Lecture 3

Two-stage stochastic linear programs

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### Introduction

Consider the two-stage stochastic linear program defined as

<table>
<thead>
<tr>
<th>SLP</th>
<th>minimize ( c^T x + Q(x) )</th>
<th>subject to ( Ax = b, ) ( x \geq 0, )</th>
</tr>
</thead>
</table>

where \( Q(x) \) is defined as follows:

\[
Q(x) \triangleq \mathbb{E}[Q(x, \xi)].
\]

If \( \xi(\omega) \triangleq (T(\omega), W(\omega), q(\omega), b(\omega)) \), then the integrand of this expectation, denoted by \( Q(x, \xi) \), is the optimal value of the following second-stage
problem:

\[
\text{SecLP}(\xi) \quad \begin{array}{ll}
\text{minimize} & q(\omega)^T y(\omega) \\
\end{array}
\]
subject to
\[
T(\omega)x + W(\omega)y(\omega) = b(\omega) \quad y(\omega) \geq 0.
\]

- \(T(\omega)\) is referred to as the tender

- \(W(\omega)\) is called the recourse matrix
Discrete random variables

- Suppose $\xi$ is a discrete random variable taking on values in $\Xi$.
- Suppose $K_1$ and $K_2(\xi)$ (elementary feasibility set) are defined as

\[ K_1 \triangleq \{ x : Ax = b, x \geq 0 \} \]

and

\[ K_2(\xi) \triangleq \{ x : y \geq 0 \quad \text{subject to} \quad W(\omega)y = h(\omega) - T(\omega)x \} . \]

- Moreover, $K_2$ is defined as

\[ K_2 \triangleq \bigcap_{\xi \in \Xi} K_2(\xi). \]
Next, we prove some useful properties of $K_2(\xi)$ but need to define the positive hull:

**Definition 1 (Positive hull)** The positive hull of $W$ is defined as

$$\text{pos } W \triangleq \{ t : W y = t, y \geq 0 \}.$$  

In fact, $\text{pos } W$ is a finitely generated cone which is the set of nonnegative linear combinations of finitely many vectors. Note that it is convex and polyhedral.

**Proposition 1 (Polyhedrality of $K_2(\xi)$ and $K_2$)** The following hold:

1. For a given $\xi$, $K_2(\xi)$ is a convex polyhedron;
2. When $\xi$ is a finite discrete random variable, $K_2$ is a convex polyhedron.

**Proof:**
(i) Consider an \( x \not\in K_2(\xi) \) and \( \xi = (T(\omega), W(\omega), q(\omega), b(\omega)) \). Consequently there exists no nonnegative \( y \) such that \( W(\omega)y = h(\omega) - T(\omega)x \); equivalently \( h(\omega) - T(\omega)x \not\in \text{pos} \, W(\omega) \), where \( \text{pos} \, W(\omega) \) is a finitely generated convex cone. Therefore, by the separating hyperplane theorem, there exists a hyperplane, defined as \( H \), such that \( H = \{ v : u^Tv = 0 \} \) such that \( u^T(h(\omega) - T(\omega)x) < 0 \) and \( u^Tv > 0 \) where \( v \in \text{pos} \, W(\omega) \). Since \( \text{pos} \, W(\omega) \) is a finitely generated cone, there can only be finitely many such hyperplanes. It follows that \( K_2(\xi) \) can be viewed as an intersection of a finite number of halfspaces and is therefore a convex polyhedron.

(ii) Since \( K_2 \) is an intersection of a finite number of polyhedral sets, \( K_2 \) is a convex polyhedron.

\[ \square \]

- Next, we examine the properties of \( Q(x, \xi) \)
Proposition 2 (Piecewise linearity of $Q(x, \xi)$) For a given $\xi$, $Q(x, \xi)$ satisfies the following:

(i) $Q(x, \xi)$ is a piecewise linear convex function in $(h, T)$

(ii) $Q(x, \xi)$ is a piecewise linear concave function in $q$

(iii) $Q(x, \xi)$ is a piecewise linear convex function in $x$ for $x \in K_2$

Proof:

