
0 = \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} = (2 - \lambda)(4 - \lambda) + 5 = \lambda^2 - 6\lambda + 13.
\]

Completing the square (or using the quadratic formula), we see that the roots are \( \lambda_{1,2} = 3 \pm 2i \).

If we get complex eigenvalues, to find the eigenvectors we proceed just as we did in the previous example.

1. \( \lambda_1 = 3 + 2i \)

Here the matrix equation

\[
(A - \lambda I)\eta = 0
\]

becomes

\[
\begin{pmatrix} -1 - 2i & -1 \\ 5 & 1 - 2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} (-1 - 2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1 - 2i)\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

So the pair of equations we get are

\[
(-1 - 2i)\eta_1 - \eta_2 = 0
\]

\[
5\eta_1 + (1 - 2i)\eta_2 = 0.
\]

It’s not as obvious as in the last example, but these two equations are scalar multiples: if we multiply the first equation by \(-1 + 2i\), we recover the second. Now, we choose one of these equations to work with. Let’s use the first. This gives us that \( \eta_2 = (-1 - 2i)\eta_1 \), so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1 - 2i)\eta_1 \end{pmatrix}.
\]

Choosing \( \eta_1 = 1 \) gives a first eigenvector of

\[
\eta^{(1)} = \begin{pmatrix} 1 \\ -1 - 2i \end{pmatrix}.
\]

2. \( \lambda_1 = 3 - 2i \)

Here the matrix equation

\[
(A - \lambda I)\eta = 0
\]

becomes

\[
\begin{pmatrix} -1 + 2i & -1 \\ 5 & 1 + 2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} (-1 + 2i)\eta_1 - \eta_2 \\ 5\eta_1 + (1 + 2i)\eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

So the pair of equations we get are

\[
(-1 + 2i)\eta_1 - \eta_2 = 0
\]

\[
5\eta_1 + (1 + 2i)\eta_2 = 0.
\]
Let’s use the first equation again. This gives us that \( \eta_2 = (-1 + 2i)\eta_1 \), so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ (-1 + 2i)\eta_1 \end{pmatrix}.
\]

Choosing \( \eta_1 = 1 \) gives a second eigenvector of

\[
\eta^{(2)} = \begin{pmatrix} 1 \\ -1 + 2i \end{pmatrix}.
\]

To sum, \( A \) has the following eigenvalue/eigenvector pairs.

\[
\lambda_1 = 3 - 2i \quad \eta^{(1)} = \begin{pmatrix} 1 \\ -1 - 2i \end{pmatrix} \\
\lambda_2 = 3 + 2i \quad \eta^{(2)} = \begin{pmatrix} 1 \\ -1 + 2i \end{pmatrix}
\]

**Remark.** Notice that the eigenvalues came in complex conjugate pairs; that is, they were of the form \( a \pm bi \). This is always the case for complex roots, as we can easily see from the quadratic formula (we also saw this in the sections on second and higher order equations). Moreover, the complex entries in the eigenvectors were also complex conjugate, and the real entries were the same (all up to a multiplicative constant, of course). This is always the case so long as \( A \) doesn’t have any complex entries.
LECTURE 31

Systems With Real Eigenvalues

1. Solutions to Systems of DEs

Now that we have the required background, we can return our attention to homogeneous systems of first order differential equations (i.e., there shouldn’t be any terms involving just the independent variable \(t\)). A two-dimensional linear system of differential equations has the form

\[
\begin{align*}
x' &= ax + by \\
y' &= cx + dy.
\end{align*}
\]

Suppose we’ve got our system written in matrix form,

\[
x' = Ax.
\] (31.1)

How do we go about solving this equation? If \(A\) were a 1 \(\times\) 1 matrix, i.e., a constant, and \(x\) were a vector with 1 component, the differential equation would be the separable equation

\[
x' = ax.
\]

We know that this is solved by

\[
x(t) = ce^{at}.
\]

We might think, then, that in the \(n \times n\) case, instead of \(a\) we have some other constant in the exponential, and instead of the constant of integration \(c\) we have some constant vector \(\eta\).

So let’s guess that the form of a solution will be

\[
x(t) = \eta e^{rt}.
\] (31.2)

Plugging this guess into Equation (31.1) gives

\[
\begin{align*}
r \eta e^{rt} &= A \eta e^{rt} \\
(A \eta - r \eta) e^{rt} &= 0 \\
(A - rI) \eta e^{rt} &= 0.
\end{align*}
\]

As \(e^{rt} \neq 0\), we end up with the requirement that

\[
(A - rI) \eta = 0.
\]

This should look familiar; it’s the condition for \(\eta\) to be an eigenvector of \(A\) with eigenvalue \(r\). Thus, we conclude that for (31.2) to be a solution to Equation (31.1), we must have \(\eta\) an eigenvalue of \(A\) with eigenvalue \(r\).

That tells us how to get some solutions to systems of differential equations: we find the eigenvalues and vectors of the coefficient matrix \(A\), then form solutions using (31.2). But how do we form the general solution?

Thinking back to the second/higher order linear case, what we need are enough linearly independent solutions to form a fundamental set. As we noticed last lecture, if we have all simple eigenvalues, then we’re fine: the eigenvectors are all linearly independent, and so the solutions formed will be as well. Things will get more complicated when we have eigenvalues of higher multiplicity, but we can deal with that later.
So, we’ll find the two fundamental solutions of the form (31.2), then take their linear combination to get our general solution.

2. The Phase Plane

We’re going to rely very heavily on qualitatively understanding what solutions to a linear system of differential equations look like; this will pay off when we move on to discussing systems of nonlinear equations. We know that the trivial solution $x = 0$ is always a solution to our homogeneous system $x' = Ax$. $x = 0$ is an example of an equilibrium solution, i.e., it satisfies $x' = Ax = 0$ and is a constant solution. We’ll assume that our coefficient matrix $A$ is nonsingular; this implies that $x = 0$ is the only equilibrium solution.

The question we want to ask is whether other solutions move towards or away from this constant solution as $t \to \pm \infty$, so that we can understand the long term behavior of the system. This is no different than what we did when we classified equilibrium solutions for first order autonomous equations; all we’re doing is generalizing those ideas to systems of differential equations.

When we drew our solution spaces then, we did so on a $t - y$ plane. This was suitable at that point, but it would be very difficult for us to draw now: to do something analogous, we would require three dimensions, since we would have to sketch both $x_1$ and $x_2$ versus $t$. Instead, what we will do is ”ignore” $t$ and think of our solutions as trajectories on the $x_1 - x_2$ plane. Then our equilibrium solution is the origin. The $x_1 - x_2$ plane is called the phase plane. We’ll see examples of how to sketch trajectories of solutions on the phase plane as we learn how to solve systems of differential equations. Such a sketch of solutions is called the phase portrait of the system.

3. Real, Distinct Eigenvalues

Let’s get back to the matrix system (31.1). At this point, we know that if $\lambda_1$ and $\lambda_2$ are real and distinct eigenvalues of the $2 \times 2$ coefficient matrix $A$ associated with eigenvectors $\eta^{(1)}$ and $\eta^{(2)}$, respectively. We know that $\eta^{(1)}$ and $\eta^{(2)}$ are linearly independent, as $\lambda_1$ and $\lambda_2$ are simple. Thus the solutions obtained from them using (31.2) will also be linearly independent, and in fact will form a fundamental set of solutions. The general solution, then, is

$$x(t) = c_1 e^{\lambda_1 t} \eta^{(1)} + c_2 e^{\lambda_2 t} \eta^{(2)}.$$ 

So, if we have real and distinct eigenvalues, all that we have to do is find the eigenvectors, form the general solution as above, and use any initial conditions that may exist. Let’s do some examples. We’ll also see how to sketch the phase portrait of the examples.

**Example 31.1.** Solve the following initial value problem.

$$x' = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

The first thing we need to do is to find the eigenvalues of the coefficient matrix.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -2 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix}$$

$$= \lambda^2 + \lambda - 6$$

$$= (\lambda - 2)(\lambda + 3)$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. Next, we find the eigenvectors.
(1) \(\lambda_1 = 2\)

\[
\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[-4\eta_1 + 2\eta_2 = 0 \\
2\eta_1 - \eta_2 = 0.\]

Using either equation, we find \(\eta_2 = 2\eta_1\), and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix}.
\]

Choosing \(\eta_1 = 1\), we obtain a first eigenvector

\[
\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

(2) \(\lambda_2 = -3\)

\[
\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[\eta_1 + 2\eta_2 = 0 \\
2\eta_1 + 4\eta_2 = 0.\]

Using either equation, we find \(\eta_1 = -2\eta_2\), and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2\eta_2 \\ \eta_2 \end{pmatrix}.
\]

Choosing \(\eta_2 = 1\), we obtain a second eigenvector

\[
\eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\]

Thus our general solution is

\[
x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\]

Now, we have our initial condition. Let’s use it to solve for \(c_1\) and \(c_2\). The condition says

\[
\begin{pmatrix} 5 \\ 0 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\]

All that’s left is to write out this matrix equation as a system of equations and then solve.

\[
c_1 - 2c_2 = 5 \\
2c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = 1, c_2 = -2
\]

Thus the particular solution is

\[
x(t) = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\]
Example 31.2. Sketch the phase portrait of the system from Example 31.1.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

$$\lambda_1 = 2 \quad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -3 \quad \eta^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The starting point for the phase portrait involves sketching solutions corresponding to the eigenvectors (i.e., with $c_1$ or $c_2 = 0$). We know that if $x(t)$ is one of these solutions,

$$x'(t) = A c_i e^{\lambda_i t} \eta^{(i)} = c_i \lambda_i e^{\lambda_i t} \eta^{(i)}.$$

This is just, for any $t$, a constant times the eigenvector, which indicates that lines in the direction of the eigenvector are these solutions to the system. These are called the eigensolutions of the system.

Next, we need to consider the direction that these solutions move in. Let’s start with the first eigensolution, which corresponds to the solution with $c_2 = 0$. The first eigenvalue is $\lambda_1 = 2 > 0$. This indicates that this eigensolution will grow exponentially, as the exponential in the solution has a positive exponent. The second eigensolution corresponds to $\lambda_2 = -3 < 0$, so the exponential in the appropriate solution is negative. Hence this solution will decay and move towards the origin.

What does a typical trajectory do (i.e., a trajectory where both $c_1, c_2 \neq 0$)? The general solution is

$$x(t) = c_1 e^{2t} \eta^{(1)} + c_2 e^{-3t} \eta^{(2)}.$$

Thus as $t \to \infty$, this solution will limit to the positive eigensolution, as the component corresponding to the negative eigensolution will decay away. On the other hand, as $t \to -\infty$, the trajectory will asymptotically reach the negative eigensolution, as the positive eigensolution component will be tiny.
The end result is the phase portrait as in Figure 31.1. When the phase portrait looks like this (which happens whenever the eigenvalues have mixed signs), the equilibrium solution at the origin is classified as a saddle point and is unstable.

Example 31.3. Solve the following initial value problem.

\[
\begin{align*}
x_1' &= 4x_1 + x_2 & x_1(0) &= 6 \\
x_2' &= 3x_1 + 2x_2 & x_2(0) &= 2
\end{align*}
\]

Before we can solve anything, we need to convert this system into matrix form. Doing so converts the initial value problem to

\[
x' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.
\]

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

\[
0 = \det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 1 \\ 3 & 2 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)
\]

So the eigenvalues are \( \lambda_1 = 1 \) and \( \lambda_2 = 5 \). Next, we find the eigenvectors.

1. \( \lambda_1 = 1 \)

\[
(A - I)\eta = 0
\]

\[
\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[
\begin{align*}
3\eta_1 + \eta_2 &= 0 \\
3\eta_1 + \eta_2 &= 0.
\end{align*}
\]

Using either equation, we find \( \eta_2 = -3\eta_1 \), and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix}.
\]

Choosing \( \eta_1 = 1 \), we obtain a first eigenvector

\[
\eta^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.
\]

2. \( \lambda_2 = 5 \)

\[
(A - 5I)\eta = 0
\]

\[
\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[
\begin{align*}
-\eta_1 + \eta_2 &= 0 \\
3\eta_1 - 3\eta_2 &= 0.
\end{align*}
\]

Using either equation, we find \( \eta_1 = \eta_2 \), and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2 \end{pmatrix}.
\]
Choosing \( \eta_2 = 1 \), we obtain a second eigenvector
\[
\eta^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Thus our general solution is
\[
x(t) = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Now, we have our initial condition. Let’s use it to solve for \( c_1 \) and \( c_2 \). The condition says
\[
\begin{pmatrix} 6 \\ 2 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

All that’s left is to write out this matrix equation as a system of equations and then solve.
\[
c_1 + c_2 = 6 \quad \Rightarrow \quad c_1 = 1, c_2 = 5
\]

Thus the particular solution is
\[
x(t) = e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

\[\square\]

**Example 31.4.** Sketch the phase portrait of the system from Example 31.3.
In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were
\[
\begin{align*}
\lambda_1 &= 1 & \eta^{(1)} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\
\lambda_2 &= 5 & \eta^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{align*}
\]
We begin by sketching the eigensolutions (again, these are the straight lines in the directions of the eigenvectors). Both of these trajectories move away from the origin, though, as the eigenvalues are both positive.

Since $|\lambda_2| > |\lambda_1|$, we call the second eigensolution the *fast eigensolution* and the first one the *slow eigensolution*. The terminology comes from the fact that the eigensolution corresponding to the eigenvalue with larger magnitude will either grow or decay more quickly than the other one.

As both grow in forward time, asymptotically, as $t \to \infty$, the fast eigensolution will dominate the typical trajectory, as it gets larger much more quickly than the slow eigensolution does. So, in forward time, other trajectories will get closer and closer to the eigensolution corresponding to $\eta^{(2)}$. On the other hand, as $t \to -\infty$, the fast eigensolution will decay more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(1)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 31.2. Whenever we have two positive eigenvalues, every solution moves away from the origin. We call the equilibrium solution at the origin, in this case, a *node* and classify it as being *unstable*.

**Example 31.5.** Solve the following initial value problem.

\[
\begin{align*}
x_1' &= -5x_1 + x_2 & x_1(0) &= 2 \\
x_2' &= 2x_1 - 4x_2 & x_2(0) &= -1
\end{align*}
\]

First, we convert the system intro matrix form.

\[
x' = \begin{pmatrix} -5 & 1 \\ 2 & -4 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\]

To solve, the first thing we need to do is to find the eigenvalues of the coefficient matrix.

\[
0 = \det(A - \lambda I) = \begin{vmatrix} -5 - \lambda & 1 \\ 2 & -4 - \lambda \end{vmatrix} = \lambda^2 + 9\lambda + 18 = (\lambda + 3)(\lambda + 6)
\]

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$. Next, we find the eigenvectors.

1. $\lambda_1 = -3$

\[
(A + 3I)\eta = 0
\]

\[
\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[
-2\eta_1 + \eta_2 = 0 \\
2\eta_1 - \eta_2 = 0.
\]

Using either equation, we find $\eta_2 = 2\eta_1$, and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ 2\eta_1 \end{pmatrix}.
\]

Choosing $\eta_1 = 1$, we obtain a first eigenvector

\[
\eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]
(2) \( \lambda_2 = -6 \)

\[
(A + 6I) \eta = 0
\]

\[
\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So we’ll want to find solutions to the system

\[
\eta_1 + \eta_2 = 0
\]

\[
2\eta_1 + 2\eta_2 = 0.
\]

Using either equation, we find \( \eta_1 = -\eta_2 \), and so any eigenvector has the form

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix}.
\]

Choosing \( \eta_2 = 1 \), we obtain a second eigenvector

\[
\eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Thus our general solution is

\[
x(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Now, we have our initial condition. Let’s use it to solve for \( c_1 \) and \( c_2 \). The condition says

\[
\begin{pmatrix} 2 \\ -1 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

All that’s left is to write out this matrix equation as a system of equations and then solve.

\[
c_1 - c_2 = 2 \quad \Rightarrow \quad c_1 = \frac{1}{3}, \quad c_2 = -\frac{5}{3}
\]

Thus the particular solution is

\[
x(t) = \frac{1}{3} e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{5}{3} e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Example 31.6. Sketch the phase portrait of the system from Example 31.5.

In the last example, we saw that the eigenvalue/eigenvector pairs for the coefficient matrix were

\[
\lambda_1 = -3 \quad \eta^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

\[
\lambda_2 = -6 \quad \eta^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

We begin by sketching the eigensolutions. Both of these trajectories decay towards from the origin, though, as the eigenvalues are both negative.

Since \( |\lambda_2| > |\lambda_1| \), the second eigensolution is the fast eigensolution and the first one the slow eigensolution.

In the general solution, both exponentials are negative, and so every solution will decay towards the origin. Asymptotically, as \( t \to \infty \) and the trajectory gets closer and closer to the origin, the slow eigensolution will dominate the typical trajectory, as dies out less quickly than the fast eigensolution. So, in forward time, other trajectories will get closer and closer to the eigensolution corresponding to \( \eta^{(1)} \). On the other hand, as \( t \to -\infty \), the fast eigensolution will
Figure 31.3. Phase portrait of the stable node in Example 31.5.

grow more quickly than the slow one, and so the eigensolution corresponding to $\eta^{(2)}$ will dominate in backwards time.

Thus the phase portrait will look like Figure 31.3. Whenever we have two negative eigenvalues, every solution moves towards the origin. We call the equilibrium solution at the origin, in this case, a node and classify it as being asymptotically stable.
LECTURE 32

Complex Eigenvalues

Last lecture, we looked at solutions to the equation
\[ x' = Ax, \]
where the eigenvalues of the matrix \( A \) were real and distinct. What happens if they are complex?

This isn’t dissimilar to the complex roots case when we were considering second order equations (for reasons we’ve discussed already). We still have solutions of the form
\[ x = \eta e^{\lambda t}, \]
, where \( \eta \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). However, we want real-valued solutions, which we won’t have if we leave them in that form.

Our strategy will be similar in this case: we’ll use Euler’s formula to rewrite
\[ e^{(a+ib)t} = e^{at} \cos(bt) + ie^{at} \sin(bt), \]
then we’ll write out one of our solutions fully into real and imaginary parts. It will turn out that each of these parts gives us a solution, and in fact they’ll also form a fundamental set of solutions.

Let’s illustrate this by example.

Example 32.1. Solve the following initial value problem.

\[ x' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \]

We begin by finding the eigenvalues of \( A \).

\[ 0 = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 6 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda^2 + 3 \]

Thus the two eigenvalues are \( \lambda_{1,2} = \pm \sqrt{3}i \).

Next, we need to find an eigenvector. It turns out we’ll only need one.

We’ll use \( \lambda_1 = \sqrt{3}i \).

\[ (A - \sqrt{3}i I) \eta^* = 0 \]
\[ \begin{pmatrix} 3 - \sqrt{3}i & 6 \\ -2 & -3 - \sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

The system of equations to solve is
\[ (3 - \sqrt{3}i)\eta_1 + 6\eta_2 = 0 \]
\[ -2\eta_1 + (-3 - \sqrt{3}i)\eta_2 = 0. \]

We can use either equation to find solutions, but let’s use the second one. This gives \( \eta_1 = \frac{1}{2}(-3 - \sqrt{3}i)\eta_2 \). Thus any eigenvector has the form
\[ \eta = \begin{pmatrix} \frac{1}{2}(-3 - \sqrt{3}i)\eta_2 \\ \eta_2 \end{pmatrix}, \]
and choosing \( \eta_2 = 2 \) yields a first eigenvector
\[
\eta^{(1)} = \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix}.
\]

Thus we have a solution
\[
x_1(t) = e^{\sqrt{3}it} \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix}.
\]

Unfortunately, this is complex-valued, and we’d like a real-valued solution. We had a similar problem back when we talked about second order linear equations. What did we do then? We used Euler’s formula to expand this imaginary exponential into cosine and sine terms, then split the solution into real and imaginary parts. This then gave us the two fundamental solutions we needed.