(i) Suppose $f(b)$ is defined as $f(b) = \min \{q^T y : W y = b, y \geq 0\}$. We proceed to show that $f(b)$ is convex in $b$. Consider $b_1, b_2$ such that $b_\lambda = \lambda b_1 + (1 - \lambda)b_2$ where $\lambda \in (0, 1)$. Let $y_1$ and $y_2$ denote solutions of $\{q^T y : W y = b, y \geq 0\}$ for $b = b_1$ and $b_2$ respectively. Furthermore, suppose $y_\lambda$ denotes the solution of $\min \{q^T y : W y = b_\lambda, y \geq 0\}$ and it follows that

$$f(b_\lambda) = q^T y_\lambda \leq q^T y_1, q^T y_2.$$
Consequently,

\[ f(b_\lambda) = q^T y_\lambda \leq q^T (\lambda y_1 + (1 - \lambda)y_2) \]

\[ = \lambda q^T y_1 + (1 - \lambda) q^T y_2 \]

\[ = f(b_1) \]

\[ = f(b_2) \]

\[ = \lambda f(b_1) + (1 - \lambda) f(b_2), \]

where \( \lambda y_1 + (1 - \lambda)y_2 \) is feasible with respect to the constraint \( \{ y : Wy = b_\lambda \} \). The convexity of \( f \) in \( (T, h) \) follows.

(ii) To show the concavity of \( Q(x, \xi) \) in \( q \), we may write \( f(q) \) as

\[ f(q) = \max \{ \pi^T b : W^T \pi \leq q \}. \]

Proceeding as earlier, it is relatively easy to show that \( f(q) \) is concave
in \( q \).

(iii) Next, we show the piecewise linearity of \( Q(x, \xi) \) in \((h, T)\). Suppose \( y \) is a solution of \( \min \{ q^T y : Wy = b, y \geq 0 \} \). Then if \( y_B \) and \( y_N (\equiv 0) \) denote the basic and non-basic variables and \( B(\omega) \) denotes the basis and \( N(\omega) \) represents the remaining columns, we have

\[
B(\omega)y_B + N(\omega)y_N = h(\omega) - T(\omega)x
\]

\[
B(\omega)y_B = h(\omega) - T(\omega)x
\]

\[
y_B = (B(\omega))^{-1}(h(\omega) - T(\omega)x).
\]
Recall that when a feasible basis is optimal, we have that

\[ q(\omega)^T \tilde{y} \geq q_B(\omega)^T y_B \]
\[ = q_B(\omega)^T (B(\omega))^{-1}(h(\omega) - T(\omega)x) \]
\[ = Q(x, \xi), \]

implying that \( Q(x, \xi) \) is linear in \((h, T, q, x)\) on a domain prescribed by feasibility and optimality conditions. Piecewise linearity follows from noting the existence of a finite number of different optimal bases for the second-stage program.
Support functions

The support function $s_C(\bullet)$ of a nonempty convex set $C$ is defined as

$$s(h) \triangleq \sup_{z \in C} z^T h.$$ 

The support function $s(\bullet)$ is convex, positively homogeneous,* and lower semicontinuous†

If $s_1(\bullet)$ and $s_2(\bullet)$ are support functions of $C_1$ and $C_2$ respectively, then

$$s_1(\bullet) \leq s_2(\bullet) \iff C_1 \subset C_2.$$ 

*A function $f(x)$ is positively homogeneous if $f(\lambda x) = \lambda f(x)$ where $\lambda \geq 0$ and $x \in \mathbb{R}^n$.

†A function $f(x)$ is lower semicontinuous at $x_0$ if $\lim_{x \to x_0} \inf f(x) \geq f(x_0)$. 

Stochastic Optimization
Furthermore,
\[ s_1(\bullet) = s_2(\bullet) \iff C_1 = C_2. \]

**Example 1 (Support function of unit ball)** Consider the unit ball \( \mathcal{B} \) defined with the Euclidean norm as follows

\[ \mathcal{B} \triangleq \{ \pi : \|\pi\|_2 \leq 1 \}. \]

Then \( s_\mathcal{B}(\cdot) = \|\cdot\|_2 \).

By definition, we have that

\[ s_\mathcal{B}(\chi) = \sup_{\pi \in \mathcal{B}} \pi^T \chi. \]

Since \( \mathcal{B} \) is a compact set, by the continuity of the objective, the supremum is achieved. Suppose the maximizer is denoted by \( \pi^* \). Then, \( \|\pi^*\| \leq 1 \) and
it follows that $\pi^* = \frac{\chi}{\|\chi\|}$. Consequently,

$$(\pi^*)^T \chi = \frac{\chi^T}{\|\chi\|} \chi = \frac{\|\chi\|^2}{\|\chi\|} = \|\chi\|.$$
Subgradients of $Q(x, \xi)$

Consider the second-stage linear program given by

\[
\text{SecLP}(\xi) \quad \text{minimize} \quad y(\omega) q(\omega)^T y(\omega) \\
\text{subject to} \quad T(\omega)x + W(\omega)y(\omega) = b(\omega) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad y(\omega) \geq 0.
\]

Then the associated dual problem is given by

\[
\text{D-SecLP}(\xi) \quad \text{maximize} \quad \pi(\omega) b(\omega)^T \pi(\omega) \\
\text{subject to} \quad W^T(\omega)\pi(\omega) \leq q(\omega).
\]
Let \( s_q(\chi) \triangleq \inf \{ q^T y : Wy = \chi, y \geq 0 \} \). Suppose \( \Pi(q) \) is defined as

\[ \Pi(q) \triangleq \{ \pi : W^T \pi \leq q \}. \]

It follows that

\[ s_q(x) = \sup_{\pi \in \Pi(q)} \pi^T \chi. \]

This implies that \( s_q(\chi) \) is the support function of \( \Pi(q) \), a closed and convex polyhedral set. If \( \Pi(q) \) is empty, then \( s_q(\chi) = +\infty \) or \(-\infty\).

Next we examine the subdifferential of \( Q(x, \xi) \) and need some definitions.

**Definition 2 (Differentiability)** Given a mapping \( g : \mathbb{R}^n \to \mathbb{R}^m \). It is said that \( g \) is directionally differentiable at a point \( x_0 \in \mathbb{R}^n \) in a direction
$h \in \mathbb{R}^n$ if the following limit exists:

$$g'(x_0; h) := \lim_{t \to 0} \left( \frac{g(x_0 + th) - g(x_0)}{t} \right).$$

If $g$ is directionally differentiable at $x_0$ for every $h \in \mathbb{R}^n$, then it is said to be directionally differentiable at $x_0$. Note that whenever this limit exists, $g'(x_0; h)$ is said to be positively homogeneous in $h$ or $g'(x_0; th) = tg'(x_0; h)$ for any $t \geq 0$.

If $g(x)$ is directionally differentiable at $x_0$ and $g'(x_0; h)$ is linear in $h$, then $g(x)$ is said to be Gâteaux differentiable at $x_0$. This limit can also be written as follows:

$$g(x_0 + h) = g(x_0) + g'(x_0; h) + r(h),$$
where $r(h)$ is such that $\frac{r(th)}{t} \to 0$ as $t \to 0$ for any fixed $h \in \mathbb{R}^n$.

If $g'(x_0; h)$ is linear in $h$ and $\frac{r(h)}{||h||} \to 0$ as $h \to 0$ (or $r(h) = o(h)$), then $g(x)$ is said to be Fréchet differentiable at $x_0$ or merely differentiable at $x_0$.

Note that Fréchet implies Gâteaux differentiability and when the functions are locally Lipschitz, both notions coincide. Recall that a mapping $g$ is said to be locally Lipschitz on a set $X \subset \mathbb{R}^n$, if it is Lipschitz continuous on a neighborhood of every point of $X$ (with possibly different constants).

A vector $z \in \mathbb{R}^n$ is said to be a subgradient of $f(x)$ at $x_0$ if

$$f(x) - f(x_0) \geq z^T(x - x_0), \quad \forall x \in \mathbb{R}^n.$$

The set of all subgradients of $f(x)$ at $x_0$ is referred to as a subdifferential.
and is denoted by $\partial f(x_0)$. The subdifferential $\partial f(x_0)$ is a closed and convex subset of $\mathbb{R}^n$ and $f$ is subdifferentiable at $x_0$ if $\partial f(x_0) \neq \emptyset$.

**Proposition 3 (Convexity and polyhedrality of $Q(x, \xi)$)** For any given $\xi$, the function $Q(., \xi)$ is convex. Moreover, if the set $\Pi(q)$ is nonempty and $D$-SecLP($\xi$) is feasible for at least one $x$, then the function $Q(., \xi)$ is polyhedral.