We’ll do the same thing here (in reality, doing it here is why we do it there, as we’ve discussed). We’ll use Euler’s formula to expand
\[
e^{\sqrt{3}it} = \cos(\sqrt{3}t) + i\sin(\sqrt{3}t),
\]
then multiply it through the eigenvector. After separating into real and complex parts using the basic matrix arithmetic operations, it’ll turn out that each of these parts is a solution. More to the point, they’re linearly independent and give us a fundamental set of solutions.

\[
x_1(t) = \left( \cos(\sqrt{3}t) + i\sin(\sqrt{3}t) \right) \begin{pmatrix} -3 - \sqrt{3}i \\ 2 \end{pmatrix}
\]
\[
= \begin{pmatrix} -3 \cos(\sqrt{3}t) + 3i\sin(\sqrt{3}t) - \sqrt{3}i\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) + 2i\sin(\sqrt{3}t) \end{pmatrix}
\]
\[
= \begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + i\begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix}
\]
\[
= u(t) + iv(t).
\]

As noted earlier, both \( u(t) \) and \( v(t) \) are real-valued solutions to the differential equation (this isn’t hard to check: it follows from linearity of the derivative). Moreover, they’re linearly independent. Our general solution is then
\[
x(t) = c_1u(t) + c_2v(t)
\]
\[
= c_1\begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} + c_2\begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix}.
\]

Finally, we’ll use the initial condition to get \( c_1 \) and \( c_2 \). It says
\[
\begin{pmatrix} -2 \\ 4 \end{pmatrix} = x(0) = c_1\begin{pmatrix} -3 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}.
\]

This translates into the system
\[
\begin{align*}
-3c_1 - \sqrt{3}c_2 &= -2 \\
2c_1 &= 4
\end{align*} \quad \Rightarrow \quad c_1 = 2 \quad c_2 = -\frac{4}{\sqrt{3}}.
\]

Hence our particular solution is
\[
x(t) = 2\begin{pmatrix} -3 \cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) \\ 2\cos(\sqrt{3}t) \end{pmatrix} - \frac{4}{\sqrt{3}}\begin{pmatrix} -3 \sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) \\ 2\sin(\sqrt{3}t) \end{pmatrix}.
\]
\[
\square
\]
Example 32.2. Sketch the phase portrait of the system in Example 32.1.

The general solution to the system in Example 32.1 is

\[
x(t) = c_1 \left(-3 \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t)\right) + c_2 \left(-3 \sin(\sqrt{3}t) - \sqrt{3} \cos(\sqrt{3}t)\right).
\]

Every term in this solution (other than the constants \(c_1\) and \(c_2\)) is periodic; we have \(\cos(\sqrt{3}t)\) and \(\sin(\sqrt{3}t)\). Thus both \(x_1\) and \(x_2\) are periodic functions for any initial conditions. On the phase plane, this translates to trajectories which are closed; that is, they form circles or ellipses. As a result, the phase portrait looks like Figure 32.1.

This is always the case when we have purely imaginary eigenvalues, as the exponential turns into strictly a combination of sines and cosines. In this case, the equilibrium solution is called a center and is neutrally stable or just stable (note: not asymptotically stable).

The only work that needs to be done in these cases is to figure out the eccentricity and direction that the trajectory is traveled. The former is a bit difficult, and we usually don’t care that much, but the latter is a lot easier. We can determine whether the trajectories orbit the origin in a clockwise or counterclockwise direction by calculating the tangent vector \(\mathbf{x}'\) at a single point. For example, at the point \((1, 0)\) in the previous example, we have

\[
\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.
\]

Thus at \((1, 0)\), the tangent vector points down and to the right. This can only happen if the trajectories circle the origin in a clockwise direction. \(\square\)

Example 32.3. Solve the following initial value problem.

\[
x' = \begin{pmatrix} 6 & -4 \\ 7 & -2 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}
\]
We begin by finding the eigenvalues of $A$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 6 - \lambda & -4 \\ 7 & -2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 16 = (\lambda - 2)^2 + 16$$

Thus the two eigenvalues are $\lambda_{1,2} = 2 \pm 4i$.

Next, we need to find an eigenvector. We’ll use $\lambda_1 = 2 + 4i$.

$$(A - (2 + 4i)I)\eta^* = 0$$

$$\begin{pmatrix} 4 - 4i & -4 \\ 7 & -4 - 4i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system of equations to solve is

$$(4 - 4i)\eta_1 - 4\eta_2 = 0$$
$$7\eta_1 + (-4 - 4i)\eta_2 = 0.$$

We can use either equation to find solutions, but let’s use the first one. This gives $\eta_2 = (1 - i)\eta_1$. Thus any eigenvector has the form

$$\eta = \begin{pmatrix} \eta_1 \\ (1 - i)\eta_1 \end{pmatrix},$$

and choosing $\eta_1 = 1$ yields a first eigenvector

$$\eta^{(1)} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}.$$

Thus we have a solution

$$x_1(t) = e^{(2+4i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$

and using Euler’s formula gives

$$= e^{2t} e^{4it} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$
$$= e^{2t} \left( \cos(4t) + i \sin(4t) \right) \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$$
$$= e^{2t} \left( \cos(4t) + i \sin(4t) - i \cos(4t) + \sin(4t) \right)$$
$$= e^{2t} \left( \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + i \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix} \right)$$
$$= u(t) + iv(t).$$

Our general solution is

$$x(t) = c_1 u(t) + c_2 v(t)$$
$$= c_1 e^{2t} \begin{pmatrix} \cos(4t) \\ \cos(4t) + \sin(4t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(4t) \\ \sin(4t) - \cos(4t) \end{pmatrix}.$$

Finally, we’ll use the initial condition to get $c_1$ and $c_2$. It says

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$
This translates into the system

\begin{align*}
c_1 &= 1 \\
c_1 - c_2 &= 3 \\Rightarrow \quad c_1 &= 1 \quad c_2 = -2.
\end{align*}

Hence our particular solution is

\[ x(t) = e^{2t} \left( \frac{\cos(4t)}{\cos(4t) + \sin(4t)} \right) - 2e^{2t} \left( \frac{\sin(4t)}{\sin(4t) - \cos(4t)} \right). \]

\[ \square \]

**Example 32.4.** Sketch the phase portrait of the system in Example 32.3.

The only (qualitative) difference between the general solution to this example,

\[ x(t) = c_1 e^{2t} \left( \frac{\cos(4t)}{\cos(4t) + \sin(4t)} \right) + c_2 e^{2t} \left( \frac{\sin(4t)}{\sin(4t) - \cos(4t)} \right), \]

and the one in Example 32.1 is the exponential sitting out front of the periodic terms. This will make the solution quasi-periodic, rather than actually periodic. The exponential, having a positive exponent, will cause the solution to grow as \( t \to \infty \) away from the origin. The solution will still rotate, however, as the trig terms will cause oscillation. Thus, rather than forming closed circles or ellipses, the trajectories will spiral out of the origin.

As a result, when we have complex (not just imaginary) eigenvalues \( \lambda_{1,2} = a \pm bi \), we call the situation a *spiral*. In this case, as the real part \( a \) (which affects the exponent) is positive, and the solution grows, the equilibrium at the center is unstable. If \( a \) had been negative, the spiral would decay into the origin, and the equilibrium would have been asymptotically stable.

So, what’s there to calculate if we recognize we have a stable/unstable spiral? We still need to know the direction of rotation. This requires, as with the center, that we calculate tangent vectors
at a point or two. In this case, the tangent vector at the point \((1,0)\) is

\[
x' = \begin{pmatrix} 6 & -4 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.
\]

Thus the tangent vector at \((1,0)\) points up and to the right. Combined with the knowledge that
the solution is leaving the origin, this can only happen if the direction of rotation of the spiral is
counterclockwise. We obtain a picture as in Figure 32.2. 

\[\square\]
LECTURE 33

Repeated Eigenvalues

The final case we need to consider involves repeated eigenvalues. In the previous two cases, we had distinct eigenvalues, which, as Theorem 30.3 a few lectures ago told us, have linearly independent eigenvectors. Thus, all we had to do was to calculate those eigenvectors and write down solutions of the form

\[ x_i(t) = \eta_i(t)e^{\lambda_i t}. \]

In the real case, the linear independence of the eigenvectors and the differing eigenvectors assured us that these two solutions formed a fundamental set, and so our general solution was just a linear combination of the two. In the complex case, we could have done something similar, but instead we were able to form our real-valued general solution out of a single complex-valued solution.

When we have an eigenvalue of multiplicity 2, however, Theorem 30.3 tells us that we could have either one or two eigenvectors up to linear independence. If we have two, that’s fine; if we have one, we’ll clearly have more work to do. Let’s consider each of these situations.

1. Two Linearly Independent Eigenvectors

This case is actually quite simple. Suppose our repeated eigenvalue \( \lambda \) has two linearly independent eigenvectors \( \eta^{(1)} \) and \( \eta^{(2)} \). Then we can proceed as before, and our general solution is

\[ x(t) = c_1 e^{\lambda t} \eta^{(1)} + c_2 e^{\lambda t} \eta^{(2)} = e^{\lambda t} \left( c_1 \eta^{(1)} + c_2 \eta^{(2)} \right). \]

It’s a basic fact from linear algebra that given two linearly independent vectors such as \( \eta^{(1)} \) and \( \eta^{(2)} \), we can form any other two-dimensional vector out of a linear combination of these two. So what we have is a situation where any vector function of the form \( x(t) = e^{\lambda t} \eta \) is a solution. As discussed earlier, this can only happen if \( \eta \) is an eigenvector of the coefficient matrix with eigenvalue \( \lambda \). The conclusion, then, is that if \( \lambda \) has two linearly independent eigenvectors, every vector is an eigenvector.

This can only happen if the coefficient matrix \( A \) is a scalar multiple of the identity matrix, as we need

\[ A \eta = \lambda \eta = \lambda I \eta \]

for every vector \( \eta \). Thus, this case only arises when

\[ A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \]

or when the original system of equations is

\[ x_1 = \lambda x_1 \]
\[ x_2 = \lambda x_2. \]

What does the phase portrait look like in this case? Since every vector is an eigenvector, every trajectory that isn’t the constant solution at the origin is an eigensolution and hence a straight line. We call such the equilibrium solution in this case a star node and its stability is determined
by the sign of $\lambda$: if $\lambda > 0$, all of the eigensolutions grow away from the origin and the origin is unstable, but if $\lambda < 0$, every solution decays to the origin and the origin is asymptotically stable. The case of an unstable star node is shown in Figure 33.1; a stable star node would just have the directions reversed.

This is a fairly degenerate (and isolated) situation that won’t come up in further discussion, but it’s important to keep in mind that this can happen.

2. One Linearly Independent Eigenvector

The other possibility is that $\lambda$ only has a single eigenvector $\eta$ up to linear independence. In this case, we have a bit of a problem: to form a general solution we need two linearly independent solutions, but we only have one, namely

$$x_1(t) = e^{\lambda t} \eta.$$ 

In this case, we say that $\lambda$ is defective or incomplete. What can we do?

We had a similar problem in the second order linear case. When we ran into this situation there, we were able to work around it by multiplying the solution by a $t$. Let’s try to mimic that and see if

$$x(t) = te^{\lambda t} \eta$$

is a solution.

$$x' = Ax$$

$$\eta e^{\lambda t} + \lambda \eta t e^{\lambda t} = A\eta t e^{\lambda t}$$
Matching coefficients, in order for this guess to be a solution, we require
\[ \eta e^{\lambda t} = 0 \quad \Rightarrow \quad \eta = 0 \]
\[ \lambda \eta e^{\lambda t} = A \eta e^{\lambda t} \quad \Rightarrow \quad (A - \lambda I)\eta = v_z. \]
Thus we need \( \eta \) to be an eigenvector of \( A \), which we knew it was, but also for \( \eta = 0 \), which can’t be. So this guess didn’t work.

What was the problem with our earlier calculation? We ended up with a term that didn’t have a \( t \), but rather just an exponential in it, and this term caused us to require \( \eta = 0 \). A possible fix might be to add in another term to our guess that only involves an exponential and some other vector \( \rho \). Let’s guess that the form of the solution might be
\[ x(t) = te^{\lambda t} \eta + e^{\lambda t} \rho \]
and see what conditions on \( \rho \) we can derive.

\[
x' = Ax
\]
\[
\lambda \eta e^{\lambda t} + \eta e^{\lambda t} + \lambda \rho e^{\lambda t} = A \left( \eta e^{\lambda t} + \rho e^{\lambda t} \right)
\]
\[
(\eta + \lambda \rho) e^{\lambda t} + \eta e^{\lambda t} = A \eta e^{\lambda t} + A \rho e^{\lambda t}
\]
Thus, setting coefficients equal again, we have
\[ A \eta = \lambda \eta \quad \Rightarrow \quad (A - \lambda I) \eta = 0 \]
\[ \eta + \lambda \rho = A \rho \quad \Rightarrow \quad (A - \lambda I) \rho = \eta. \]
This first condition only tells us that \( \eta \) is an eigenvector of \( A \), which we already knew. But the second condition is more useful. It tells us that if \( (A - \lambda I) \rho = \eta \), then
\[ x(t) = te^{\lambda t} \eta + \rho e^{\lambda t} \]
will be a solution to the differential equation.

A vector \( \rho \) satisfying
\[
(A - \lambda I) \rho = \eta
\]
is called a **generalized eigenvector**, because while \( (A - \lambda I) \rho \neq 0 \), it’s not hard to verify that
\[
(A - \lambda I)^2 \rho = 0.
\]
So as long as we can produce a generalized eigenvector \( \rho \), this formula will give us a second solution and we can form a general solution.

**Example 33.1.** Find the general solution to the following system.
\[
x' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} x
\]

We begin by finding the eigenvalues.
\[
0 = \det(A - \lambda I) = \begin{vmatrix} -6 - \lambda & -5 \\ 5 & 4 - \lambda \end{vmatrix}
= \lambda^2 + 2\lambda + 1
= (\lambda + 1)^2
\].
\( A \) has a repeated eigenvalue of \( \lambda = -1 \). Due to the form of the matrix, our previous discussion implies that \( \lambda \) will only have one eigenvector up to linear dependence, which we now calculate by solving \( (A + I)\eta = x \).
\[
\begin{pmatrix} -5 & -5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
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So the system of equations that we want to solve is:
\[\begin{align*}
-5\eta_1 - 5\eta_2 &= 0 \\
5\eta_1 + 5\eta_2 &= 0.
\end{align*}\]

This is solved by any vector of the form \(\eta_1 = -\eta_2\). So if we choose \(\eta_2 = 1\), we get an eigenvector \(\eta = (-1/1)\).

This isn’t enough, as we have discussed. We also need to find a generalized eigenvector \(\rho\). So we need to solve \((A + I)\rho = \eta\), or
\[
\begin{pmatrix}
-5 & -5 \\
5 & 5
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix}
= \begin{pmatrix}
-1 \\
1
\end{pmatrix}
\Rightarrow \rho_1 = \frac{1}{5} - \rho_2.
\]

So a generalized eigenvector has the form \(\rho = \begin{pmatrix} 1/5 - \rho_2 \\ \rho_2 \end{pmatrix}\).

We can choose any value of \(\rho_2\) that we want, and choosing \(\rho_2 = 0\) yields
\(\rho = \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}\).

Thus our generalized solution is
\[
\begin{aligned}
x(t) &= c_1 e^{\lambda t} \eta + c_2 \left(t e^{\lambda t} \eta + e^{\lambda t} \rho\right) \\
&= c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} \right).
\end{aligned}
\]

**Example 33.2.** Sketch the phase portrait for the system in Example 33.1.

We begin by drawing the eigensolution. Note that in this case, there is only one, unlike the (nondegenerate) nodes from Lecture 31. The eigensolution is the straight line in the direction of \(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\), as indicated in Figure 33.2. As the eigenvalue is negative, this solution will decay towards the origin.

But what happens along other trajectories? Let’s consider the general solution
\[
\begin{aligned}
x(t) &= c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} \right)
\end{aligned}
\]

with \(c_2 \neq 0\). All three terms have the same exponential, but as \(t \to \pm \infty\) the \(t e^{-t}\) term will have a larger magnitude than the other two. Thus, in both forward and backward time, the trajectories will become parallel to the single eigensolution (this is also shown in Figure 33.2).

Now, since \(\lambda < 0\) and exponentials decay faster than polynomials grow, we see that as \(t \to \infty\) every solution will decay to the origin. So the origin is asymptotically stable. We call the origin a degenerate node in this case, as it behaves similarly to how a node behaves but has only a single eigensolution.

Here’s one way to think about how a degenerate node should behave. Consider a node with two close eigenvalues. Then try to imagine what happens to Figure 31.3 as we bring the eigenvalues together. The eigensolutions will collapse together, but the non-eigensolution trajectories would keep their asymptotic behavior with regard to this collapsed eigensolution — they will be parallel to it in both the positive and negative infinite time limits.

Notice that, as illustrated in Figure 33.2, we end up with a large degree of rotation to the solution. The solution has to turn around to be asymptotic to the solution in both forwards and
backwards time. In another manner of thinking, degenerate nodes are the borderline case between nodes and spirals. Suppose our characteristic equation is
\[ 0 = \lambda^2 + b\lambda + c. \]
The eigenvalues are then, by the quadratic formula,
\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]
The discriminant of this equation is positive in the node case and negative in the spiral/center cases. Degenerate nodes occur when the solutions transition between these two cases and the discriminant becomes zero. So a way of thinking about the degenerate node is that the solutions are trying to wind around in a spiral, but they don’t quite make it due to the lack of complexity of the eigenvalue.

But how do we know the direction of rotation? We do the same thing that we did in the spiral case: compute the tangent at a point or two. That, combined with our knowledge of the stability of the origin, will tell us how non-eigensolutions must turn.

Let’s start by considering the point \((1, 0)\). At this point,
\[ \mathbf{x}' = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}. \]
As we know the origin is asymptotically stable, the tangent vector can only point up and to the left if the solutions rotate counterclockwise as they start to approach the origin. \(\square\)
Example 33.3. Find the general solution to the following system.

\[ x = \begin{pmatrix} 12 & 4 \\ -16 & -4 \end{pmatrix} x. \]

We begin by finding the eigenvalues.

\[
0 = \det(A - \lambda I) = \begin{vmatrix} 12 - \lambda & 4 \\ -16 & -4 - \lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2
\]

So we have a repeated eigenvalue of \( \lambda = 4 \). Let’s calculate an eigenvector by solving \((A - 4I)\eta = 0\).

\[
\begin{pmatrix} 8 & -4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

The system of equations we want to solve is

\[
8\eta_1 + 4\eta_2 = 0 \\
-16\eta_1 - 8\eta_2 = 0.
\]

This is solved by any vector of the form \( \eta_2 = -2\eta_1 \). So if we choose \( \eta_1 = 1 \), we get an eigenvector of

\[
\eta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

Now let’s find a generalized eigenvalue \( \rho \). We need to solve \((A - 4I)\rho = \eta\), or

\[
\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \rho_2 = \frac{1}{4} - 2\rho_1.
\]

So the generalized eigenvector has the form

\[
\rho = \begin{pmatrix} \rho_1 \\ \frac{1}{4} - 2\rho_1 \end{pmatrix},
\]

and choosing \( \rho_1 = 0 \) gives

\[
\rho = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}.
\]

Thus the general solution is

\[
x(t) = c_1 e^{4t} \eta + c_2 \left( t e^{4t} \eta + e^{4t} \rho \right)
\]

\[
= c_1 e^{4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left( t e^{4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{4t} \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} \right).
\]

Example 33.4. Sketch the phase portrait of the system in Example 33.3.

Everything in this problem is analogous to that in Example 33.2. We sketch the eigensolution, and as it will grow away from the origin as \( t \to \infty \), the origin will be an unstable degenerate node.

Typical trajectories will once again come out of the origin parallel to the eigensolution and rotate around to be parallel to it again, and all we need to do is to calculate the direction of rotation by calculating the tangent vector at a point. At \( (1, 0) \), we get

\[
x' = \begin{pmatrix} 16 \\ 12 \end{pmatrix},
\]

which can only occur for solutions that are growing if the direction of rotation is clockwise. Thus we get Figure 33.3. \( \square \)
Figure 33.3. Phase portrait for the asymptotically stable degenerate node in Example 33.3.
Nonlinear Systems

It’s time to turn our attention to systems of equations which are the most likely to show up in applications. These systems, unlike the ones we’ve been discussing to this point, are nonlinear, which means that \( x_1' \) and \( x_2' \) aren’t linear combinations of \( x_1 \) and \( x_2 \). As we’ve seen when looking at single differential equations, nonlinear equations can be difficult in general to deal with. In fact, we won’t usually be able to obtain solutions to these systems. Instead, we’ll focus more on qualitative analyses of these systems.

We could easily spend an entire semester on this topic. Instead, we’ll try to get a bit of the flavor of how these systems differ from the linear systems we’ve been learning about.

1. Introduction to Nonlinear Systems

The general form of a nonlinear two-dimensional system of differential equations is

\[
\begin{align*}
x_1' &= f_1(x_1, x_2) \\
x_2' &= f_2(x_1, x_2).
\end{align*}
\]

We could rewrite this more compactly in vector notation as

\[
x' = f(x),
\]

where \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) and \( f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \). For systems like this, there is generally no hope of finding trajectories analytically, as we did for the linear systems we discussed earlier. Thus, as mentioned earlier, our attention will be focused on the qualitative behavior of these solutions.

There are some features of nonlinear phase portraits that are especially salient:

1. The fixed or critical points, which are the equilibrium or steady-state solutions. These correspond to points \( x \) satisfying \( f(x) = 0 \); in other words, \( x_1 \) and \( x_2 \) are zeroes for both \( f_1 \) and \( f_2 \). In this course, we will mostly be focused on learning about the role fixed points play in determining the phase portrait of a nonlinear system.

2. The closed orbits, which correspond to solutions that are periodic for both \( x_1 \) and \( x_2 \). We’ll briefly discuss some techniques here, but for the most part this is a topic that will be focused on in a future differential equations course you might take.

3. How trajectories are arranged near fixed points and closed orbits; again, well primarily look at what happens near fixed points.

4. The stability or instability of fixed points and closed orbits; which of these attract nearby trajectories, and which repel them?

How do we even know we have solutions to our general nonlinear system? As we’ve seen, existence and uniqueness questions can be tricky for nonlinear equations.

**Theorem 34.1 (Existence and Uniqueness).** Consider the initial value problem

\[
x' = f(x) \quad x(0) = x_0.
\]

If \( f \) is continuous and so are its derivatives \( \partial f_j / \partial x_j \) on some region in the plane containing \( x_0 \), then the initial value problem has a unique solution \( x(t) \) on some time interval near \( t = 0 \).
The upshot for our purposes is that if we have sufficiently nice $f_1$ and $f_2$, so that they and their partial derivatives are continuous for all $x_1, x_2$, then any point can be taken as an initial condition for our system.

An important consequence of the existence and uniqueness theorem is that, for the nice systems we be considering, different trajectories can never intersect. If they did, that point of intersection would be an initial condition corresponding to two different solutions, which can’t happen. This means that phase portraits look very polished: they just seem to fit together.

2. Linearization Around Critical Points

The starting point for just about any qualitative analysis of nonlinear systems is determining the critical points. These are points $(x_1^*, x_2^*)$ that correspond to equilibrium solutions $x_1(t) = x_1^*$ and $x_2(t) = x_2^*$. As we’ve discussed, if our system is linear, the only critical point is the origin, $(0, 0)$. Nonlinear systems, however, can have many fixed points, and our goal is to try to determine what we can about the trajectories close to these points.

Consider the system

$$
x' = f(x, y)$$
$$y' = g(x, y).$$

How do we find the critical points? As they are constant solutions for both $x$ and $y$, they are the points where both $x' = 0$ and $y' = 0$. Thus “all” we have to do is to find the values of $x$ and $y$ that are zeroes of both $f$ and $g$. For the examples that we will be looking at, this will be fairly straightforward, though you can probably imagine examples where it would be a difficult process.

Now suppose $(x_0, y_0)$ is a fixed point. Then we know that $f(x_0, y_0) = g(x_0, y_0) = 0$.

The goal of the linearization technique is to use our knowledge of linear systems to try to conclude what we can about the phase portrait near $(x_0, y_0)$. To do this, we’ll try to approximate our nonlinear system by a linear system, which we can then classify using the methods that we discussed in the previous lectures. Since $(x_0, y_0)$ is a fixed point, and the only fixed point of a linear system is the origin, well want to change variables so that $(x_0, y_0)$ becomes the origin of the new coordinate system. Thus, let

$$u = x - x_0$$
$$v = y - y_0.$$

We need to rewrite our differential equations in terms of $u$ and $v$.

$$u' = x'$$
$$= f(x, y)$$
$$= f(x_0 + u, y_0 + v)$$

The natural thing to do is to take the Taylor expansion of $f$ near $(x_0, y_0)$.

$$= f(x_0, y_0) + u \frac{\partial f}{\partial x}(x_0, y_0) + v \frac{\partial f}{\partial y}(x_0, y_0) + \text{higher order terms}$$

$$= u \frac{\partial f}{\partial x}(x_0, y_0) + v \frac{\partial f}{\partial y}(x_0, y_0) + \text{H.O.T.}$$

To simplify notation, we’ll sometimes suppress writing explicitly that $\partial f/\partial x$ and $\partial f/\partial y$ are evaluated at $(x_0, y_0)$, but it’s important to keep this in mind. For our purposes, these partial derivatives are numbers, not functions.
Another important observation is that, as we are considering what happens very close to our fixed point, \( u \) and \( v \) are both small; hence the higher order terms are smaller still and will be disregarded in our computations.

Now, by a similar computation,

\[
v' = u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + \text{H.O.T.}
\]

Ignoring these very small higher order terms, we can write this system of rewritten differential equations in matrix form. The linearized system near \((x_0, y_0)\) is

\[
\begin{pmatrix}
  u' \\
  v'
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
  \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\begin{pmatrix}
  u \\
  v
\end{pmatrix}.
\]

We will use, from this point on, the notation \( f_x = \frac{\partial f}{\partial x} \). The matrix

\[
A = \begin{pmatrix}
  f_x(x_0, y_0) & f_y(x_0, y_0) \\
  g_x(x_0, y_0) & g_y(x_0, y_0)
\end{pmatrix}
\]

is called the Jacobian matrix at \((x_0, y_0)\) of the vector-valued function \( f(x) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \). In multivariable calculus, the Jacobian matrix is the appropriate generalization of the single-variable derivative.

We can then study the linearized system using our standard techniques.
Linearization Near Critical Points

Last lecture, we began considering the nonlinear system

\[ \begin{align*}
    x' &= f(x, y) \\
    y' &= g(x, y).
\end{align*} \tag{35.1} \]

We saw that near the critical points \((x_0, y_0)\) satisfying \(f(x_0, y_0) = g(x_0, y_0) = 0\), which correspond to the equilibrium solutions of the system (35.1), we can approximate the nonlinear system by the linear system

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{35.2} \]

We can then analyze these linear systems to conclude what we can about the behavior of solutions near the equilibria. Notice that this is entirely a local method: we don’t obtain information about what happens far away from the critical points. That this works in general is the content of the Hartman-Grobman Theorem, which we will not formulate. However, we will look at some examples, as well as observing when the theorem fails and linearization does not suffice to characterize the behavior near an equilibrium solution.

Example 35.1. Find the critical points of the following systems, then determine the linearized approximations near them.

(i) \( x' = y^2 - 3x + 2 \quad y' = x^2 - y^2 \)

The first task is to find the critical points. We do this by setting \(x' = f(x, y)\) and \(y' = g(x, y)\) and looking for values of \(x\) and \(y\) that make both \(f\) and \(g\) equal to zero.

Looking at the two equations, we might decide to start looking at \(x^2 - y^2 = 0\). This shows that any fixed point must satisfy \(x^2 = y^2\). Next, the equation \(f(x, y) = y^2 - 3x + 2\) becomes

\[ x^2 - 3x + 2 = 0 \]
\[ (x - 2)(x - 1) = 0. \]

So fixed points can only occur at \(x = 2\) and \(x = 1\). Using that \(x^2 = y^2\). So if \(x = 2\), \(y^2 = 4\), and so \(y = \pm 2\). Similarly, if \(x = 1\), \(y^2 = 1\), and \(y = \pm 1\).

We then have four fixed points: \((1, 1), (1, -1), (2, 2), (2, -2)\). We want to find the linearizations of the system near these points. The Jacobian matrix of the nonlinear system at a general point \((x, y)\) is

\[ \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}. \]
To get the coefficient matrix of our linearized system near one of the critical points, all we have to do is to evaluate these terms at the critical point. Thus we get the following linearizations:

\[(1, 1) : \quad x' = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} x\]

\[(1, -1) : \quad x' = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} x\]

\[(2, 2) : \quad x' = \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix} x\]

\[(2, -2) : \quad x' = \begin{pmatrix} -3 & -4 \\ 4 & 4 \end{pmatrix} x\]

\[(ii) \quad x' = y \quad y' = -x + x^3\]

Again, the first task is to find the critical points. For \(x' = 0\), we need to require that \(y = 0\). Then for \(y' = 0\), we need \(x = x^3\). This means \(x = 0, \pm 1\). So the critical points are \((0, 0)\), \((1, 0)\), and \((-1, 0)\).

The Jacobian matrix of the system at \((x, y)\) is

\[
\begin{pmatrix}
0 & 1 \\
-1 + 3x^2 & 0
\end{pmatrix},
\]

so we end up with the following linearizations at each of the critical points.

\[(0, 0) : \quad x' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x\]

\[(1, 0) : \quad x' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} x\]

\[(-1, 0) : \quad x' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} x\]

So, we’ve found the critical points and the linearizations of the original nonlinear system near them. Now what? Normally, we would go through our linear systems analysis to determine the type and stability of each critical point: remember, we can do this from just determining the eigenvalues. Ideally, the type and stability of the origin in these linear systems would correspond to the type and stability of the associated critical point in the nonlinear system.

**Exercise.** For each of the critical points in both systems given in Example 35.1, determine the type and stability of the origin in the linearized system.

However, there’s...

### 1. A Caveat, or A Term’s A Term No Matter How Small

In turning our nonlinear system near a critical point into the linear system (35.2), we disregarded terms of order two or higher on the grounds that they were very small. Is it really safe to do that? In principle, after all, those terms could influence the behavior of the system in ways that make this linearization method unreliable. It turns out that as long as linearized system’s critical point is not one of several borderline cases, this doesn’t affect the qualitative type of the critical point.
1.1. Simple Eigenvalues. If the linearized system near \((x_0, y_0)\) has simple (i.e., distinct) eigenvalues, the only problems occur when the linearization predicts we will have a center. Spirals, nodes, and saddles are all preserved. Moreover, stability is preserved: we can’t go from having an unstable spiral in the linearized system to having a stable spiral in the nonlinear system. Thus if our linearization predicts we have a spiral, node, or saddle, we can conclude that the critical point will be of the same type.

Here’s an example of how this fails when the linearization predicts a center.

**Example 35.2.** Show that the two systems
\[
\begin{align*}
x' &= -y + x(x^2 + y^2) \\
y' &= x + y(x^2 + y^2)
\end{align*}
\]
\[
\begin{align*}
x' &= -y - x(x^2 + y^2) \\
y' &= x - y(x^2 + y^2)
\end{align*}
\]
have the same linearized systems at the critical point \((0, 0)\), but different phase portraits.

The Jacobian of the first system is
\[
\begin{pmatrix}
3x^2 + y^2 & -1 + 2xy \\
1 + 2xy & x^2 + 3y^2
\end{pmatrix}
\]
The Jacobian of the second system is
\[
\begin{pmatrix}
-3x^2 - y^2 & -1 - 2xy \\
1 - 2xy & -x^2 - 3y^2
\end{pmatrix}
\]
At \((0, 0)\), then, both have the same linearization, namely
\[
x' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x.
\]
The eigenvalues of this matrix are both \(\pm i\), so the linearized system has a center at the origin.

However, we can analyze this particular system more directly by using polar coordinates. If we do that, the two systems become
\[
\begin{align*}
r' &= r^3 \\
\theta' &= 1
\end{align*}
\]
\[
\begin{align*}
r' &= -r^3 \\
\theta' &= 1
\end{align*}
\]
Thus, non-zero trajectories for the first system will expand, as \(r' > 0\) for all \(r > 0\), while non-zero trajectories for the second system will decay as \(r' < 0\) for \(r > 0\). We actually have an unstable spiral for the first system and an asymptotically stable spiral for the second system, even though the linearization predicted a center. These spirals, however, rotate in the same direction as the predicted center.

The previous example demonstrated the sort of process we would need to go through if we actually wanted to determine the phase portrait near a predicted center. In general, this will require a more direct computation rather than a linearization. For our purposes, though, it will generally suffice to be aware that we can’t trust the prediction of a center; even small higher order terms can throw a center to a spiral of either stability.

1.2. Repeated Eigenvalues. In the case of a repeated eigenvalue \(\lambda\), we ended up getting either a star node or a degenerate node depending on whether \(\lambda\) was complete or defective. This is also a delicate case; we will, however, be able to conclude something.

If \(\lambda\) is complete, and the linearization predicts a star node, we can conclude only that the nonlinear system will have a node (possibly degenerate or star) at the associated critical point. On the other hand, if \(\lambda\) is defective, we could get a node (again, it could possibly be degenerate or star) or even a spiral point. The stability of the linearized system, however, will be preserved.
Remark. Throughout this entire discussion of systems, we’ve assumed that our coefficient matrices are nonsingular; that is, they don’t have any zero eigenvalues. This is an interesting case, though, since it’s the only case where you get substantially different behavior between the linearization and the nonlinear system. This is a topic for a higher level course on nonlinear differential equations.
Applications of Nonlinear Systems

In today’s lecture, we’ll look at some particular nonlinear systems, generally inspired from physics or biology. We’ll then use the linearization method we’ve been discussing for the last few lectures to try to understand the behavior of these systems.

1. Competing Species

One of the classical examples of a nonlinear system is the Lotka-Volterra model of competition between two species (let’s take them to be rabbits and kangaroos). Both species are competing for the same food supply (the grass) and there’s a limited amount of this resource. We’ll assume these populations exist in an environment without predators.

What are the main considerations we’ll need to keep in mind to write down this system?

(1) When we discussed the logistic equation as an example of an autonomous first order equation, we mentioned the existence of an environmental carrying capacity: that is, given a species which consumes resources at a certain rate, there is a certain upper limit for the population of this species that the environment can support. Here, since we’re assuming that we’ve got a kangaroo population and a rabbit population and no others, we can assume that in the absence of the other population, each would grow to its carrying capacity. Thus we’ll incorporate logistic growth into the equations for each species. We’ll also assign rabbits a higher intrinsic growth rate due to the well-known ability of rabbits to reproduce.

In particular, if \( x(t) \) is the rabbit population and \( y(t) \) is the kangaroo population at time \( t \), the model

\[
\begin{align*}
x' &= x(3 - x - 2y) \\
y' &= y(2 - x - y)
\end{align*}
\]  

(36.1)

incorporates these assumptions.

Our first task is to find the fixed points of the system. We solve \( x' = y' = 0 \) and obtain four fixed points: \((0, 0), (0, 2), (3, 0), \) and \((1, 1)\). Next, we use the linearization method to try to classify them. The Jacobian of the system (36.1) is

\[
A = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.
\]

Let’s now consider each fixed point to determine the nearby behavior.

• \((0, 0)\): Here \( A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \). This has eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \), so we can conclude that \((0, 0)\) is an unstable node, as non-degenerate or star nodes are preserved in the original nonlinear system. Recall that right at a node, typical trajectories are tangential to the slow
Figure 36.1. Phase portrait of the rabbits ($x$) vs. kangaroo ($y$) model (36.1). Observe the fixed points and their stability.

eigensolution, which in this case is the eigensolution corresponding to $\lambda_2 = 2$. The eigenvector corresponding to $\lambda_2$ is $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so trajectories will leave the node along the $y$-axis.

- $(0, 2)$: Here $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$. This matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$; hence the fixed point is a stable node. A typical trajectory approaches it along the eigensolution corresponding to $\lambda_1$, which is in the direction of the eigenvector $\eta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

- $(3, 0)$: Here $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$. The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$, so this is a stable node. The slow eigensolution is in the direction of the eigenvector $\lambda_2$, namely $\eta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

- $(1, 1)$: Here $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ and $\lambda_{1,2} = -1 \pm \sqrt{2}$. This is a saddle point.

We can further note that the $x$-axis is a straight-line trajectory, as $x' = 0$ when $x = 0$. Similarly, the $y$-axis is a straight-line trajectory. Lines where $x' = 0$ or $y' = 0$ are called nullclines, and in this case these nullclines are also solutions. Putting this all together and using some common-sense to fill in the rest of the trajectories, we obtain a phase portrait that looks something like Figure 36.1.

This has an interesting biological interpretation: it shows that, in general, one species drives the other to extinction. If we start above the stable solution of the saddle point, the kangaroos drive the rabbits to extinction, but if we start below it, the rabbits drive the kangaroos to extinction. This phenomenon shows up in more complex models of competing species as well, and it has led to the formulation of the principle of competitive exclusion, which states that, in general, two species competing for the same resources cannot coexist. This is why releasing pets into the wild is a bad idea; its species might drive native populations competing for the same resources out.
2. Nonlinear Pendulum

In the absence of any damping or external driving, the motion of a pendulum is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin(\theta) = 0,$$

where $\theta(t)$ is the angle from the downward vertical, $g$ is the acceleration due to gravity, and $L$ is the length of the pendulum. This is derived from the rotational formulation of Newton’s Second Law,

$$\tau = I \alpha,$$

where $\tau = -mgL \sin(\theta)$ is the torque, $I = mL^2$ is the moment of inertia, and $\alpha = d^2\theta/dt^2$ is the rotational acceleration. We write $\omega_0 = \sqrt{g/L}$. Then the equation becomes

$$\theta'' + \omega_0^2 \sin(\theta) = 0. \quad (36.2)$$

For very small angles, this equation will be linearized using $\sin(\theta) \approx \theta$ (this is generally done in high school, for example), but using phase plane methods, we can study the nonlinear equation even for large angles.

Writing Equation (36.2) as a system yields

$$\theta' = \nu$$

$$\nu' = -\omega_0^2 \sin(\theta), \quad (36.3)$$

where $\nu$ is the angular velocity.

The fixed points are of the form $(n\pi, 0)$, where $n$ is any integer. The Jacobian matrix of (36.3) is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 \cos(\theta) & 0 \end{pmatrix}.$$ 

Notice that there is no difference between angles that differ by $2\pi$, either physically or formally, so we’ll focus on the fixed points $(0, 0)$ and $(\pi, 0)$.

Near $(0, 0)$, the linearized system is

$$x = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} x.$$ 

The eigenvalues of the coefficient matrix are $\lambda_{1,2} = \pm \omega_0 i$, which says that the linearized system has a center. As we discussed in Lecture 35, however, that doesn’t mean that the system have a nonlinear center at the fixed point; it could be a spiral. So what do we do?

It turns out that the system (36.3) is an example of a reversible system; that is, if we “reverse time” by changing $t$ to $-t$ and $\nu$ to $-\nu$ (since reversing time will reverse the direction of velocity), the system stays the same. Physically, this should make sense: if we tape a pendulum’s motion and play it backwards, we won’t see any physical absurdities.

It turns out that for reversible systems, if the origin is a linear center, it will be a nonlinear center, as well. The idea is that we take a trajectory close to the origin which swirls around the origin (as the origin will have to be a spiral or a center). Reversing time and velocity reflects this to a twin trajectory with the same endpoints but with the arrow reversed, which closes the orbit. This is illustrated in Figure 36.2. Thus $(0, 0)$, and hence $(2k\pi, 0)$, are all centers.