**Proof:** Recall that the set $\Pi(q)$ is closed, convex, and polyhedral. If $\Pi(q)$ is nonempty, then $s_q(\chi)$, its support function, is polyhedral\(^\dagger\) where $\chi = h - Tx$. Furthermore, $s_q(\chi)$ is positively homogeneous. If $\Pi(q)$ is empty, the infimum given by

$$\inf\{q^T y : Wy = \chi, y \geq 0\}$$

\(^\dagger\)Recall that an extended real-valued function is called polyhedral, if it is proper,\(^\S\) convex, and lower semicontinuous, its domain is a closed and convex polyhedron, and it is piecewise linear on its domain.
can be $+\infty$ or $-\infty$. Finally, since $Q(x, \xi) = s_q(\chi)$, the result follows.

Next we characterize the subdifferential of $Q(x, \xi)$ but require the Fenchel-Moreau theorem:

**Theorem 4 (Fenchel-Moreau)** Suppose that $f$ is a proper, convex, and lower semicontinuous function. Then $f^{**} = f$, where $f^*$, the conjugate function of $f$, is defined as $f^*(z) = \sup_{x \in \mathbb{R}^n} \{ z^T x - f(x) \}$.

Note that the conjugate function $f^* : \mathbb{R}^n \to \bar{\mathbb{R}}$ is always convex and lsc. Furthermore, we have that

$$
\partial f^*(x) = \arg\max_{z \in \mathbb{R}^n} \{ z^T x - f(z) \}.
$$

**Proposition 5 (Subdifferential of $Q(x, \xi)$)** Suppose that for a given $\xi \in \Xi$, the function $Q(x, \xi)$ is finite. Then $Q(x, \xi)$ is subdifferentiable at $x_0$ and

$$
\partial Q(x_0, \xi) = -T^T D(x_0, \xi),
$$
where $\mathcal{D}(x, \xi)$ the set of dual solutions is defined as

$$
\mathcal{D}(x, \xi) := \arg \max_{\pi \in \Pi(q)} \pi^T (h - Tx).
$$

**Proof:** Since $Q(x_0, \xi)$ is finite, then $\Pi(q)$ is nonempty and $s_q(\chi)$ is its support function. By the definition of a conjugate function, it can be seen that $s_q(\chi)$ is the conjugate of the indicator function $1_{l_q}(\pi)$, defined as

$$
1_{l_q}(\pi) := \begin{cases} 
0, & \text{if } \pi \in \Pi(q) \\
+\infty, & \text{otherwise.}
\end{cases}
$$

Since $\Pi(q)$ is closed and convex, the function $1_{l_q}(\pi)$ is convex and lower semicontinuous. By the Fenchel-Moreau theorem, the conjugate of $s_q(.)$ is
\[ \mathbb{1}_{l_q}(.) \text{ and for } \chi_0 = h - T x_0, \text{ we have} \]

\[ \partial s_q(x_0) = \arg \max_{\pi} \left\{ \pi^T \chi_0 - \mathbb{1}_q(\pi) \right\} = \arg \max_{\pi \in \Pi(q)} \pi^T \chi_0. \]

Since \( \Pi(q) \) is polyhedral and \( s_q(\chi_0) \) is finite, it follows that \( \partial s_q(\chi_0) \) is nonempty. Moreover, \( s_q(.) \) is piecewise linear (earlier in this lecture) and by the chain rule

\[ \partial_x Q(x, \xi) = (\partial_x \chi_0)^T \partial \chi_0 s_q(\chi_0) = -T^T D(x_0, \xi). \]
**Expected recourse for discrete distributions**

In this section, we consider the expected cost of recourse, denoted by $Q(x)$, where

$$Q(x) \triangleq \mathbb{E}[Q(x, \xi)].$$

Suppose, the distribution of $\xi$ has finite support, implying that $\xi$ can take on a finite number of realizations given by $\xi_1, \ldots, \xi_K$ with a mass function denoted by $p_1, \ldots, p_K$. For a given $x$, $y_j$ is given by the solution to

$$\text{SecLP}(\xi_j) \quad \text{minimize} \quad q_j^T y_j$$
$$\text{subject to} \quad T_j x + W_j y_j = b_j$$
$$y_j \geq 0,$$

where $(T(\xi_j), W(\xi_j), q(\xi_j), b(\xi_j)) = (T_j, W_j, q_j, b_j)$. Given that the
problems in $y$ are separable, we may specify the expected cost of recourse as the following problem in $(y_1, \ldots, y_K)$ as follows:

\[
\text{SecLP}(\xi_1, \ldots, \xi_K) \quad \text{minimize} \quad \sum_{j=1}^{K} p_j q_j^T y_j \\
\text{subject to} \quad T_j x + W_j y_j = b_j, \quad j = 1, \ldots, K \\
y_j \geq 0, \quad j = 1, \ldots, K
\]
Note that the entire two-stage problem can then be specified as follows:

<table>
<thead>
<tr>
<th>SLP</th>
<th>minimize</th>
<th>( c^T x + \sum_{j=1}^{K} p_j q_j^T y_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>subject to</td>
<td>( Ax = b )</td>
</tr>
<tr>
<td></td>
<td>subject to</td>
<td>( T_jx + W_jy_j = b_j, \quad j = 1, \ldots, K )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( x, y_j \geq 0, \quad j = 1, \ldots, K )</td>
</tr>
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</table>

The subdifferential of \( Q(x) \) is specified by the following proposition.

**Proposition 6 (Subdifferential of \( \partial Q(x) \))** Suppose that the probability distribution function for \( \xi \) has finite support or \( \Xi = \{ \xi_1, \ldots, \xi_K \} \) and the expected recourse cost has a finite value for at least some \( \tilde{x} \in \mathbb{R}^n \).
Then $Q(x)$ is polyhedral and

$$\partial Q(x_0) = \sum_{j=1}^{K} \partial Q(x, \xi_j).$$

**Proof:** This follows as a consequence of the Moreau-Rockafellar theorem.

**Theorem 7 (Moreau-Rockafellar)** Let $f_i : \mathbb{R}^n \to \bar{\mathbb{R}}$ be proper convex functions for $i = 1, \ldots, N$. Let $f(.) = \sum_{i=1}^{N} f_i(.)$ and $x_0$ be a point such that $f_i(x_0)$ are finite or $x_0 \in \cap_{i=1}^{N} \text{dom } f_i$. Then

$$\partial f_1 + \ldots + \partial f_N \subset \partial f.$$
Furthermore, we have equality in this relationship or

\[ \partial f_1 + \ldots + \partial f_N = \partial f, \]

if one of the following hold:

1. The set \( \cap_{i=1}^m \text{ri}(\text{dom} f_i) \) is nonempty;

2. the functions \( f_1, \ldots, f_k \) for \( k \leq m \) are polyhedral and the intersection of the sets \( \cap_{i=1}^k \text{dom} f_i \) and \( \cap_{i=k+1}^m \text{ri}(\text{dom} f_i) \) is nonempty;

3. there exists a point \( \bar{x} \in \text{int}(\text{dom} f_i), i = 1, \ldots, m. \)
Expected recourse for general distributions

Consider \( Q(x, \xi) \) where \( \xi : \Omega \to \mathbb{R}^d \). The cost of taking recourse given \( \xi \) is the minimum value of \( q(\omega)^T y(\omega) \).

**Lemma 1** \( Q(x, \xi) \) is a random lower semicontinuous function.

**Proof:** This follows from noting that \( Q(.,.) \) is measurable with respect to the Borel sigma algebra on \( \mathbb{R}^n \times \mathbb{R}^d \). Furthermore, \( Q(.,\xi) \) is lower semicontinuous. It follows that \( Q(x, \xi) \) is a random lower semicontinuous function. \( \blacksquare \)

**Proposition 8** Suppose either \( \mathbb{E}[Q(x, \xi)]^+ \) or \( \mathbb{E}[Q(x, \xi)]^- \) is finite. Then, \( Q(x) \) is well defined.

**Proof:** This requires verifying whether \( Q(x, .) \) is measurable with respect to the Borel sigma algebra on \( \mathbb{R}^d \). This follows by directly employing Theorem 7.37 [SDR09].
**Theorem 9** Let $F : \mathbb{R}^n \times \Omega \rightarrow \overline{\mathbb{R}}$ be a random lower semicontinuous function. Then the optimal value function $\vartheta(\omega)$ and the optimal solution multifunction $X^*(\omega)$ are both measurable where

$$\vartheta(\omega) \triangleq \inf_{y \in \mathbb{R}^n} F(y, \omega) \text{ and } X^*(\omega) \triangleq \arg \min_{x \in \mathbb{R}^n} F(x, \omega).$$

Next, we consider several settings where either $\mathbb{E}[Q(x, \xi)^+]$ or $\mathbb{E}[Q(x, \xi)^-]$ is finite. However, this requires taking a detour in discussing various notions of recourse.
Notions of recourse

- The two-stage problem is said to have fixed recourse if $W(\omega) = W$ for every $\omega \in \Omega$.

- The problem is said to have complete recourse if there exists a solution $y$ satisfying $W y = \chi$ and $y \geq 0$ for every $\chi$. In effect, for any first-stage decision, the primal second-stage problem is feasible. More formally, this can be stated as follows:

$$\text{pos} W = \mathbb{R}^m,$$

where $h - Tx \in \mathbb{R}^m$.

- By employing duality, the fixed recourse is said to be complete if and only if the feasible set $\Pi(q)$ of the second-stage dual problem is bounded.
Complete recourse ⇔ Primal second-stage is feasible
⇔ Dual second-stage is bounded (could be empty).

• Given a set $X$, then $X_{\infty}$ refers to the recession cone of $X$ and is defined as follows:

$$X_{\infty} \triangleq \{ y : \forall x \in X, x + \lambda y \in X, \lambda \geq 0 \} .$$

• Boundedness of $\Pi(q)$ implies that its recession cone denoted by $\Pi_0 = \Pi(0)$ contains only the zero element; $\Pi_0 = \{0\}$, provided that $\Pi(q)$ is nonempty.
• **Claim:** $\Pi(q)$ is nonempty and bounded implies that $\Pi_0$ contains only the zero element.

**Proof sketch:** Suppose this claim is false. Then $\Pi_0 \neq \emptyset$ and suppose $u \in \Pi_0$. This implies that $W^T u \leq 0$ where $u \neq 0$. Since $\Pi(q)$ is nonempty, then it contains at least one element, say $\hat{\pi}$. It follows that

$$W^T(\hat{\pi} + \lambda u) \leq W^T \hat{\pi} \leq q,$$

for all $\lambda \geq 0$. This implies that $\Pi(q)$ is unbounded since it contains a ray given by $\hat{\pi} + \lambda u$ where $\lambda \geq 0$ and $\hat{\pi} \in \Pi(q)$.

• A subclass of problems with fixed and complete recourse are **simple recourse** problems where the recourse decisions are either surplus or shortage decisions determined by the first-stage decisions. Furthermore, $T$ and $q$ are deterministic and $q > 0$. 

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Stochastic Optimization 30
Specifically, \( y(\omega) = y^+(\omega) + y^-(\omega) \). If \([h - Tx]_i \geq 0\), \( y^+_i(\omega) = [h - Tx]_i \) and \( y^-_i(\omega) = 0 \). Similarly, if \([h - Tx]_i \leq 0\), \( y^-_i(\omega) = [h - Tx]_i \) and \( y^+_i(\omega) = 0 \). This is compactly captured by defining \( W = (I - I) \), leading to the following system:

\[
\begin{align*}
y^+(\omega) - y^-(\omega) &= h(\omega) - Tx \\
y^+(\omega), y^-(\omega) &\geq 0.
\end{align*}
\]

- The recourse is said to be **relatively complete** if for every \( x \) in the set 
  \( X \triangleq \{ x : Ax = b, x \geq 0 \} \), the feasible set of the primal second-stage
  problem is nonempty for a.e. \( \omega \in \Omega \). Note that there may be events that
  occur with zero probability (measure) that may have \( Q(x, \xi) = +\infty \). However, these do not affect the overall expectation.
  
- A sufficient condition for **relatively complete recourse** is the following:

  for every \( x \in X, Q(x, \xi) < +\infty \) for all \( \xi \in \Xi \).
• Note that this condition becomes necessary and sufficient in two cases:
  • The vector $\xi$ has finite support
  • Fixed recourse

• Next we consider an instance of a problem with random recourse and discuss some concerns.

**Example 2 (Random recourse)** Suppose $Q(x, \xi) := \inf\{y : \xi y = x, y \geq 0\}$, with $x \in [0, 1]$ and $\xi$ having a density function given by $p(z) = 2z, 0 \leq z \leq 1$. Then for $\xi > 0$, $x \in [0, 1]$, $Q(x, \xi) = x/\xi$ and $\mathbb{E}[Q(x, \xi)] = 2x$. For $\xi = 0$ and $x > 0$, $Q(x, \xi) = +\infty$. It can be seen that for almost every $\xi \in [0, 1]$, $Q(x, \xi) < +\infty$.