What happens near $(\pi, 0)$? The linearized system is

$$x = \begin{pmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{pmatrix} x.$$ 

The coefficient matrix has eigenvalues $\lambda_{1,2} = \pm \omega_0$, and so $(\pi, 0)$ (and any other fixed point of the form $(2(k + 1)\pi, 0)$ is a saddle point.
Differential Equations Lecture 36: Applications of Nonlinear Systems

(A) Trajectory in the vicinity of a linearized center. (B) For a reversible system, the trajectory in \(36.2a\) must have its mirror image as a trajectory as well, completing the periodic orbit and making the fixed point a nonlinear center.

Figure 36.2. Why an origin that is a linear center is also a nonlinear center in a reversible system.

Now we can draw the phase portrait near the fixed points. How do we physically interpret the classification we just found? The centers correspond to a state of neutrally stable equilibrium; the pendulum is at rest and is hanging straight down (as \(\theta = 2k\pi\)). If we move the pendulum slightly away from there (and possibly give it a little initial velocity), we’ll get oscillation back and forth. The saddles, on the other hand, correspond to the cases where the pendulum is at rest but is balanced perfectly up. They’re unstable, as if we move the pendulum slightly from this balance, they’ll swing back down.

What happens away from the fixed points? This corresponds to giving the pendulum a lot of initial velocity. We’ll actually end up with the pendulum rotating around and around the axis, so that the phase portrait looks like Figure 36.3.

Exercise (Free Damped Pendulum). If we add a damper with damping coefficient \(\gamma > 0\) to the nonlinear pendulum modeled by Equation (36.2), the new equation governing the motion of the pendulum is

\[
\theta'' + \gamma \theta' + \omega_0^2 \sin(\theta) = 0.
\]

After writing this equation as a system of first order equations, find and classify the fixed points for all \(\gamma > 0\) and plot the phase portraits for each qualitatively different case.

3. Next Steps

We’ve only dipped our toes into the theory of nonlinear systems, but hopefully we’ve gotten the sense that there’s a lot of interesting phenomena occurring here. Let’s briefly list some of the directions there are go in from here; most of these would get at least a brief discussion in a second course in differential equations.

3.1. Limit Cycles and Periodic Solutions. After equilibria, the next most important feature of nonlinear systems is the possibility of periodic solutions. These also very strongly affect the overall phase portrait. There are several global methods that can be brought to bear on the problem of detecting and understanding periodic solutions.

There are many questions that can be asked here. If we have a closed trajectory, must it always circle a fixed point? What kinds of fixed points are permitted inside of one? Is there a restriction on the number of closed trajectories we can have or where they can be located?
In fact, we’ve seen examples where closed trajectories occur in nested families. They can also be isolated: such a solution is called a limit cycle. Nearby solutions, which aren’t closed, can limit to or away from the limit cycle. Stable limit cycles are very important scientifically, as they model self-sustained oscillations, such as in the human heart. These are, as we’ve seen, inherently nonlinear phenomena; if a linear system has closed solutions, so are nearby solutions.

3.2. Bifurcations. We’ve only looked at what happens if we have a single system of nonlinear differential equations. What if we have a family of them, differing by some parameter? For example, what if we have a nonlinear pendulum or an electrical system driven by a constant external force, which we proceed to increase? Can the behavior change as we vary the parameter?

The answer is yes. Many different things can happen here: the number or stability of fixed points or closed orbits can change, for example. Such a change is called a bifurcation. This is very important if you think about an externally driven electrical circuit, for example: understanding where and what bifurcations can occur will tell us how much force we can safely apply to the circuit.

Bifurcations can also occur for one-dimensional equations, as well, and some very interesting ones occur for discrete systems, where we take some function and iterate it repeatedly, looking for stable and unstable fixed points.

3.3. Chaos. It can be shown that for a two-dimensional system, things behave generally nicely and limit cycles are “typical” in some sense. However, in three dimensions, all bets are off. In 1963, while modeling “convection rolls” in the upper atmosphere, Edward Lorenz wrote down the following system of differential equations:

\[
\begin{align*}
x' &= -ax + by \\
y' &= -xz + rx - y \\
z' &= xy - bz,
\end{align*}
\]
where $a, b$ are constants and $r > a/b$. This looks like a relatively simple system, but it turns out to be surprisingly complicated. The solutions don’t ever settle down to a periodic orbit or fixed point, but they also don’t run off to infinity. Instead, they wrap around two equilibria in a fairly crazy manner.

The manner in which they wrap around these equilibria depends very strongly on the initial conditions; small changes lead to solutions which behave very differently. This is known as sensitivity to initial conditions, and it’s one of the hallmarks of a chaotic system. It’s become known as the “butterfly effect”: a butterfly flaps its wings in Beijing, and in New York it rains a few days later instead of being sunny. It’s why we can’t predict the weather very far in advance, since instruments and computers only have a certain number of decimal places they can measure/compute to, but down the line, the small bits that are lost in the process become very important.

\[1\] There’s another possibility for why this is called the butterfly effect. There’s a classic science fiction story (and an episode of The Simpsons parodying it) called “A Sound of Thunder” by Ray Bradbury. It’s about a time traveler who goes back in time to hunt dinosaurs and steps on a butterfly. When he returns to his time, everything has changed in a substantial way. Killing that one butterfly changed the initial conditions of the universe and upon moving forward a significant amount of time, everything evolved in a different way.
Part 7

Partial Differential Equations
Boundary Value Problems and Eigenfunctions

1. Boundary Value Problems

At this point, we’re essentially finished discussing ordinary differential equations. We’re going to begin discussing partial differential equations. Partial differential equations are much more complicated than ordinary differential equations: we’ll need to specify the way we want solutions to behave on the boundary of the region our equation is valid on. Such data are called *boundary values* or *boundary conditions*, and a combination of a differential equation and some boundary conditions is called a *boundary value problem*.

Boundary conditions depend on the region our equation is valid on: if we have an ordinary differential equation, this will be some interval; for a partial differential equation, this might also be an interval, or it might be on a square in the plane. To try to get a better feel for how boundary values work, let’s see how they affect solutions to ordinary differential equations.

**Example 37.1.** Let’s consider the second order ordinary differential equation $y'' + y = 0$. Specifying boundary conditions for this equation involves specifying the values of the solution (or its derivatives) at two points. Let’s say we fix $y(0) = 0$ and $y(2\pi) = 0$. We know that solutions to this equation have the form $y(x) = A \cos(x) + B \sin(x)$.

Applying the first boundary condition tells us that $0 = y(0) = A$. Applying the second condition gives $0 = y(2\pi) = B \sin(2\pi)$. Now, $\sin(2\pi) = 0$, so this condition doesn’t tell us anything about $B$. Thus the solutions to the boundary value problem are any functions of the form $y(x) = B \sin(x)$.

**Example 37.2.** Consider $y'' + y = 0$ with boundary values $y(0) = y(6) = 0$. This seems similar to the previous problem; the solutions have the form $y(x) = A \cos(x) + B \sin(x)$

and the first condition gives $A = 0$. The second condition tells us that $0 = y(6) = B \sin(6)$. As $\sin(6) \neq 0$, $B = 0$ and the only possible solution is $y(x) = 0$.

Boundary value problems are very natural. We can interpret Examples 37.1 and 37.2 physically. We know that the equation $y'' + y = 0$ models an oscillator; something like a rock hanging from a Slinky\(^1\). This rock oscillates with natural frequency $1/2\pi$. The condition $y(0) = 0$ means that when we start observing, the rock should be in the equilibrium position. If we then specify $y(2\pi) = 0$, this is no problem for any solution, because the natural motion of the rock is $2\pi$-periodic. On the other hand, this also makes it impossible for the rock to return to the equilibrium position after 6 time units if it ever left, as it can then only return every multiple of $2\pi$. Thus to satisfy $y(6) = 0$, the rock must remain at the equilibrium position for all time, which means $y(t) = 0$.

\(^1\)It’s Slinky, it’s Slinky, for fun it’s a wonderful toy. It’s Slinky, it’s Slinky, it’s fun for a a girl and a boy!
Examples 37.1 and 37.2 are examples of homogeneous boundary value problems. We say that a boundary value problem is homogeneous if the equation is homogeneous and the two boundary conditions involve zero. That is, homogeneous boundary conditions might be of the following types:

\[
\begin{align*}
 y(x_1) &= 0 \\
 y'(x_1) &= 0 \\
 y(x_2) &= 0 \\
 y'(x_2) &= 0
\end{align*}
\]

On the other hand, if the equation is nonhomogeneous or any of the boundary conditions don’t involve zero, we say that the boundary value problem is nonhomogeneous. Let’s look at some examples of nonhomogeneous boundary value problems.

**Example 37.3.** Take \( y'' + 9y = 0 \) with boundary conditions \( y(0) = 2 \) and \( y(\pi/6) = 1 \). The general solution to the differential equation is

\[ y(x) = A \cos(3x) + B \sin(3x). \]

The two conditions give

\[
\begin{align*}
 2 &= y(0) = A \\
 1 &= y \left( \frac{\pi}{2} \right) = B,
\end{align*}
\]

so the solution is

\[ y(x) = 2 \cos(3x) + \sin(3x). \]

□

**Example 37.4.** Take \( y'' + 9y = 0 \) with boundary conditions \( y(0) = 2 \) and \( y(2\pi) = 2 \). The general solution to the differential equation is

\[ y(x) = A \cos(3x) + B \sin(3x). \]

The two conditions give

\[
\begin{align*}
 2 &= y(0) = A \\
 2 &= y(6\pi) = A.
\end{align*}
\]

In other words, this time the second condition didn’t give any new information, like in Example 37.1, and \( B \) doesn’t affect whether or not the solution satisfies the boundary conditions. We then have infinitely many solutions,

\[ y(x) = 2 \cos(3x) + B \sin(3x). \]

□

**Example 37.5.** Take \( y'' + 9y = 0 \) with boundary conditions \( y(0) = 2 \) and \( y(2\pi) = 4 \). The general solution to the differential equation is

\[ y(x) = A \cos(3x) + B \sin(3x). \]

The two conditions give

\[
\begin{align*}
 2 &= y(0) = A \\
 4 &= y(6\pi) = A,
\end{align*}
\]

So on the one hand, we must have \( A = 2 \), but on the other, \( A = 4 \). This is impossible, and this boundary value problem has no solutions. □

These examples hopefully illustrate that a small change to the boundary conditions can dramatically change the problem, unlike with initial value problems.
2. Eigenfunctions

All of the examples of boundary value problems that we’ve been looking at had the form \( y'' + \lambda y = 0 \) for some \( \lambda \). We’ve seen how a boundary value problem can change if we change the conditions slightly, so what happens if we modify the equation slightly?

For example, we saw that for the equation \( y'' + y = 0 \) with boundary conditions \( y(0) = y(2\pi) = 0 \), we had infinitely many solutions \( y(x) = B \sin(x) \). What happens if we look at \( y'' + 3y = 0 \) with the same conditions? The general solution is

\[
y(x) = A \cos \left( \sqrt{3}x \right) + B \sin \left( \sqrt{3}x \right) .
\]

The first condition says \( A = 0 \), while the second condition says that \( 0 = y(2\pi) = B \sin (2\sqrt{3}\pi) \). However, \( \sin (2\sqrt{3}\pi) \neq 0 \), so we get \( B = 0 \) and the only solution is \( y = 0 \).

Thus, when considering the boundary value problems \( y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2\pi) = 0 \), we can sometimes get nontrivial (non-zero) solutions, but other times we only get the trivial solution \( y = 0 \). We say that a \( \lambda \) producing a nontrivial solution is an \textit{eigenvalue} and the nontrivial solutions are \textit{eigenfunctions} corresponding to the associated eigenvalue. This terminology comes from the similarity of the equations

\[
A \eta = \lambda \eta
\]

and

\[
\frac{d^2}{dx^2} y = -\lambda y.
\]

In both cases, (some operation) acting on (something) produces a constant times that original (something), so the operation merely scales the original (something).

Let’s see if we can figure out the eigenvalues and eigenfunctions of this problem.

**Example 37.6.** Find all of the eigenvalues and eigenfunctions of the boundary value problem

\[
y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2\pi) = 0 .
\]

We have to be a little bit careful: the general solution of \( y'' + \lambda y = 0 \) will change depending on whether \( \lambda \) is positive, negative, or zero. We need to consider each case.

(\( \lambda > 0 \)): If \( \lambda > 0 \), we can write \( \lambda = \beta^2 \), where \( \beta \neq 0 \). The characteristic equation of \( y'' + \beta^2 y = 0 \) has roots \( \pm \beta i \), so the general solution is then

\[
y(x) = A \cos(\beta x) + B \sin(\beta x).
\]

\( y(0) = 0 \) implies that \( A = 0 \). \( y(2\pi) = 0 \) becomes

\[
B \sin(2\beta \pi) = 0.
\]

If \( \beta \) is such that \( \sin(2\beta \pi) \neq 0 \), then \( B = 0 \) and we only have trivial solutions. On the other hand, if \( \sin(2\beta \pi) = 0 \), then we have infinitely many nontrivial solutions as \( B \) can take any value. So we get nontrivial solutions if and only if \( 2\beta \pi = n\pi \) for any positive integer \( n \), or when \( \beta = n/2 \). Thus, in this case we have eigenvalues

\[
\lambda_n = \left( \frac{n}{2} \right)^2
\]

and eigenfunctions

\[
y_n(x) = \sin \left( \frac{nx}{2} \right) .
\]

(\( \lambda < 0 \)): We write \( \lambda = -\beta^2 \), again with \( \beta > 0 \). The characteristic equation has roots \( \pm \beta \), and the general solution is

\[
y(x) = Ae^{\beta x} + Be^{-\beta x}.
\]
Applying the first boundary condition gives \( A + B = 0 \), so we have \( A = -B \) and the solution must have the form \( y(x) = Ae^{\beta x} - Ae^{-\beta x} \). Using \( y(2\pi) = 0 \),

\[
Ae^{2\beta \pi} - Ae^{-2\beta \pi} = 0,
\]

so we must have \( 2\beta \pi = -2\beta \pi \) as the exponential is a one-to-one function. This can only happen if \( \beta = 0 \), but that isn’t covered by this case, and so we can only conclude that there are no negative eigenvalues.

(\( \lambda = 0 \)): The equation here is \( y'' = 0 \), which has general solution

\[
y(x) = Ax + B.
\]

The first boundary condition says that \( B = 0 \), while the second says \( 2\pi A = 0 \), which can only occur if \( A = 0 \). So the only solution to the boundary value problem when \( \lambda = 0 \) is \( y(x) = 0 \), and so 0 cannot be an eigenvalue.

\[\square\]
LECTURE 38

The Heat Equation

The world of partial differential equations is much more complicated than that of ordinary differential equations. For linear ODEs, there is a general theory which stems from the theory of systems of first order linear equations that we’ve been discussing. However, for PDEs, there is no general theory: methods need to be adopted for smaller groups of equations. Thus, we will only be dipping our toes into the vast area of PDEs. We’ll also be focusing only on a single solution method, called separation of variables, which will help us understand some of the phenomena that can occur.

The first partial differential equation we will work with is called the heat equation, as it models temperature distribution in some object. We’ll be working exclusively with the one-dimensional heat equation: that is, we’re trying to find temperature distributions in a one-dimensional bar of length $l$. In particular, we’ll assume that our bar corresponds to the interval $(0, l)$ on the real line.

This assumption is made for reasons of simplicity, but it’s also not that bad. If we assume we have a “real” bar, the one-dimensional assumption boils down to assuming that at every lateral cross-section and every instant in time, the temperature is a constant: unrealistic, to be sure, but not terrible. Moreover, we’re assuming that the lateral surface of the bar is perfectly insulated, so that the only way heat could enter or leave the bar is through the ends $x = 0$ and $x = l$. Thus any heat transfer is essentially one-dimensional.

Many PDEs come from fairly basic physics. Here, we’ll derive the heat equation from two physical principles. Let’s get some notation down before we start deriving the heat equation. We’ll let $u(x, t)$ denote the temperature at a point $x$ and time $t$. $c$ will be the specific heat of the material the bar is made from (that is, the amount of heat needed to raise one unit of mass of this material by one temperature unit) and $\rho$ is the density of the rod. Note that in general, the specific heat and density of the rod don’t have to be constant; they can vary with $x$. However, it greatly simplifies the problem if they are constant.

Let’s consider a small slab of length $\Delta x$. We’ll let $H(t)$ be the amount of heat contained in this slab. The mass of the slab is $\rho \Delta x$. Then the heat energy contained in this small region is given by

$$H(t) = cu\rho \Delta x.$$  \hfill (38.1)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure38.1}
\caption{Heat flux across the boundary of a small slab with length $\Delta x$. The graph is the graph of temperature ($u(x, t)$) at a given time $t$. In accordance with Fourier’s law, the heat enters or leaves the boundary by flowing from hot to cold; hence at $x$ the flux is opposing the sign of $u_x$, while at $x + \Delta x$ it agrees.}
\end{figure}
Differential Equations  

Lecture 38: The Heat Equation

Figure 38.2. Temperature versus position on a bar. The arrows show time-dependence in accordance with the heat equation. On the left, the temperature graph is concave up, so the left side of the bar is warming up. On the right, the temperature graph is concave down, so the right side of the bar is cooling down.

On the other hand, within the slab, heat will flow from hot to cold (this is called Fourier’s law. Additionally, the only way heat can leave the region is by moving across the boundary points, which are at $x$ and $x + \Delta x$ (this follows from the law of conservation of energy). So the change of heat energy of the slab is equal to the heat flux across the boundary. More precisely, if $\kappa$ is the conductivity of the bar’s material,

$$\frac{dH}{dt} = \kappa u_x(x + \Delta x, t) - \kappa u_x(x, t). \tag{38.2}$$

This is illustrated in Figure 38.1. Setting the time-derivative of Equation (38.1) equal to Equation (38.2) yields

$$cu(x, t)\rho \Delta x = \kappa u_x(x + \Delta x, t) - \kappa u_x(x, t),$$

or

$$c\rho u_t(x, t)\rho = \frac{\kappa u_x(x + \Delta x, t) - \kappa u_x(x, t)}{\Delta x}.$$  

If we take the limit as $\Delta x \to 0$, the right-hand side is just the $x$-derivative of $\kappa u_x$ at $(x, t)$, or

$$c\rho u_t(x, t)\rho = \kappa u_{xx}(x, t).$$

Setting $k = \kappa/c\rho$, we have the heat equation

$$u_t = ku_{xx}. \tag{38.3}$$

Notice that the heat equation is a linear PDE: none of the derivatives of $u$ are undergoing any operations aside from multiplication by a constant. What is the constant $k$? It’s called the thermal diffusivity of the bar and is a measure of how quickly heat spreads through the given material.

How do we interpret the heat equation? Say we graph the temperature of the bar at a fixed time. Suppose it looks like the temperature graph in Figure 38.2. On the left side of the bar, the graph is concave up. If the graph is concave up, that means that the second derivative of the temperature (with respect to position $x$) is positive. The heat equation then tells us that the time derivative of the temperature at any of the points on the left side of the bar will be increasing; in other words, the left side of the bar will be warming up. Similarly, on the right side of the bar, the graph is concave down. Thus the second $x$-derivative of the temperature is negative, and so will be the first $t$-derivative, and we can conclude that the right side of the bar is cooling down.
LECTURE 39

Separation of Variables and the Heat Equation

1. Initial Value Problems for PDEs

Partial differential equations generally have lots of solutions. To specify a unique one, we’ll need some additional conditions. These conditions are usually motivated by the physics and come in two varieties: initial conditions and boundary conditions. An initial value problem for a PDE consists of the equation, initial conditions, and boundary conditions.

An initial condition specifies the physical state at a given time \( t_0 \). For example, an initial condition for the heat equation would be the starting temperature distribution \( u(x,0) = f(x) \).