However, a small perturbation in the distribution may change that. For instance, a discretization of the distribution with the first point at
\( \xi = 0 \), immediately leads to \( \mathbb{E}[Q(x, \xi)] = +\infty \) for \( x > 0 \). In effect, this problem is unstable and this issue cannot occur if the recourse is fixed.

- We now consider the support function \( s_q \) associated with \( \Pi(q) \). Recall that this is defined as

\[
s_q(\chi) = \sup_{\pi \in \Pi(q)} \pi^T \chi.
\]

Our goal lies in finding sufficiency conditions for the existence of the Expectation \( \mathbb{E}[s_q(\chi)] \). We will employ Hoffman’s Lemma [SDR09, Theorem 7.11] which is defined as follows:

**Lemma 2 (Hoffman’s lemma)** Consider the multifunction \( \mathcal{M}(b) := \{x : Ax \leq b\} \), where \( A \in \mathbb{R}^{m \times n} \). Then there exists a positive constant
\( \kappa, \) depending on \( A, \) such that for any given \( x \in \mathbb{R}^n \) and any \( b \in \text{dom} \mathcal{M}, \)

\[
\text{dist}(x, \mathcal{M}(b)) \leq \kappa \| (Ax - b)_+ \|.
\]

- By Hoffman’s Lemma, there exists a constant \( \kappa \) depending on \( W \) such that if for some \( q_0, \) the set \( \Pi(q_0) \) is nonempty, then for every \( q, \) the following inclusion holds:

\[
\Pi(q) \subset \Pi(q_0) + \kappa \| q - q_0 \| B,
\]

where \( B := \{ \pi : \| \pi \| \leq 1 \} \) and \( \| . \| \) denotes the Euclidean norm. This inclusion allows for deriving an upper bound for the support function \( s_q(\cdot). \)

- Since the support function of the unit ball is the Euclidean norm, when \( \Pi(q_0) \) is nonempty, and that

\[
 s_1(\cdot) \leq s_2(\cdot) \Leftrightarrow C_1 \subset C_2,
\]
we have that

\[ s_q(\bullet) \leq s_{q_0}(\bullet) + \kappa \| q - q_0 \| \| \bullet \|. \]  \hspace{1cm} (1)

- Consider \( q_0 = 0 \) and the set \( \Pi(0) \) is defined as

\[ \Pi(0) \triangleq \{ \pi : W^T \pi \leq 0 \} . \]

This set is a cone and suppose its support function is denoted by \( s_0 = s_{q_0} \).

- The support function is defined as follows (proof omitted):

\[ s_0(\xi) = \begin{cases} 
0, & \text{if } \chi \in \text{pos } W \\
+\infty, & \text{otherwise}
\end{cases} \]
Therefore, if $q_0 = 0$ in (1) allows one to claim that if $\Pi(q)$ is nonempty, then $s_q(\chi) \leq \kappa \|q\| \|\chi\|$ for all $\chi \in \text{pos } W$, and $s_q(\chi) = +\infty$ if $\chi \not\in \text{pos } W$.

By the polyhedrality of $\Pi(q)$, if $\Pi(q)$ is nonempty, we have that $s_q(\cdot)$ is piecewise linear on its $\text{dom}(s_q(\cdot))$ which is given by $\text{pos } W$. In fact, $s_q(\cdot)$ is Lipschitz continuous on its domain or

$$|s_q(\chi_1) - s_q(\chi_2)| \leq \kappa \|q\| \|\chi_1 - \chi_2\|. $$

We now provide a necessary and sufficient condition for a fixed recourse problem having a finite $\mathbb{E}[Q(x, \xi)_+]$.

**Proposition 10 (Nec. and suff. condition for finiteness of $\mathbb{E}(Q(x, \xi)_+)$)**

Suppose that recourse is fixed and $\mathbb{E}[\|q\| \|h\|] < +\infty$ and $\mathbb{E}[\|q\| \|T\|] < +\infty$. Consider a point $x \in \mathbb{R}^n$. Then $\mathbb{E}[Q(x, \xi)_+]$ is finite if and only if $h(\xi) - T(\xi)x \in \text{pos } W$ for almost every $\xi$. 