This is the only initial condition required because the heat equation is first order with respect to time. The wave equation, which we will look at later, is second order with respect to time, and so needs two initial conditions.

PDEs are also generally only valid on a certain domain. Boundary conditions specify how the solution is to behave on the boundary of this domain. These need to be specified, because the solution doesn’t exist on the other side of the boundary, meaning we might have problems with differentiability there.

Our heat equation was derived for a one-dimensional bar of length \( l \), so the relevant domain in question can be taken to be the interval \( 0 < x < l \) and the boundary consists of the two points \( x = 0 \) and \( x = l \). We could have derived a two-dimensional heat equation, for example, in which case the domain would be some region in the \( xy \)-plane with boundary some closed curve.

It will usually be clear from a physical description of the problem what the appropriate boundary conditions are. We might know that, at the endpoints \( x = 0 \) and \( x = l \), the temperatures \( u(0,t) \) and \( u(l,t) \) are fixed. Boundary conditions that give the value of the solution are called Dirichlet conditions. Or we might insulate the ends, meaning there would be no heat flow out of the boundary; this would yield the boundary conditions \( u_x(0,t) = u_x(l,t) = 0 \). If the boundary conditions specify the derivative of the solution, they’re called Neumann conditions. We could also specify that we have one insulated end and at the other, we control the temperature; this is an example of a mixed boundary condition.

As we’ve seen, changing boundary conditions can significantly change the character of a problem. Initially, to get a feel for our solution method, we’ll work with the homogeneous Dirichlet conditions \( u(0,t) = u(l,t) = 0 \), giving us the following initial value problem:

\[
\begin{align*}
\text{(DE)} & : \ u_t = ku_{xx} \\
\text{(BC)} & : \ u(0,t) = u(l,t) = 0 \\
\text{(IC)} & : \ u(x,0) = f(x).
\end{align*}
\]

After we have seen the general method, we’ll see what happens with homogeneous Neumann conditions, leaving nonhomogeneous conditions for a later discussion.
2. Separation of Variables

Suppose we have a heat equation IVP such as (39.1). How do we proceed? In these types of problems, we will try to build up a general solution from smaller solutions that are easier to find.

A typical approach is to assume that “smaller” solutions have a nice form. For example, we might suppose that we have a separated solution, where

\[ u(x, t) = X(x)T(t). \]  

(39.2)

In other words, we assume that our solution is the product of a function that depends only on \( x \) and a function that depends only on \( t \). We can then try to convert our PDE into two ODEs, one involving \( x \) and one involving \( t \), and then use our knowledge of ODEs to solve them.

It should be noted that having a solution in this form is a very special situation and should not be expected in general. In fact, this method (called separation of variables) cannot always be used, and even when it can be used it’s often hard to move past the first step. However, it will work for the PDEs we will study in this course and is a good starting point.

We begin by plugging our separated solution (39.2) into the heat equation (39.1a).

\[
\begin{align*}
\frac{\partial}{\partial t} [X(x)T(t)] &= k \frac{\partial^2}{\partial x^2} [X(x)T(t)] \\
X(x)T'(t) &= kX''(x)T(t)
\end{align*}
\]

Now we can move everything involving \( x \) to one side and everything involving \( t \) to the other.

\[
\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.
\]

(39.3)

We’ve written the minus sign in front of \( \lambda \) for convenience. It will turn out that \( \lambda > 0 \), but that these expressions should be negative.

Equation (39.3) really contains a pair of separate ordinary differential equations:

\[
\begin{align*}
X'' + \lambda X &= 0 \\
T' + \lambda kT &= 0.
\end{align*}
\]

(39.4)

Equation (39.4b) is straightforward enough to solve: \( T' = -k\lambda T \), which is a separable linear first order ODE, so

\[ T(t) = Ae^{-\lambda kt}. \]

(39.5)

Meanwhile, Equation (39.4a) along with the boundary conditions (39.1b) is the boundary value problem

\[
\begin{align*}
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X(l) &= 0.
\end{align*}
\]

(39.6)

This should look familiar: it’s very similar to Example 37.6. As in that example, the only eigenvalues for the boundary value problem (39.6) will be positive. Letting \( \lambda = \beta^2 \), the general solution is

\[ X(x) = B \cos(\beta x) + C \sin(\beta x). \]
The first boundary condition says $B = 0$. The second condition says that $X(l) = C \sin(\beta l) = 0$. To avoid having only the trivial solution, we must have $\beta l = n\pi$. In other words,

$$
\lambda_n = \left( \frac{n\pi}{l} \right),
$$

$$
X_n(x) = \sin \left( \frac{n\pi x}{l} \right)
$$

for $n = 1, 2, 3, \ldots$.

So we have found an infinite number of solutions to the heat equation boundary value problem given by (39.1a) and (39.1b), one for each positive integer $n$. They are

$$
u_n(x, t) = A_n e^{(n\pi/l)^2 kt} \sin \left( \frac{n\pi x}{l} \right).
$$

The heat equation is linear and homogeneous. As such, the Principle of Superposition still holds: any linear combination of solutions is still a solution. So any function of the form

$$
u(x, t) = \sum_{n=1}^{N} A_n e^{(n\pi/l)^2 kt} \sin \left( \frac{n\pi x}{l} \right)
$$

is also a solution to (39.1a) and (39.1b).

Notice that we haven’t used the initial condition (39.1c) yet, which is why we referred to (39.7) as a solution to only the boundary value problem. How does the initial data factor into the solution?

We have

$$f(x) = u(x, 0) = \sum_{n=1}^{N} A_n \sin \left( \frac{n\pi x}{l} \right).
$$

(39.8)

So if the initial condition (39.1c) has this form, the solution to the initial value problem will be given by (39.7), with the coefficients $A_n$ coming from the initial condition as in Equation (39.8).

Example 39.1. Find the solution to the following heat equation problem on a rod of length 2.

$$
u_t = \nu_{xx}
$$

$$u(0, t) = u(2, t) = 0
$$

$$u(x, 0) = \sin \left( \frac{3\pi x}{2} \right) - 5 \sin(3\pi x)
$$

In this problem, $k = 1$. We know our solution will have the form (39.7). We just need to figure out which index values of $n$ are represented and what the corresponding coefficients $A_n$ are.

This isn’t too hard to do in this case. Our initial condition is

$$f(x) = \sin \left( \frac{3\pi x}{2} \right) - 5 \sin(3\pi x).
$$

Looking at (39.7) with $l = 2$, we can see that the first term corresponds to $n = 3$ and the second $n = 6$, and that these are the only terms. Thus we have $A_3 = 1$, $A_6 = -5$, and all other $A_n = 0$. Our solution is then

$$u(x, t) = e^{-9\pi^2 t/4} \sin \left( \frac{3\pi x}{2} \right) - 5e^{-9\pi^2 t} \sin(3\pi x).
$$

□

It would not be very satisfying if we could only solve problems where the initial conditions were a finite combination of sine terms, as in Example 39.1. There’s no reason to suppose that an initial distribution would be represented by such an expression. What do we do if our initial distribution has a more general form?
Let’s consider what happens if we take an infinite sum of separated solutions. Then the solution to the boundary value problem (39.1a) and (39.1b) is given by

\[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{(n\pi/l)^2kt} \sin \left( \frac{n\pi x}{l} \right). \]

Now the initial condition specifies that the coefficients must satisfy

\[ f(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right). \]

This idea of expressing a general function as an infinite linear combination of wave functions is due to the French mathematician Joseph Fourier, and (39.9) is called the Fourier sine series for \( f(x) \).

There are several important questions raised by the idea of a Fourier series, however. Why should we believe that the initial distribution \( f(x) \) ought to be able to be written as an infinite sum of sines? Why should such a sum have to converge to anything reasonable? We’ll address these concerns over the next few lectures, but keep them in mind.

**Neumann boundary conditions.** Now let’s consider a heat equation problem with homogeneous Neumann conditions.

\[
\begin{aligned}
(\text{DE}) & : \quad u_t = ku_{xx} \\
(\text{BC}) & : \quad u_x(0, t) = u_x(l, t) = 0 \\
(\text{IC}) & : \quad u(x, 0) = f(x).
\end{aligned}
\]

We’ll start by again supposing that we have a separated solution \( u(x, t) = X(x)T(t) \), and as the PDE (39.10a) is identical to the Dirichlet boundary conditions case, we obtain the same pair of ODEs (39.4b) and (39.4a). The solution to Equation (39.4b) is still \( T(t) = Ae^{\lambda kt} \).

Now we need to determine the boundary conditions for Equation (39.4a). The Neumann boundary conditions (39.10b) are \( u_x(0, t) = u_x(l, t) = 0 \); thus they are conditions on \( X'(0) \) and \( X'(l) \), as the \( t \)-derivative isn’t controlled at all. So the boundary value problem becomes

\[
\begin{aligned}
X'' + \lambda X &= 0 \\
X'(0) &= 0 \\
X'(l) &= 0.
\end{aligned}
\]

Along the lines of the analogous computation from Example 37.6, the eigenvalues and eigenfunctions of the boundary value problem (39.11) are

\[
\lambda_n = \left( \frac{n\pi}{l} \right)^2 \\
X_n(x) = \cos \left( \frac{n\pi x}{l} \right)
\]

Fourier (1768–1830) was a big promoter of the French Revolution, and traveled with Napoleon to Egypt, where he was made governor of Lower Egypt. Fourier has been credited with being one of the first to understand the greenhouse effect and, more generally, planetary energy balance.

There’s a great (though almost certainly apocryphal) story about what led Fourier to study the heat equation. Reportedly, what deeply concerned Fourier was being able to find the ideal depth to build his wine cellar so that the wine would be stored at the perfect temperature year-round, and so he proceeded to try to understand the way heat propagated through the ground.
for \( n = 0, 1, 2, 3, \ldots \) (this time \( \lambda = 0 \) is an eigenvalue). So individual separable solutions to (39.10a) and (39.10b) have the form

\[
 u_n(x, t) = A_n e^{(n\pi/l)^2kt} \cos \left( \frac{n\pi x}{l} \right).
\]

Taking finite linear combinations of these solutions works similarly to the Dirichlet case (and is the solution to (39.10) when \( f(x) \) is a finite linear combination of constants and cosines, in direct analogy to Example 39.1), but in general we are interested in knowing when we can take infinite sums, i.e.

\[
 u(x, t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{(n\pi/l)^2kt} \cos \left( \frac{n\pi x}{l} \right).
\]

Notice how we wrote the \( n = 0 \) case as \( \frac{1}{2}A_0 \); the reason for this will be made clear in the future.

The initial condition (39.10c) means that we require

\[
 f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{l} \right). \quad (39.12)
\]

An expression of the form (39.12) is called the Fourier cosine series of \( f(x) \).

**Other boundary conditions.** It’s also possible for certain boundary conditions to require the full Fourier series, or just the Fourier series of the initial data; this is an expression of the form

\[
 f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n\pi x}{l} \right) + B_n \sin \left( \frac{n\pi x}{l} \right) \right), \quad (39.13)
\]

but for most of our purposes, we will be interested in solving problems with Dirichlet or Neumann boundary conditions. However, in the process of learning about Fourier sine and cosine series, we’ll also learn how to compute the full Fourier series of a function.
Fourier Series Coefficients

There are two important issues that we need to discuss before we can feel free to use Fourier sine and cosine series to solve heat equation initial value problems of the forms (39.1) and (39.10).

- For what functions $f(x)$ is it possible to find coefficients for series of the form (39.9) and (39.12)?
- For which functions $f(x)$ will the Fourier series be convergent, if any? And what exactly will they converge to?

We’ll leave the issue of converge for a few lectures down the road. First, we’ll learn how to find the Fourier series coefficients for appropriate functions $f(x)$. Fortunately, there is a very beautiful and conceptually elegant formula for the Fourier coefficients, called the Euler-Fourier formula.

1. Fourier sine series

We start off with the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right).$$

The key observation is that sine functions have the property that

$$\int_{0}^{l} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi x}{l} \right) \, dx = 0 \quad (40.1)$$

if $m \neq n$ are both positive integers. This can be seen by direct integration. We start by recalling the trig identity

$$\sin(a) \sin(b) = \frac{1}{2} \cos(a - b) - \frac{1}{2} \cos(a + b).$$

Then the integral on the left-hand side of Equation (40.1) evaluates to

$$\left. \frac{1}{2(m-n)\pi} \sin \left( \frac{(m-n)\pi x}{l} \right) \right|_{0}^{l} = \left. \frac{1}{2(m-n)\pi} \sin \left( \frac{(m+n)\pi x}{l} \right) \right|_{0}^{l}$$

so long as $m \neq n$. But these terms are just linear combinations of $\sin((m \pm n)\pi)$ and $\sin(0)$ and so everything vanishes.

Now, let’s fix $m$ and multiply Equation (39.9) by $\sin(m\pi x/l)$. Integrating term-by-term\(^1\), we get

$$\int_{0}^{l} f(x) \sin \left( \frac{m\pi x}{l} \right) \, dx = \int_{0}^{l} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi x}{l} \right) \, dx$$

$$= \sum_{n=1}^{\infty} A_n \int_{0}^{l} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi x}{l} \right) \, dx.$$ 

Due to Equation (40.1), all but one of these terms (with $n = m$) vanishes. So we are left with

$$\int_{0}^{l} f(x) \sin \left( \frac{m\pi x}{l} \right) \, dx = A_m \int_{0}^{l} \sin^2 \left( \frac{m\pi x}{l} \right) \, dx = \frac{1}{2} l A_m.$$

---

\(^1\)I know, I know...we can’t always integrate infinite series term-by-term. We can in this case, however.
and so

\[
A_m = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{m\pi x}{l} \right) \, dx.
\]  

(40.2)

To summarize what we’ve computed: if \( f(x) \) has a Fourier sine expansion (39.9), the coefficients must be given by Equation (40.2). These are the only possible coefficients for such a series, but we have not yet shown that (39.9) is a valid expression for \( f(x) \). We’ll take that for granted in the following examples.

**Example 40.1.** Compute the Fourier sine series for \( f(x) = 1 \) on \( 0 \leq x \leq l \).

By Equation (40.2), the coefficients must be given by

\[
A_m = \frac{2}{l} \int_0^l \sin \left( \frac{m\pi x}{l} \right) \, dx
= - \left[ \frac{2}{m\pi} \cos \left( \frac{m\pi x}{l} \right) \right]_0^l
= \frac{2}{m\pi} (1 - \cos(m\pi))
= \frac{2}{m\pi} (1 - (-1)^m).
\]

So we have \( A_m = 4/m\pi \) if \( m \) is odd and \( A_m = 0 \) if \( m \) is even. Thus on \( (0,l) \),

\[
1 = \frac{4}{\pi} \left( \sin \left( \frac{\pi x}{l} \right) + \frac{1}{3} \sin \left( \frac{3\pi x}{l} \right) + \frac{1}{5} \sin \left( \frac{5\pi x}{l} \right) + \ldots \right)
= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \left( \frac{(2n-1)\pi x}{l} \right).
\]

□

**Example 40.2.** Compute the Fourier sine series for \( f(x) = x \) on \( 0 \leq x \leq l \).

In this case, Equation (40.2) tells us that the coefficients must be given by

\[
A_m = \frac{2}{l} \int_0^l x \sin \left( \frac{m\pi x}{l} \right) \, dx
= - \left[ - \frac{2x}{m\pi} \cos \left( \frac{m\pi x}{l} \right) + \frac{2l}{m^2\pi^2} \sin \left( \frac{m\pi x}{l} \right) \right]_0^l
= - \frac{2l}{m\pi} \cos(m\pi) + \frac{2l}{m^2\pi^2} \sin(m\pi)
= (-1)^{m+1} \frac{2l}{m\pi}.
\]

So on \( (0,l) \),

\[
x = \frac{2l}{\pi} \left( \sin \left( \frac{\pi x}{l} \right) - \frac{1}{2} \sin \left( \frac{2\pi x}{l} \right) + \frac{1}{3} \sin \left( \frac{3\pi x}{l} \right) - \ldots \right)
= \frac{2l}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} \sin \left( \frac{(2n-1)\pi x}{l} \right) - \frac{1}{2n} \sin \left( \frac{2n\pi x}{l} \right) \right).
\]

□

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2. Fourier cosine series

Now let’s consider the Fourier cosine series
\[ f(x) = \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{l} \right). \]

Here we have the fact (analogously derived as for Equation (40.1)) that
\[ \int_0^l \cos \left( \frac{n\pi x}{l} \right) \cos \left( \frac{m\pi x}{l} \right) \, dx = 0 \quad (40.3) \]
if \( n \neq m \).

**Exercise.** Verify Equation (40.3).

By a similar computation as for sines, if \( m \neq 0 \) we get
\[ \int_0^l f(x) \cos \left( \frac{m\pi x}{l} \right) \, dx = A_m \int_0^l \cos^2 \left( \frac{m\pi x}{l} \right) \, dx = \frac{1}{2} l A_m. \]

If \( m = 0 \),
\[ \int_0^l f(x) \cdot 1 \, dx = \frac{1}{2} A_0 \int_0^l 1^2 \, dx = \frac{1}{2} l A_0. \]

Thus, for all \( m \geq 0 \), we have
\[ A_m = \frac{2}{l} \int_0^l f(x) \cos \left( \frac{m\pi x}{l} \right) \, dx. \quad (40.4) \]

This is why we explicitly multiplied \( A_0 \) by \( 1/2 \) in Equation (39.12).

**Example 40.3.** Compute the Fourier cosine series for \( f(x) = 1 \) on \( 0 \leq x \leq l \).

By Equation (40.4), the coefficients for \( m > 0 \) are given by
\[ A_m = \frac{2}{l} \int_0^l \cos \left( \frac{m\pi x}{l} \right) \, dx \]
\[ = \frac{2}{m\pi} \sin \left( \frac{m\pi l}{l} \right) \bigg|_0^l \]
\[ = \frac{2}{m\pi} \sin (m\pi) = 0. \]

If \( m = 0 \),
\[ A_0 = \frac{2}{l} \int_0^l 1 \, dx = 2. \]

So the only non-zero coefficient occurs for the constant coefficient \( A_0 \), and the Fourier cosine series is
\[ 1 = 1 + 0 \cos \left( \frac{\pi x}{l} \right) + 0 \cos \left( \frac{2\pi x}{l} \right) + \ldots \]
This should make sense: the Fourier cosine expansion is unique, and the above sum is obvious.  

Example 40.4. Compute the Fourier cosine series for \( f(x) = x \) on \( 0 \leq x \leq l \).

For \( m > 0 \), Equation (40.4) becomes

\[
A_m = \frac{2}{l} \int_0^l x \cos \left( \frac{m \pi x}{l} \right) \, dx = \left[ \frac{2x}{m \pi} \sin \left( \frac{m \pi x}{l} \right) + \frac{2l}{m^2 \pi^2} \cos \left( \frac{m \pi x}{l} \right) \right]_0^l
\]

\[
= \frac{2l}{m \pi} \sin(m \pi) + \frac{2l}{m^2 \pi^2} (\cos(m \pi) - 1)
\]

\[
= \frac{2l}{m^2 \pi^2} ((-1)^m - 1)
\]

\[
= \begin{cases} 
-\frac{4l}{m^2 \pi^2} & \text{if } m \text{ odd} \\
0 & \text{if } m \text{ even}.
\end{cases}
\]

If \( m = 0 \),

\[
A_0 = \frac{2}{l} \int_0^l x \, dx = \frac{l}{2}.
\]

So on \((0, l)\), we have the Fourier cosine series

\[
x = \frac{l}{2} - \frac{4l}{\pi^2} \left( \cos \left( \frac{\pi x}{l} \right) + \frac{1}{9} \cos \left( \frac{3\pi x}{l} \right) + \frac{1}{25} \cos \left( \frac{5\pi x}{l} \right) + \ldots \right)
\]

\[
= \frac{l}{2} + \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \left( \frac{(2n-1)\pi x}{l} \right).
\]

\[\square\]

3. Full Fourier series

The full Fourier series, given by Equation (39.13), is defined on the interval \(-l \leq x \leq l\). Be careful: now the interval we’re working on is twice as long!

The computation of coefficients for the series is similar to that for the Fourier sine and cosine series. We need the following set of identities:

\[
\int_{-l}^l \cos \left( \frac{n \pi x}{l} \right) \sin \left( \frac{m \pi x}{l} \right) \, dx = 0 \quad \text{for all } n, m
\]

\[
\int_{-l}^l \cos \left( \frac{n \pi x}{l} \right) \cos \left( \frac{m \pi x}{l} \right) \, dx = 0 \quad \text{for } n \neq m
\]

\[
\int_{-l}^l \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{m \pi x}{l} \right) \, dx = 0 \quad \text{for } n \neq m
\]

\[
\int_{-l}^l 1 \cdot \cos \left( \frac{n \pi x}{l} \right) \, dx = 0 = \int_{-l}^l 1 \cdot \sin \left( \frac{n \pi x}{l} \right) \, dx.
\]

Using the same procedure as for the sine and cosine series, we can derive an expression for the coefficients of the full Fourier series by fixing \( m \) and multiplying by \( \cos(m \pi x/l) \) before doing the same for \( \sin(m \pi x/l) \). So we need to calculate the integrals of the squares

\[
\int_{-l}^l \cos^2 \left( \frac{m \pi x}{l} \right) \, dx = 1 = \int_{-l}^l \sin^2 \left( \frac{m \pi x}{l} \right) \, dx \quad \text{and} \quad \int_{-l}^l 1^2 \, dx = 2l.
\]
EXERCISE. Verify these integrals.

So we get the formulas

\[
A_m = \frac{1}{l} \int_{-l}^{l} f(x) \cos \left( \frac{m \pi x}{l} \right) \, dx \quad \text{(40.5a)}
\]

\[
B_m = \frac{1}{l} \int_{-l}^{l} f(x) \sin \left( \frac{m \pi x}{l} \right) \, dx \quad \text{(40.5b)}
\]

for the coefficients of the full Fourier series. Notice the similarities between Equations (40.5a) and (40.5b) and Equations (40.4) and (40.2).

**Example 40.5.** Compute the full Fourier series of \( f(x) = x + 1 \).

Using Equations (40.5) (and treating \( A_0 \) separately from the general \( A_m \) term, which as we have seen is typical),

\[
A_0 = \frac{1}{l} \int_{-l}^{l} (1 + x) \, dx = 2
\]

\[
A_m = \frac{1}{l} \int_{-l}^{l} (1 + x) \cos \left( \frac{m \pi x}{l} \right) \, dx
\]

\[
= \left[ \frac{1 + x}{m \pi} \sin \left( \frac{m \pi x}{l} \right) + \frac{l}{m^2 \pi^2} \cos \left( \frac{m \pi x}{l} \right) \right]_{-l}^{l}
\]

\[
= \frac{l}{m^2 \pi} (\cos(m \pi) - \cos(-m \pi)) = 0, \quad m > 0
\]

\[
B_m = \frac{1}{l} \int_{-l}^{l} (1 + x) \sin \left( \frac{m \pi x}{l} \right) \, dx
\]

\[
= \left[ -\frac{1 + x}{m \pi} \cos \left( \frac{m \pi x}{l} \right) + \frac{l}{m^2 \pi^2} \sin \left( \frac{m \pi x}{l} \right) \right]_{-l}^{l}
\]

\[
= -\frac{2l}{m \pi} \cos(m \pi) = (-1)^{m+1} \frac{2l}{m \pi}
\]

So the full Fourier series of \( f(x) = x + 1 \) is

\[
1 + x = 1 + \frac{2l}{\pi} \left( \sin \left( \frac{\pi x}{l} \right) - \frac{1}{2} \sin \left( \frac{2 \pi x}{l} \right) + \frac{1}{3} \sin \left( \frac{3 \pi x}{l} \right) - \ldots \right)
\]

\[
= 1 + \frac{2l}{\pi} \sum_{i=1}^{\infty} \left( \frac{1}{2n-1} \sin \left( \frac{(2n-1) \pi x}{l} \right) - \frac{1}{2n} \sin \left( \frac{2n \pi x}{l} \right) \right)
\]

\[\square\]

**Example 40.6.** Compute the Fourier series for \( f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1 - x & -1 \leq x < 2 \end{cases} \) on the interval \((-2, 2)\).

We start by using the Euler-Fourier formulas (40.5). For the cosine terms, (40.5a) gives

\[
A_0 = \frac{1}{2} \int_{-2}^{2} f(x) \, dx
\]

\[
= \frac{1}{2} \left( \int_{-2}^{-1} 2 \, dx + \int_{-1}^{2} 1 - x \, dx \right)
\]

\[
= \frac{1}{2} \left( 2 + \frac{3}{2} \right) = \frac{7}{4}
\]
Differential Equations  

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and

\[ A_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \left( \frac{n\pi x}{2} \right) \, dx \]

\[ = \frac{1}{2} \left( \int_{-2}^{-1} 2 \cos \left( \frac{n\pi x}{2} \right) \, dx + \int_{-1}^{2} (1 - x) \cos \left( \frac{n\pi x}{2} \right) \, dx \right) \]

\[ = \frac{1}{2} \left( \frac{4}{n\pi} \sin \left( \frac{n\pi x}{2} \right) \bigg|_{-2}^{1} + \frac{2(1 - x)}{n\pi} \sin \left( \frac{n\pi x}{2} \right) \bigg|_{-1}^{2} - \frac{4}{n^2\pi^2} \left( \cos \left( \frac{n\pi x}{2} \right) \right)^2 \right) \]

\[ = \frac{1}{2} \left( -\frac{4}{n\pi} \sin \left( \frac{n\pi}{2} \right) + \frac{4}{n\pi} \sin \left( \frac{n\pi}{2} \right) - \frac{4}{n^2\pi^2} \left( \cos \left( \frac{n\pi}{2} \right) - \cos \left( \frac{n\pi}{2} \right) \right) \right) \]

\[ = \begin{cases} 
\frac{2n\pi^2}{n^2\pi^2} & n \text{ odd} \\
0 & n = 4m \\
-\frac{4}{n^2\pi^2} & n = 4m + 2 
\end{cases} \]

Also, (40.5b) gives

\[ B_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \left( \frac{n\pi x}{2} \right) \, dx \]

\[ = \frac{1}{2} \left( \int_{-2}^{-1} 2 \sin \left( \frac{n\pi x}{2} \right) \, dx + \int_{-1}^{2} (1 - x) \sin \left( \frac{n\pi x}{2} \right) \, dx \right) \]

\[ = \frac{1}{2} \left( -\frac{4}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \bigg|_{-2}^{1} - \frac{2(1 - x)}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \bigg|_{-1}^{2} - \frac{4}{n^2\pi^2} \sin \left( \frac{n\pi x}{2} \right) \bigg|_{-1}^{2} \right) \]

\[ = \frac{1}{2} \left( \frac{6}{n\pi} \cos(n\pi) - \frac{4}{n^2\pi^2} \sin \left( \frac{n\pi}{2} \right) \right) \]

\[ = \begin{cases} 
\frac{3n\pi}{n\pi} & n \text{ even} \\
-\frac{3}{n\pi} - \frac{2}{n^2\pi^2} & n = 4m + 1 \\
-\frac{3}{n\pi} + \frac{2}{n^2\pi^2} & n = 4m + 3 
\end{cases} \]

So we have

\[ f(x) = \frac{7}{8} + \sum_{m=1}^{\infty} \frac{2}{(4m + 1)^2\pi^2} \cos \left( \frac{(4m + 1)\pi x}{2} \right) + \left( -\frac{3}{(4m + 1)\pi} - \frac{2}{(4m + 1)^2\pi^2} \right) \sin \left( \frac{(4m + 1)\pi x}{2} \right) \]

\[ - \frac{4}{(4m + 2)^2\pi^2} \cos \left( \frac{(4m + 2)\pi x}{2} \right) + \frac{3}{(4m + 2)\pi} \sin \left( \frac{(4m + 2)\pi x}{2} \right) \]

\[ + \frac{2}{(4m + 3)^2\pi^2} \cos \left( \frac{(4m + 3)\pi x}{2} \right) + \left( -\frac{3}{(4m + 3)\pi} + \frac{2}{(4m + 3)^2\pi^2} \right) \sin \left( \frac{(4m + 3)\pi x}{2} \right) \]

\[ + \frac{3}{4m\pi} \sin \left( \frac{4m\pi x}{2} \right) . \]

\[ \square \]

This example served as sort of a “worst case scenario” example: there are a lot of Fourier coefficients to keep track of. Notice that for each value of \( m \), though, the summand specifies four different Fourier terms, for \( 4m, 4m + 1, 4m + 2, \) and \( 4m + 3 \). This can often happen; in some cases (depending on \( l \)), even more will be required.
LECTURE 41

Convergence of Fourier Series

1. Fourier Series Convergence

So, we know that if a function \( f(x) \) is to have a Fourier series on an appropriate interval, the coefficients have to be given by Equation (40.5b) for a sine series (39.9) on \((0,l)\), Equation (40.5a) for a cosine series (39.12) on \((0,l)\), and (40.5) for a full series (39.13) on \((-l,l)\). But what, if anything, do these series converge to? We’ll think about Fourier convergence with respect to the full Fourier series first, and then see how this relates to the sine and cosine series.

First, we have to require that \( f(x) \) is piecewise smooth. This is stronger than the notion of piecewise continuity that we saw earlier when discussing Laplace transforms; we need to be able to divide up \((-l,l)\) into a finite number of subintervals so that both \( f(x) \) and its derivative \( f'(x) \) are continuous on each interval. Moreover, the only discontinuities we allow for either (at the boundary points of the subintervals) are jump discontinuities.

**Example 41.1.** Any continuous function with continuous derivative on the desired interval is automatically piecewise smooth. \( \square \)

**Example 41.2.** Consider the function from Example 40.6,

\[
f(x) = \begin{cases} 
  2 & -2 \leq x < -1 \\ 
  1 - x & -1 \leq x \leq 2
\end{cases}.
\]

\( f(x) \) is continuous for all \( x \) in \((-2,2)\), but the derivative \( f'(x) \) has a discontinuity at \( x = -1 \). However, this is a jump discontinuity, with \( \lim_{x \to -1^-} f'(x) = 0 \) and \( \lim_{x \to -1^+} = -1 \). Thus \( f(x) \) is piecewise smooth. \( \square \)

The next thing to notice is that, even though we only need \( f(x) \) to be defined on \((-l,l)\) to compute its Fourier series, the Fourier series itself is defined for all \( x \). Moreover, all of the terms in a Fourier series are \( 2l \)-periodic; they’re all either constants or of the form \( \sin \left( \frac{n\pi x}{l} \right) \) or \( \cos \left( \frac{n\pi x}{l} \right) \). Thus, if the Fourier series converges to \( f(x) \) on the interval \((-l,l)\), it will converge to a “shifted” \( f(x) \) on any interval of the form \((-l + 2kl, l + 2kl)\), where \( k \) is some integer. Let’s define a function

\[
f_{\text{per}}(x) = f(x - 2kl), \quad \text{for } -l + 2kl < x < l + 2kl.
\]

\( f_{\text{per}}(x) \) is called the periodic extension of \( f(x) \), as it’s now a piecewise smooth function whose graph on the interval \((-l + 2kl, l + 2kl)\) looks like that \( f(x) \) on \((-l,l)\).

Now we can say what the Fourier series of \( f(x) \) converges to.

**Theorem 41.1 (Fourier Series Convergence).** Suppose \( f(x) \) is piecewise smooth on \((-l,l)\). Then, at \( x = x_0 \), the Fourier series of \( f(x) \) will converge to

- \( f_{\text{per}}(x_0) \) if \( f_{\text{per}} \) is continuous at \( x_0 \) or
- the average of the one-sided limits \( \frac{1}{2} \left[ f_{\text{per}}(x_0^+) + f_{\text{per}}(x_0^-) \right] \) if \( f_{\text{per}} \) has a jump discontinuity at \( x = x_0 \).

\(^1\)If you haven’t noticed by now, I have no reluctance whatsoever about ending sentences with prepositions.
Theorem 41.1 tells us that, in particular, on the interval \((-l, l)\) the Fourier series will converge to the original function \(f(x)\) except at any jump discontinuities of \(f(x)\), where it will converge “halfway”. We may have similar issues at any points \(x = l + 2kl\) if the periodic extension of \(f\) has a jump discontinuity (i.e. if \(f(-l) \neq f(l)\)).

**Example 41.3.** What does the Fourier series of \(f(x) = \begin{cases} 1 & -3 \leq x < 0 \\ 2x & 0 < x \leq 3 \end{cases}\) converge to at \(x = -2, x = 0, x = 3, x = 5, \text{ and } x = 6?\)

The first two points are inside the original interval of definition of \(f(x)\), so we can just directly consider that instead of having to consider \(f_{\text{per}}(x)\). The only discontinuity of \(f(x)\) occurs at \(x = 0\). So at \(x = -2\), \(f(x)\) is nice and continuous, so the Fourier series will converge to \(f(-2) = 1\). On the other hand, \(f(x)\) is a jump discontinuity, so the Fourier series will converge to the average of the one-sided limits. \(f(0^+) = \lim_{x \to 0^+} f(x) = 0\) while \(f(0^-) = \lim_{x \to 0^-} f(x) = 1\), so the Fourier series will converge to \(\frac{1}{2} [f(0^+) + f(0^-)] = \frac{1}{2}\).

What happens at the other three points? Here we have to consider \(f_{\text{per}}(x)\) and where it has jump discontinuities. These can only occur either at \(x = x_0 + 2kl\) where \(-l < x_0 < l\) is a jump discontinuity of \(f(x)\) or at “boundary points” \(x = \pm l + 2kl\), since the periodic extension might not “sync up” at these points, producing a jump discontinuity.

We have

\[
f_{\text{per}}(x) = f(x - 6k),
\]

where \(k\) is given by \(-2 + 4k < x < 2 + 4k\). So at \(x = 3\), we’re at one of these “boundary points,” and the left-sided limit is 6 while the right-sided limit is 1. Thus the Fourier series will converge here to \(\frac{7}{2}\). \(x = 5\), on the other hand, is a point of continuity for \(f_{\text{per}}(x)\), and so the Fourier series will converge to \(f_{\text{per}}(5) = f(-1) = 1\). \(x = 6\), though, is a jump discontinuity (corresponding to \(x = 0\)), and so the Fourier series will converge to \(\frac{1}{2}\).

The convergence of the Fourier series for this example can be visualized in Figure 41.1.

**Example 41.4.** Where does the Fourier series for \(f(x) = \begin{cases} 2 & -2 \leq x < -1 \\ 1 - x & -1 \leq x \leq 2 \end{cases}\) converge at \(x = -7, x = -1, \text{ and } x = 6?\)

Our first observation is that there are no points inside \((-2, 2)\) where \(f(x)\) is discontinuous. Thus the only points where the periodic extension might be discontinuous are the “boundary points” \(x = \pm 2 + 4k\). In fact, as \(f(-2) \neq f(2)\), these will be points of discontinuity. So, \(f_{\text{per}}(x)\) is continuous at \(x = -7\), as it’s not one of these boundary points and we have \(f_{\text{per}}(-7) = f(1) = 0\), which is what the Fourier series will converge to. The same goes for \(x = -1\): the Fourier series will converge to \(f(-1) = 2\).

On the other hand, \(x = 6\) is one of these “boundary” jump discontinuities. The left-sided limit is -1, while the right-sided limit is 2, so the Fourier series will converge to their average, \(\frac{1}{2}\).

The convergence of the Fourier series for this example can be visualized in Figure 41.2.

**2. Sine and Cosine Series**

Before we can apply the discussion from Section 1 to Fourier sine and cosine series, we need to review some facts about even and odd functions.

**2.1. Even and Odd Functions.** Recall that an even function is a function satisfying

\[
g(-x) = g(x).
\]

This means that the graph of \(y = g(x)\) is symmetric with respect to the \(y\)-axis. An odd function satisfies

\[
g(-x) = -g(x),
\]
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**Figure 41.1.** Fourier series convergence for Example 41.3. The black curve is the periodic extension of $f(x)$.

**Figure 41.2.** Fourier series convergence for Example 41.4. The black curve is the periodic extension of $f(x)$. 
so the graph of \( y = g(x) \) is symmetric with respect to the origin.

**Example 41.5.** A monomial \( x^n \) is even if \( n \) is even and odd if \( n \) is odd. \( \cos(x) \) is even while \( \sin(x) \) is odd. \( \tan(x) \) is odd. \( \square \)

There are some rules for how products and sums of even and odd functions behave (none are especially difficult, just keep track of how many negatives pop out of the arguments and cancel).

- If \( g(x) \) is odd and \( h(x) \) is even, their product \( g(x)h(x) \) is odd.
- If \( g(x) \) and \( h(x) \) are both even or both odd, their product \( g(x)h(x) \) is even.
- The sums of two even or two odd functions is again even or odd, respectively.
- The sum of an even and an odd function can be anything, including neither even nor odd. In fact, any function on \((-l, l)\) can be written as the sum of an even function, called the even part, and an odd function, called the odd part.
- Differentiation and integration change the parity of a function. That is, if \( f(x) \) is even, \( df/dx \) and \( \int_0^x f(s) ds \) are both odd, and vice versa.

The graph of an odd function \( g(x) \) must pass through the origin by definition. This implies that if \( g(x) \) is even and if \( g'(0) \) exists, \( g'(0) = 0 \).

Definite integrals on symmetric intervals of odd functions and even functions have very useful properties.

\[
\int_{-l}^{l} (\text{odd}) \, dx = 0 \quad \text{and} \quad \int_{-l}^{l} (\text{even}) \, dx = 2 \int_{0}^{l} (\text{even}) \, dx \quad (41.1)
\]

Given a function \( f(x) \) defined on \((0, l)\), there is only one way to extend it to \((-l, l)\) as either an even or an odd function. The *even extension* of \( f(x) \) is

\[
f_{\text{even}}(x) = \begin{cases} 
  f(x) & \text{for } 0 < x < l \\
  f(-x) & \text{for } -l < x < 0
\end{cases} \quad (41.2)
\]

This is just the reflection of \( f(x) \) across the \( x \)-axis. Notice that the even extension is not necessarily defined at the origin, as it may be impossible to define \( f_{\text{even}}(x) \) in such a way that \( f'(0) = 0 \).

The *odd extension* of \( f(x) \) is

\[
f_{\text{odd}}(x) = \begin{cases} 
  f(x) & \text{for } 0 < x < l \\
  -f(-x) & \text{for } -l < x < 0 \\
  0 & \text{for } x = 0
\end{cases} \quad (41.3)
\]

This is just the reflection of \( f(x) \) through the origin.

### 3. Fourier Sine Series

Each of the terms in the Fourier sine series (39.9) for \( f(x) \) is odd. As with the full Fourier series, each of these terms also has period 2\( l \). So we can think of the Fourier sine series as the expansion of an odd function with period 2\( l \) defined on the entire line which coincides with \( f(x) \) on \((0, l)\).

In fact, (41.1) can be used to show that the full Fourier series of \( f_{\text{odd}}(x) \) is precisely the same as the Fourier sine series of \( f(x) \). Let

\[
\frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n\pi x}{l} \right) + B_n \sin \left( \frac{n\pi x}{l} \right) \right]
\]

be the Fourier series for \( f_{\text{odd}}(x) \). The coefficients \( A_n \) are given by

\[
A_n = \frac{1}{l} \int_{-l}^{l} f_{\text{odd}}(x) \cos \left( \frac{n\pi x}{l} \right) \, dx.
\]
But \( f_{\text{odd}}(x) \) is odd and \( \cos \) is even, so their product is odd and \( A_n = 0 \). Meanwhile, the coefficients \( B_n \) are given by

\[
B_n = \frac{2}{l} \int_0^l f_{\text{odd}}(x) \sin \left( \frac{n\pi x}{l} \right) \, dx.
\]

As \( \sin \) is even, the product in the integrand is even, so

\[
B_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx,
\]

which are the Fourier sine coefficients of \( f(x) \). Thus, as the Fourier sine series of \( f(x) \) is just the full Fourier series of \( f_{\text{odd}}(x) \), the \( 2l \)-periodic odd function that the Fourier sine series expands is the periodic extension of \( f_{\text{odd}}(x) \).

This goes both ways, of course. If we want to compute a Fourier series for an odd function on \((0, l)\), we can just compute the Fourier sine series of the function restricted to \((0, l)\); it will almost converge to the original function on \((l, l)\), with the only issues occurring at any jump discontinuities. This only works for odd functions, however. Do not use the formula for the coefficients of the sine series unless you’re working with an odd function.

**Example 41.6.** Work down the odd extension of \( f(x) = l - x \) on \((0, l)\) and compute its Fourier series.

To get the odd extension of \( f(x) \), we’ll need to reflect it across the origin. We end up with

\[
f_{\text{odd}}(x) = \begin{cases} 
  l - x & 0 < x < l \\
  -l - x & -l < x < 0 \\
  0 & x = 0.
\end{cases}
\]

What is the Fourier series of \( f_{\text{odd}}(x) \)? From the previous discussion, we know it will be equivalent to the Fourier sine series of \( f(x) \), as this will converge on \((-l, 0)\) to \( f_{\text{odd}}(x) \). So

\[
f_{\text{odd}}(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right),
\]

where

\[
A_n = \frac{2}{l} \int_0^l (l - x) \sin \left( \frac{n\pi x}{l} \right) \, dx = \frac{2}{l} \left[ -\frac{l(l-x)}{n\pi} \cos \left( \frac{n\pi x}{l} \right) - \frac{l^2}{n^2\pi^2} \sin \left( \frac{n\pi x}{l} \right) \right]_0^l = \frac{2l}{n\pi}.
\]

Thus the desired Fourier series is

\[
f_{\text{odd}}(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x}{l} \right).
\]

You might be asking right now how we were able, a few lectures ago, to compute the Fourier sine series of a constant function like \( f(x) = 1 \), which is even. However, it’s important to remember that if we were computing the Fourier sine series for \( f(x) \), it only needs to converge to \( f(x) \) on \((0, l)\), where issues of evenness and oddness don’t occur, and the Fourier sine series will converge to the odd extension of \( f(x) \) on \((l, l)\). Let’s do another example.
**Example 41.7.** Find the Fourier series for the odd extension of

\[
f(x) = \begin{cases} 
3/2 & 0 < x < 3/2 \\
x - 3/2 & 3/2 < x < 3 
\end{cases}
\]

on \((-3, 3)\).

The Fourier series for \(f_{\text{odd}}(x)\) on \((-3, 3)\) will be just the Fourier sine series for \(f(x)\) on \((0, 3)\). The Fourier sine coefficients for \(f(x)\) are

\[
A_n = \frac{2}{3} \int_0^3 f(x) \sin \left(\frac{n\pi x}{3}\right) \, dx
\]

\[
= \frac{2}{3} \left( \int_0^{3/2} \frac{3}{2} \sin \left(\frac{n\pi x}{3}\right) \, dx + \int_{3/2}^3 \left(x - \frac{3}{2}\right) \sin \left(\frac{n\pi x}{3}\right) \, dx \right)
\]

\[
= \frac{2}{3} \left( -\frac{9}{2n\pi} \cos \left(\frac{n\pi x}{2}\right) \bigg|_0^{3/2} + \frac{3(x - 3/2)}{n\pi} \cos \left(\frac{n\pi x}{3}\right) \bigg|_{3/2}^3 + \frac{9}{n^2\pi^2} \sin \left(\frac{n\pi x}{3}\right) \bigg|_{3/2}^3 \right)
\]

\[
= \frac{2}{3} \left( -\frac{9}{2n\pi} \left( \cos \left(\frac{n\pi}{2}\right) - 1 \right) - \frac{9}{2n\pi} \cos(n\pi) - \frac{9}{n^2\pi^2} \sin \left(\frac{n\pi}{2}\right) \right)
\]

\[
= \frac{3}{n\pi} \left( 1 - \cos \left(\frac{n\pi}{2}\right) + (-1)^{n+1} - \frac{2}{n\pi} \sin \left(\frac{n\pi}{2}\right) \right),
\]

and the Fourier series is

\[
f_{\text{odd}}(x) = \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[ 1 - \cos \left(\frac{n\pi}{2}\right) + (-1)^{n+1} - \frac{2}{n\pi} \sin \left(\frac{n\pi}{2}\right) \right] \sin \left(\frac{n\pi x}{3}\right).
\]

\[\Box\]

**Exercise.** Sketch the odd extension of \(f(x)\) given in Example 41.7 and write down its formula.

### 4. Fourier Cosine Series

Now, let’s consider what happens for the Fourier cosine series of \(f(x)\) on \((0, l)\). This is directly analogous to the sine series case. Every term in the cosine series has the form

\[
A_n \cos \left(\frac{n\pi x}{l}\right)
\]

and hence is even, and so the entire cosine series is even. So the cosine series must converge on \((l, l)\) to an even function which coincides on \((0, l)\) with \(f(x)\). This must be the even extension given by Equation (41.2).

It’s straightforward enough to show that the Fourier coefficients of \(f_{\text{even}}(x)\) coincide with the Fourier cosine coefficients of \(f(x)\). The Euler-Fourier formulas give for the cosine coefficients

\[
A_n = \frac{1}{l} \int_{-l}^{l} f_{\text{even}}(x) \cos \left(\frac{n\pi x}{l}\right) \, dx
\]

\[
= \frac{2}{l} \int_{0}^{l} f_{\text{even}}(x) \cos \left(\frac{n\pi x}{l}\right) \, dx \quad \text{as } f_{\text{even}}(x) \cos \left(\frac{n\pi x}{l}\right) \text{ is even}
\]

\[
= \frac{2}{l} \int_{0}^{l} f(x) \cos \left(\frac{n\pi x}{l}\right) \, dx,
\]

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and for the sine coefficients,
\[
B_n = \frac{1}{l} \int_{-l}^{l} f_{\text{even}}(x) \sin \left( \frac{n\pi x}{l} \right) \, dx = 0
\]
as \( f_{\text{even}}(x) \sin \left( \frac{n\pi x}{l} \right) \) is odd.

Thus the Fourier cosine series of \( f(x) \) on \((0, l)\) can be considered as the Fourier expansion of \( f_{\text{even}}(x) \) on \((-l, l)\) and therefore also as an expansion of the periodic extension of \( f_{\text{even}}(x) \). It will converge as per Theorem 41.1 to this periodic extension.

This also means that if we want to compute the Fourier series of an even function, we can just compute the Fourier cosine series of its restriction to \((0, l)\). It’s very important, however, that this only be attempted if the function were starting with is even.

**Example 41.8.** Write down the even extension of \( f(x) = lx \) on \((0, l)\) and compute its Fourier series.

The even extension, by (41.2), will be
\[
f_{\text{even}}(x) = \begin{cases} 
  l - x & 0 < x < l \\
  l + x & -l < x < 0
\end{cases}
\]

Its Fourier series is the same as the Fourier cosine series of \( f(x) \) by the previous discussion, so we can just compute those coefficients.

\[
f_{\text{even}}(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{l} \right),
\]

where
\[
A_0 = \frac{2}{l} \int_{0}^{l} (l - x) \, dx = l
\]
\[
A_n = \frac{2}{l} \int_{0}^{l} (l - x) \cos \left( \frac{n\pi x}{l} \right) \, dx
\]
\[
= \frac{2}{l} \left[ \frac{l(l - x)}{n\pi} \sin \left( \frac{n\pi x}{l} \right) - \frac{l^2}{n^2\pi^2} \cos \left( \frac{n\pi x}{l} \right) \right]_0^l
\]
\[
= \frac{2}{l} \left( \frac{l^2}{n^2\pi^2} (-\cos(n\pi) + \cos(0)) \right)
\]
\[
= \frac{2l}{n^2\pi^2} \left( (-1)^{n+1} + 1 \right).
\]

Thus the desired Fourier series is
\[
f_{\text{odd}}(x) = \frac{l}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( (-1)^{n+1} + 1 \right).
\]

**Example 41.9.** Find the Fourier series for the even extension of
\[
f(x) = \begin{cases} 
  3/2 & 0 < x < 3/2 \\
  x - 3/2 & 3/2 < x < 3
\end{cases}
\]
on \((-3, 3)\).
Using (41.2), we see that the even extension is
\[
f_{\text{even}}(x) = \begin{cases} 
  x - 3/2 & 3/2 < x < 3 \\
  3/2 & 0 \leq x \leq 3/2 \\
  -3/2 < x < 0 \\
  -x - 3/2 & -3 < x \leq -3/2.
\end{cases}
\]

We just need to compute the Fourier cosine coefficients of the original \( f(x) \) on \((0, 3)\).

\[
A_0 = \frac{2}{3} \int_0^3 f(x) \, dx
\]
\[
= \frac{2}{3} \left( \int_0^{3/2} \frac{3}{2} \, dx + \int_{3/2}^3 \left( x - \frac{3}{2} \right) \, dx \right)
= \frac{2}{3} \left( \frac{9}{4} + \frac{9}{8} \right) = \frac{9}{4}
\]

\[
A_n = \frac{2}{3} \int_0^3 f(x) \cos \left( \frac{n\pi x}{3} \right) \, dx
\]
\[
= \frac{2}{3} \left( \int_0^{3/2} \frac{3}{2} \cos \left( \frac{n\pi x}{3} \right) \, dx + \int_{3/2}^3 \left( x - \frac{3}{2} \right) \cos \left( \frac{n\pi x}{3} \right) \, dx \right)
\]
\[
= \frac{2}{3} \left( \frac{9}{2n\pi} \sin \left( \frac{n\pi x}{3} \right) \bigg|_0^{3/2} + \frac{3(x - 3/2)}{n\pi} \sin \left( \frac{n\pi x}{3} \right) \bigg|_{3/2}^3 + \frac{9}{n^2\pi^2} \cos \left( \frac{n\pi x}{3} \right) \bigg|_{3/2}^3 \right)
\]
\[
= \frac{2}{3} \left( \frac{9}{2n\pi} \sin \left( \frac{n\pi}{2} \right) + \frac{9}{n^2\pi^2} \left( \cos(n\pi) - \cos \left( \frac{n\pi}{2} \right) \right) \right)
\]
\[
= \frac{6}{n\pi} \left( \frac{1}{2} \sin \left( \frac{n\pi}{2} \right) + \frac{1}{n\pi} \left( (-1)^n - \cos \left( \frac{n\pi}{2} \right) \right) \right).
\]

So the Fourier series is
\[
f_{\text{even}}(x) = \frac{9}{8} + \frac{6}{\pi} \sum_{n=1}^\infty \frac{1}{n} \left( \frac{1}{2} \sin \left( \frac{n\pi}{2} \right) + \frac{1}{n\pi} \left( (-1)^n - \cos \left( \frac{n\pi}{2} \right) \right) \right) \cos \left( \frac{n\pi x}{3} \right).
\]

□
Heat Equation Problems

1. Review and Summary

In Lecture 38, we saw that separable solutions to the heat equation with homogeneous Dirichlet boundary conditions
\[ u_t = ku_{xx} \quad 0 < x < l, t > 0 \]
\[ u(0, t) = u(l, t) = 0 \]
\[ u(x, 0) = f(x) \quad (42.1) \]
had the form
\[ u_n(x, t) = B_n e^{-((n\pi/l)^2)kt} \sin \left( \frac{n\pi x}{l} \right) \quad n = 1, 2, 3, \ldots. \]
Taking linear combinations (over \( n \)) gives a general solution to (42.1),
\[ u(x, t) = \sum_{n=1}^{\infty} B_n e^{-((n\pi/l)^2)kt} \sin \left( \frac{n\pi x}{l} \right). \quad (42.2) \]
Setting \( t = 0 \), this implies that we must have
\[ f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{l} \right). \]
In other words, the coefficients of the general solution (42.2) are the Fourier sine coefficients for the initial data \( f(x) \) on \((0, l)\), which are given by the Euler-Fourier formula for sine series,
\[ B_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx. \quad (42.3) \]

We also saw that if instead we have a heat equation problem with homogeneous Neumann boundary conditions
\[ u_t = ku_{xx} \quad 0 < x < l, t > 0 \]
\[ u_x(0, t) = u_x(l, t) = 0 \]
\[ u(x, 0) = f(x), \quad (42.4) \]
separable solutions had the form
\[ u_n(x, t) = \frac{1}{2} A_0 + A_n e^{-(n\pi/l)^2kt} \cos \left( \frac{n\pi x}{l} \right) \quad n = 0, 1, 2, \ldots, \]
and the general solution to (42.4) was
\[ u(x, t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2kt} \cos \left( \frac{n\pi x}{l} \right). \quad (42.5) \]
With \( t = 0 \), the coefficients must satisfy
\[ f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{l} \right), \]
and so the coefficients in Equation (42.5) are the Fourier cosine coefficients of \( f(x) \), given by the Euler-Fourier formula for cosine series,

\[
A_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \left( \frac{n\pi x}{l} \right) \, dx.
\] (42.6)

One way of thinking about this difference is that, given the initial data \( u(x,0) = f(x) \), the Dirichlet conditions in (42.1) specify the odd extension of \( f(x) \) as the basis for the required periodic solution, while the Neumann conditions in (42.4) specify the even extension. This should make some sense: as we’ve discussed, odd functions must have \( f(0) = 0 \), while even functions must have \( f'(0) = 0 \) if it exists.

So, to solve a homogeneous heat equation problem, we begin by identifying the type of boundary conditions we have. If we have Dirichlet conditions, we know our solution will have the form (42.2), and all we have to do is compute the Fourier sine coefficients of \( f(x) \), (42.3). Similarly, if we have Neumann conditions, we know the solution has the form (42.5) and we have to compute the Fourier cosine coefficients of \( f(x) \), (42.6).

**Remark.** Observe that for any homogeneous Dirichlet problem (42.1), the temperature distribution (42.2) will go to 0 as \( t \to \infty \). This should make sense: these boundary conditions have a physical interpretation where we keep the ends of our rod at freezing without regulating the heat flow in or out of the endpoints. As a result, if the interior of the rod is initially above freezing, that heat will radiate towards the endpoints and into our reservoirs at the endpoints. On the other hand, if the interior of the rod is below freezing, heat will come from the reservoirs at the endpoints and warm it up until the temperature is all uniform.

For the Neumann problem (42.4), the temperature distribution (42.5) will converge to \( \frac{1}{2} A_0 \). Again, this should make some physical sense: these boundary conditions correspond to a situation where we have insulated ends, as we’re preventing any heat from leaving the bar. Thus all the heat energy will move around inside the rod until the temperature is uniform.

### 2. Examples

**Example 42.1.** Solve the initial value problem

\[
\begin{align*}
  u_t &= 3u_{xx} & 0 < x < 2, \, t > 0 \\
  u(0, t) &= u(2, t) = 0 \\
  u(x, 0) &= 20.
\end{align*}
\]

This problem has homogeneous Dirichlet boundary conditions, so the general solution is given by Equation (42.2), or

\[
  u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\left(n\pi/2\right)^2 t} \sin \left( \frac{n\pi x}{l} \right).
\]

The coefficients are the Fourier sine coefficients of \( u(x,0) = 20 \), so we have

\[
B_n = \frac{2}{2} \int_{0}^{2} 20 \sin \left( \frac{n\pi x}{2} \right) \, dx
= \left[ \frac{40}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \right]_{0}^{2}
= -\frac{40}{n\pi} \left( \cos(n\pi) - \cos(0) \right)
= \frac{40}{n\pi} \left( 1 + (-1)^{n+1} \right)
\]

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and the solution to the problem is
\[ u(x, t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} e^{-(3n^2\pi^2t)/4} \sin \left( \frac{n\pi x}{2} \right). \]

**Example 42.2.** Solve the initial value problem
\[ u_t = 3u_{xx} \quad 0 < x < 2, t > 0 \]
\[ u_x(0, t) = u_x(2, t) = 0 \]
\[ u(x, 0) = 3x. \]

This problem has homogeneous Neumann boundary conditions, so the general solution is given by Equation (42.5), or
\[ u(x, t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2kt} \cos \left( \frac{n\pi x}{l} \right). \]

The coefficients are the Fourier sine coefficients of \( u(x, 0) = 3x \), so we have
\[ A_0 = \frac{2}{2} \int_0^2 3x \, dx = 6 \]
\[ A_n = \frac{2}{2} \int_0^2 3x \cos \left( \frac{n\pi x}{2} \right) \, dx \]
\[ = \left[ -\frac{6x}{n\pi} \cos \left( \frac{n\pi x}{2} \right) + \frac{12}{n^2\pi^2} \sin \left( \frac{n\pi x}{2} \right) \right]_0^2 \]
\[ = -\frac{12}{n\pi} \cos(n\pi) \]
\[ = \frac{12}{n\pi} (-1)^{n+1}, \]
and the solution to the problem is
\[ u(x, t) = 3 + \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-(3n^2\pi^2t)/4} \cos \left( \frac{n\pi x}{2} \right). \]

**Example 42.3.** Solve the initial value problem
\[ u_t = 4u_{xx} \quad 0 < x < 2\pi, t > 0 \]
\[ u(0, t) = u(2\pi, t) = 0 \]
\[ u(x, 0) = \begin{cases} 
1 & 0 < x \leq \pi \\
0 & \pi < x < 2\pi.
\end{cases} \]

This problem has homogeneous Dirichlet boundary conditions, so the general solution is given by Equation (42.2), or
\[ u(x, t) = \sum_{n=1}^{\infty} B_n e^{-(n\pi/l)^2kt} \sin \left( \frac{n\pi x}{l} \right). \]
The coefficients are
\[ B_n = \frac{2}{2\pi} \left( \int_0^\pi \sin \left( \frac{nx}{2} \right) \, dx + \int_\pi^{2\pi} x \sin \left( \frac{nx}{2} \right) \, dx \right) \]
\[ = -\frac{2}{n\pi} \cos \left( \frac{n\pi}{2} \right) \bigg|_0^\pi - \frac{2x}{n\pi} \cos \left( \frac{n\pi}{2} \right) \bigg|_\pi^{2\pi} + \frac{4}{n^2\pi} \sin \left( \frac{n\pi}{2} \right) \bigg|_\pi^{2\pi} \]
\[ = -\frac{2}{n\pi} \left( \cos \left( \frac{n\pi}{2} \right) - \cos(0) \right) - \frac{4}{n} \cos(n\pi) + \frac{2}{n} \cos \left( \frac{n\pi}{2} \right) - \frac{4}{n^2\pi} \sin \left( \frac{n\pi}{2} \right) \]
\[ = \frac{2}{n} \left( -\frac{1}{\pi} \left( \cos \left( \frac{n\pi}{2} \right) - 1 \right) + 2(-1)^{n+1} \cos \left( \frac{n\pi}{2} \right) - \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) \right), \]
and the solution to the problem is
\[ u(x,t) = 2 \sum_{n=1}^\infty \frac{1}{n} \left( -\frac{1}{\pi} \left( \cos \left( \frac{n\pi}{2} \right) - 1 \right) + 2(-1)^{n+1} \cos \left( \frac{n\pi}{2} \right) - \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) \right) e^{-n^2t} \sin \left( \frac{nx}{2} \right). \]

As you can see, the process can get quite messy, but we are now able to approximate solutions to whatever order is required for our application.
LECTURE 43

Other Boundary Conditions

So far, we’ve used the technique of separation of variables to produce solutions to the heat equation

\[ u_t = ku_{xx} \quad (43.1) \]

on \(0 < x < l, t > 0\), with either homogeneous Dirichlet \((u(0, t) = u(l, t) = 0)\) or Neumann \((u_x(0, t) = u_x(l, t) = 0)\) boundary conditions. Both of these sets of boundary conditions had reasonable physical interpretations, but these aren’t the only two scenarios that we might encounter in a real problem. Can we use the same method for other boundary conditions? The answer is yes for some and no for others. We will focus our attention on some classes of other boundary conditions where the separation of variables technique can still be used.

1. Mixed Homogeneous Boundary Conditions

*Mixed* homogeneous boundary conditions refer to the situation where one end of the interval has a homogeneous Dirichlet condition while the other has a homogeneous Neumann condition. Mathematically, this is the boundary condition

\[ u(0, t) = u_x(l, t) = 0. \quad (43.2) \]

The physical interpretation of this boundary condition might correspond to keeping the \(x = 0\) end of the rod in a bowl of ice water, while the other end is well-insulated.

To solve the resulting initial value problem using separation of variables, we let \(u(x, t) = X(x)T(t)\) as before. The heat equation separates into the pair of ODEs

\[ T' = -k\lambda T \]
\[ X'' = -\lambda X, \]

just as before. So we still have

\[ T(t) = Ae^{-k\lambda t}. \]

We now have to deal with the boundary value problem for \(X\). The boundary conditions \((43.2)\) become

\[ X(0) = X'(l) = 0. \]

The process of looking for eigenvalues and eigenfunctions is no different than before, other than the resulting values. There are only positive eigenvalues, and they are given by

\[ \lambda_n = \left( \frac{(2n - 1)\pi}{2l} \right)^2, \]

with eigenfunctions

\[ X_n = \sin \left( \frac{(2n - 1)\pi x}{2l} \right). \]
EXERCISE. Verify that \( \lambda_n \) and \( X_n \) are the eigenvalues and eigenfunctions for this boundary value problem. What changes if the boundary conditions are reversed: \( X'(0) = X(l) = 0 \)?

The separated solutions are then
\[
  u_n(x,t) = B_n e^{-((2n-1)\pi/2l)^2 kt} \sin \left( \frac{(2n-1)\pi x}{2l} \right),
\]
and the general solution is
\[
  u(x,t) = \sum_{n=1}^{\infty} B_n e^{-((2n-1)\pi/2l)^2 kt} \sin \left( \frac{(2n-1)\pi x}{2l} \right). \tag{43.3}
\]

EXERCISE. Notice that as \( t \to \infty \), \( u(x,\cdot) \to 0 \) for all \( x \) in \((0,l)\), just as with the homogeneous Dirichlet conditions. Why does this make physical sense given our interpretation of this set of boundary conditions?

With an initial condition \( u(x,0) = f(x) \), we have that
\[
  f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{(2n-1)\pi x}{2l} \right). \tag{43.4}
\]
This is an example of a specialized sort of Fourier series; the coefficients (derived in a similar process as earlier) are
\[
  B_n = \frac{2}{l} f(x) \sin \left( \frac{(2n-1)\pi x}{2l} \right) \, dx. \tag{43.5}
\]

REMARK. While the form of the series (43.4) and coefficients (43.5) might look similar to the Fourier series we were working with earlier, the convergence properties for this series are rather different. Recall that the standard Fourier sine and cosine series from before converged to the periodic extensions of the odd or even extensions of the original function, respectively. However, the terms in Equation (43.4) are periodic with period \( 4l \), rather than the \( 2l \)-periodic series we have been working with. To find what function the Fourier series converges to, we would first need to extend our initial data \( f(x) \), specified on \((0,l)\), to a function on \((0,2l)\) that is symmetric about \( x = l \). Then the series (43.4) is a sine series, the convergence on \((-2l,2l)\) will be to the odd extension of this extended function, and the series will converge on the entire real line to the periodic extension of this odd extension.

Example 43.1. Solve the following heat equation problem.
\[
  u_t = 25u_{xx}, \quad 0 < x < 10, t > 0
\]
\[
  u(0, t) = u_x(10, t) = 0
\]
\[
  u(x, 0) = 5
\]
By Equation (43.3), the general solution is
\[
  u(x, t) = \sum_{n=1}^{\infty} B_n e^{-25((2n-1)\pi/20)^2 t} \sin \left( \frac{(2n-1)\pi x}{20} \right).
\]
The coefficients are given by

\[ B_n = \frac{2}{10} \int_0^{10} 5 \sin \left( \frac{(2n - 1)\pi x}{20} \right) \, dx \]

\[ = - \frac{10}{(2n - 1)\pi} \cos \left( \frac{(2n - 1)\pi x}{20} \right) \bigg|_0^{10} \]

\[ = - \frac{10}{(2n - 1)\pi} \left( \cos \left( \frac{(2n - 1)\pi}{2} \right) - \cos(0) \right) \]

\[ = \frac{10}{(2n - 1)\pi}. \]

Thus the solution is

\[ u(x, t) = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n - 1} e^{-25((2n-1)\pi/20)^2 t} \sin \left( \frac{(2n - 1)\pi x}{20} \right). \]

□

2. Nonhomogeneous Dirichlet Conditions

The next type of boundary conditions we will look at are nonhomogeneous Dirichlet conditions. Earlier, we had worked with homogeneous Dirichlet conditions, which fixed the temperature \( u \) at each endpoint \( x = 0 \) and \( x = l \) at 0. Nonhomogeneous Dirichlet conditions have the form

\[ u(0, t) = T_1, \quad u(l, t) = T_2, \] (43.6)

where \( T_1 \) and \( T_2 \) can be values. This problem is slightly more difficult than the homogeneous Dirichlet condition problem we’ve studied. Recall that for separation of variables to work, the differential equations and the boundary conditions must be homogeneous. The idea whenever we have nonhomogeneous conditions is to try to split the problem into one involving homogeneous conditions, which we know how to solve, and another dealing with the nonhomogeneity.

Remark. This is analogous to our technique for solving nonhomogeneous second order differential equations — we found the general solution for the associated homogeneous equation and combined it with a solution that captured the nonhomogeneous term.

How can we separate the “core” homogeneous problem from whatever’s causing the nonhomogeneity? Consider what happens as \( t \to \infty \). We should expect that, as we’re fixing the temperatures at the endpoints and allowing free heat flux at the boundary, at some point the temperature will stabilize and be at equilibrium. Such a temperature distribution would clearly not depend on time, and we can write

\[ \lim_{t \to \infty} u(x, t) = v(x). \]

Notice that \( v(x) \) must still satisfy the heat equation (43.1) and the boundary conditions (43.6), but there is no reason to expect it to satisfy the initial condition, as \( v(x) \) will only occur after a very long time. A solution such as \( v(x) \) which does not depend on \( t \) is called a steady-state or equilibrium solution.

For a steady state solution such as \( v(x) \), the boundary value problem becomes

\[ kv'' = 0, \quad v(0) = T_1, \quad v(l) = T_2. \] (43.7)

The solutions to this second order differential equation are

\[ v(x) = c_1 x + c_2. \]
and after applying the boundary conditions, we have
\[ v(x) = T_1 + \frac{T_2 - T_1}{l} x. \]
Now, let
\[ w(x, t) = u(x, t) - v(x), \]
so that
\[ u(x, t) = w(x, t) + v(x). \]
This function \( w(x, t) \) is called the \textit{transient part} of \( u(x, t) \), as opposed to \( v(x) \), which is the equilibrium part; \( w(x, t) \) is so-called because \( \lim_{t \to \infty} w(x, t) = 0 \) for all \( x \).

Taking derivatives,
\[ u_t = w_t + v_t = w_t \quad \text{and} \quad u_{xx} = w_{xx} + v_{xx} = w_{xx} \]
as \( v(x) \) is independent of \( t \) and is linear in \( x \).

Thus, \( w(x, t) \) must satisfy the heat equation, as its relevant derivatives are equivalent to those of \( u(x, t) \), which we know satisfies the equation. What are the boundary and initial conditions on \( w(x, t) \)?

\[ w(0, t) = u(0, t) - v(0) = T_1 - T_1 = 0 \]
\[ w(l, t) = u(l, t) - v(l) = T_2 - T_2 = 0 \]
\[ w(x, 0) = u(x, 0) - v(x) = f(x) - v(x), \]
where \( f(x) = u(x, 0) \) is the given initial condition for the nonhomogeneous problem. Now, even though our initial condition is a little bit messier, we have homogeneous Dirichlet conditions on \( w(x, t) \), and the general solution for \( w(x, t) \) must be
\[ w(x, t) = \sum_{n=1}^{\infty} B_n e^{-(n\pi/l)^2kt} \sin \left( \frac{n\pi x}{l} \right), \]
with coefficients given by
\[ B_n = \frac{2}{l} \int_0^l (f(x) - v(x)) \sin \left( \frac{n\pi x}{l} \right) \, dx. \]
Notice that \( \lim_{t \to \infty} w(x, t) = 0 \), so the solution is transient as desired.

Finally, the solution to the nonhomogeneous Dirichlet problem
\[ u_t = ku_{xx}, \quad 0 < x < l, t > 0 \]
\[ u(0, t) = T_1, \quad u(l, t) = T_2 \]
\[ u(x, 0) = f(x) \]
is \( u(x, t) = w(x, t) + v(x) \), or
\[ u(x, t) = T_1 + \frac{T_2 - T_1}{l} x + \sum_{n=1}^{\infty} B_n e^{-(n\pi/l)^2kt} \sin \left( \frac{n\pi x}{l} \right), \]
with coefficients
\[ B_n = \frac{2}{l} \int_0^l \left( f(x) - T_1 - \frac{T_2 - T_1}{l} x \right) \sin \left( \frac{n\pi x}{l} \right) \, dx. \]
Remark. My recommendation is that you do not try to memorize the previous two formulas. Instead, be able to work out what problem \( w(x,t) \) must solve; once you formulate the appropriate homogeneous problem, you know how to solve it. The formula for \( v(x) \) isn’t difficult, but memorizing it also isn’t required, as it can be easily recovered by observing that \( v'' = 0 \) and it satisfies the same boundary conditions as \( u(x,t) \).

Example 43.2. Solve the following heat equation problem.

\[
\begin{align*}
  u_t &= 3u_{xx}, \quad 0 < x < 40, t > 0 \\
  u(0, t) &= 20, \quad u(40, t) = 100 \\
  u(x, 0) &= 40 - 3x
\end{align*}
\]

We start by writing

\[
 u(x, t) = w(x, t) + v(x),
\]

and we know from the above discussion that the equilibrium part \( v(x) = 20 + 2x \). Additionally, \( w(x,t) \) must satisfy the problem

\[
\begin{align*}
  w_t &= 3w_{xx}, \quad 0 < x < 40, t > 0 \\
  w(0,t) &= w(40,t) = 0 \\
  w(x,0) &= 40 - 3x - (20 + 2x) = 20 - x.
\end{align*}
\]

This is a homogeneous Dirichlet problem, so the general solution is

\[
 w(x,t) = \sum_{n=1}^{\infty} B_n e^{-3(n\pi/40)^2t} \sin\left(\frac{n\pi x}{40}\right). 
\]

The coefficients are given by

\[
 B_n = \frac{2}{40} \int_0^{40} (20 - x) \sin\left(\frac{n\pi x}{40}\right) \, dx
\]

\[
 = \frac{1}{20} \left[ -\frac{40(20-x)}{n\pi} \cos\left(\frac{n\pi x}{40}\right) - \frac{1600}{n^2\pi^2} \sin\left(\frac{n\pi x}{40}\right) \right]_0^{40}
\]

\[
 = \frac{1}{20} \left( \frac{800}{n\pi} \cos(n\pi) + \frac{800}{n\pi} \cos(0) \right)
\]

\[
 = \frac{40}{n\pi} ((-1)^n + 1).
\]

Thus the solution is

\[
 u(x,t) = 20 + 2x + \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n} e^{-3(n\pi/40)^2t} \sin\left(\frac{n\pi x}{40}\right).
\]

□

3. Other Conditions

There are of course many other boundary conditions we could have used, most of which have a physical interpretation. For example, the boundary conditions

\[
 u(0, t) + \kappa u_x(0, t) = 0, \quad u(l, t) + u_x(l, t) = 0
\]

say that the heat flux at the end points should be proportional to the temperature. Separation of variables can be used here.
EXERCISE. Apply separation of variables to the boundary conditions

\[ u(0, t) = 0, \quad u(l, t) + u_x(l, t) = 0 \]

and find the eigenvalues and eigenfunctions. The condition at \( x = l \) is called a *Robin* boundary condition.

We could also have had nonhomogeneous Neumann conditions

\[ u_x(0, t) = F_1, \quad u_x(l, t) = F_2, \]

which specify the (nonzero) heat flux at the boundaries. These conditions aren’t well-suited to the method of separation of variables.
1. Derivation

Consider a completely flexible string of length \( l \) and constant density \( \rho \). We’ll assume that the string will only undergo relatively small vertical vibrations, so that points don’t move from side to side. An example might be a plucked guitar string. Thus we can let \( u(x, t) \) be its displacement from equilibrium at time \( t \). The assumption of complete flexibility means that the tension force is tangential to the string, since it string itself provides no resistance to bending. This means the tension force only depends on the slope of the string.

Take a small piece of string going from \( x \) to \( x + \Delta x \). Let \( \theta(x, t) \) be the angle from the horizontal of the string. This is shown in Figure 44.1. Our goal is to use Newton’s Second Law \( F = ma \) to describe the motion. What forces are acting on this piece of string?

(i) Tension pulling to the right, which has magnitude \( T(x + \Delta x, t) \) and acts at an angle of \( \theta(x + \Delta x, t) \) from the horizontal.

(ii) Tension pulling to the left, which has magnitude \( T(x, t) \) and acts at an angle of \( \theta(x, t) \) from the horizontal.

(iii) Any external forces, which we denote by \( F(x, t) \).

In this course, we will assume \( F(x, t) = 0 \), so that the free motion of the string is only a result of the tension forces.

![Figure 44.1](image-url)
For small enough $\Delta x$, the length of the string can be approximated by $\sqrt{(\Delta x)^2 + (\Delta u)^2}$, as seen in Figure 44.1. The vertical component of Newton’s Second Law says

$$\rho\sqrt{(\Delta x)^2 + (\Delta u)^2}u_t(x, t) = T(x + \Delta x, t)\sin(\theta(x + \Delta x, t)) - T(x, t)\sin(\theta(x, t)).$$

Dividing by $\Delta x m$ and taking the limit as $\Delta x \to 0$,

$$\rho \sqrt{1 + (u_x)^2}u_{tt}(x, t) = \frac{\partial}{\partial x} [T(x, t)\sin(\theta(x, t))].$$

(44.1)

We assumed that our vibrations were relatively small. This means that $\theta(x, t)$ is close to zero. As a result, we can approximate $\sin(\theta(x, t)) \approx \tan(\theta(x, t))$. Moreover, $\tan(\theta(x, t))$ is just the slope of the string, $u_x(x, t)$, by our assumption of complete flexibility. Therefore we can conclude that, since $\theta(x, t)$ is very small, $u_x(x, t)$ is also very small, and Equation (44.1) becomes

$$\rho u_{tt}(x, t) = (T(x, t)u_x(x, t))_x.$$  

(44.2)

We haven’t yet used the horizontal component of Newton’s Second Law. We had assumed that only vertical vibrations were allowed, so the infinitesimal string element cannot move horizontally. As a result, the net horizontal force must be zero, or

$$T(x + \Delta x, t)\cos(\theta(x + \Delta x, t)) - T(x, t)\cos(\theta(x, t)) = 0.$$

Dividing by $\Delta x$ and taking the limit as $\Delta x \to 0$ yields

$$\frac{\partial}{\partial x} [T(x, t)\cos(\theta(x, t))] = 0.$$

Since $\theta(x, t)$ is very close to zero, $\cos(\theta(x, t))$ is very close to 1, so we have that $\partial T/\partial x(x, t)$ is very close to zero, implying that $T(x, t)$ is constant along the length of the string element. We’ll also assume that $T$ is independent of $t$. Then Equation (44.2) becomes the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx},$$

(44.3)

where $c^2 = T/\rho$.

2. The Homogeneous Dirichlet Problem

Now that we have the wave equation, let’s use separation of variables to derive a solution to the homogeneous Dirichlet problem. A similar process would give solutions to the homogeneous Neumann problem, and we could attack a nonhomogeneous Dirichlet problem in the same manner as we did for the heat equation. However, notice that the wave equation is second order with respect to $t$ as well as $x$, while the heat equation was first order with respect to $t$. This means that we will need two initial conditions: one for the initial displacement, and one for the initial velocity.

A homogeneous Dirichlet problem for the wave equation has the form

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, t > 0 \quad (44.4a)$$

$$u(0, t) = u(l, t) = 0 \quad (44.4b)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \quad (44.4c)$$

The physical interpretation of the boundary conditions (44.4b) is that the ends of the string are fixed in place. They might, for example, be attached to guitar pegs.

We start by assuming our solution has the separable form

$$u(x, t) = X(x)T(t).$$

Plugging this into Equation (44.4a) gives

$$T''(t)X(x) = c^2 T(t)X''(x).$$
Separating variables, we have

\[ \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda, \]

where \( \lambda \) is a constant. This is the pair of ordinary differential equations

\[ T'' + c^2 \lambda T = 0 \]
\[ X'' + \lambda X = 0. \] (44.5a) (44.5b)

The boundary conditions (44.4b) transform into the following boundary conditions on Equation (44.5b):

\[ X(0) = 0, \quad X(l) = 0. \]

This is the same boundary value problem for \( X \) that we worked with for the heat equation with homogeneous Dirichlet conditions. The eigenvalues and eigenfunctions are

\[ \lambda_n = \left( \frac{n\pi}{l} \right)^2 \]
\[ X_n(x) = \sin \left( \frac{n\pi x}{l} \right), \]

for \( n = 1, 2, 3, \ldots \). The ODE for \( T \) is then

\[ T'' + \left( \frac{cn\pi^2}{l} \right) T = 0, \]

and as the coefficient of \( T \) is positive, this has the general solution

\[ T_n(t) = A_n \cos \left( \frac{n\pi ct}{l} \right) + B_n \sin \left( \frac{n\pi ct}{l} \right). \]

There’s no reason to think that either of these are zero, so we end up with the separated solutions

\[ u_n(x, t) = \left[ A_n \cos \left( \frac{n\pi ct}{l} \right) + B_n \sin \left( \frac{n\pi ct}{l} \right) \right] \sin \left( \frac{n\pi x}{l} \right), \]

and the general solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n\pi ct}{l} \right) + B_n \sin \left( \frac{n\pi ct}{l} \right) \right] \sin \left( \frac{n\pi x}{l} \right). \] (44.6)

We can directly apply our first initial condition to Equation (44.6), but to apply the second we’ll need to differentiate with respect to \( t \). This gives us

\[ u_t(x, t) = \sum_{n=1}^{\infty} \left[ -\frac{n\pi c}{l} A_n \sin \left( \frac{n\pi ct}{l} \right) + \frac{n\pi c}{l} B_n \cos \left( \frac{n\pi ct}{l} \right) \right] \sin \left( \frac{n\pi x}{l} \right). \]

Plugging in the initial conditions (44.4c) then yields the system of equations

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right) \]
\[ u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \left( \frac{n\pi x}{l} \right). \] (44.7a) (44.7b)

These are both Fourier sine series: Equation (44.7a) is the Fourier sine series for \( f(x) \) on \((0, l)\), while Equation (44.7b) is the Fourier sine series for \( g(x) \) on \((0, l)\) with a different (by a multiplicative
The Euler-Fourier formulas then tell us

\[ A_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx \]

\[ \frac{n\pi c}{l} B_n = \int_0^l g(x) \sin \left( \frac{n\pi x}{l} \right) \, dx, \]

or

\[ A_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) \, dx \quad (44.8) \]

\[ B_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \left( \frac{n\pi x}{l} \right) \, dx. \quad (44.9) \]

**Exercise.** Go through the process of separating variables for the wave equation with homogeneous Neumann coefficients and derive the general solution and coefficient formulas.