Stochastic Optimization
Proof:

(⇒) Suppose \( h(\xi) - T(\xi)x \not\in \text{pos } W \) with probability one. This implies that for some \( \xi \in U \subseteq \Xi \), \( h(\xi) - T(\xi)x \not\in \text{pos } W \) where \( \mu(U) = \mathbb{P}[\xi : \xi \in U] > 0 \). But for any such \( \xi \), \( Q(x, \xi) = +\infty \). But \((Q(x; \xi))_+ \triangleq \max(Q(x; \xi), 0)) \geq Q(x; \xi) = +\infty \) with positive probability. Consequently, it follows that \( \mathbb{E}[Q(x, \xi)_+] = +\infty \).

(⇐) Suppose \( h(\xi) - T(\xi)x \in \text{pos } W \) with probability one. Then \( Q(x, \xi) = s_q(h - Tx) \). By (1), we have that

\[
s_q(\chi) \leq s_0(\chi) + \kappa \|q\| \|\chi\|.
\]

Furthermore, we recall that \( s_0(\chi) = 0 \) when \( \chi \in \text{pos } W \) implying that

\[
s_q(\chi) \leq \kappa \|q\| \|\chi\|.
\]
By using the triangle inequality, we have that

\[ Q(x; \xi) = s_q(h(\xi) - T(\xi)x) \leq \kappa \| q(\xi) (\|h(\xi)\| + \|T(\xi)\|\|x\|) \text{ wp1.} \]

Since

\[ 0 \leq \kappa \| q(\xi) (\|h(\xi)\| + \|T(\xi)\|\|x\|) \text{ wp1,} \]

it follows that

\[ \max(Q(x; \xi), 0) \leq \kappa \| q(\xi) (\|h(\xi)\| + \|T(\xi)\|\|x\|) \text{ wp1.} \]

Taking expectations, we have the following:

\[ \mathbb{E}[Q(x; \xi)_+] \leq \kappa \mathbb{E}[\|q(\xi)\|\|h(\xi)\|] + \kappa \mathbb{E}[\|q(\xi)\|\|T(\xi)\|\|x\|]. \]
By assumption, we have that $E[||q||h||] < +\infty$ and $E[||q||T||] < +\infty$. As a result, $E[Q(x, \xi)_{+}] < +\infty$.

**Proposition 11** Suppose that (i) the recourse is fixed, (ii) for a.e. $q$ the set $\Pi(q)$ is nonempty, and (iii) the following holds:

$$E[||q||h||] < +\infty \text{ and } E[||q||T||] < +\infty.$$ 

Then the following hold: (a) the expectation function $Q(x)$ is well-defined and $Q(x) > -\infty$ for all $x \in \mathbb{R}^n$; (b) Moreover, $Q(x)$ is convex, lower semicontinuous, and Lipschitz continuous on $\text{dom } \phi$, and its domain is a convex closed subset of $\mathbb{R}^n$ is given by

$$\text{dom} Q = \{ x : h - Tx \in \text{pos } W \text{ w.p. } 1. \}$$

**Proof:**

(a): By (ii), $\Pi(q)$ is nonempty with probability one. Consequently, for
almost every $\xi \in \Xi$, $Q(x, \xi) = s_q(h(\xi) - T(\xi)x)$ for every $x$ and for almost every $\xi$.

Suppose $\pi(q)$ denotes the element of $\Pi(q)$ closest to zero; by the closedness of $\Pi(q)$, it exists and by Hoffman’s lemma, there exists a constant $\kappa$ such that

$$||\pi(q)|| \leq \kappa ||q||.$$

By the definition of the support function,

$$s_q(h - Tx) = \sup_{\pi(q) \in \Pi(q)} \pi(q)^T(h - Tx) \geq \pi(q)^T(h - Tx)$$
and for every $x$, the following holds with probability one:

$$-\pi(q)^T(h - Tx) \leq \|\pi(q)\| (\|h\| + \|T\|\|x\|) \quad \text{(Cauchy-Schwartz and triangle inequality)}$$

$$\leq \kappa \|q\| (\|h\| + \|T\|\|x\|)$$

$$\implies \pi(q)^T(h - Tx) \geq -\kappa \|q\| (\|h\| + \|T\|\|x\|)$$

By (iii), we have that

$$\mathbb{E}[\|q\|\|h\|] < +\infty \text{ and } \mathbb{E}[\|q\|\|T\|] < +\infty$$

and it can be concluded that $Q(\bullet)$ is well defined and $Q(x) > -\infty$ for all $x \in \mathbb{R}^n$.

(b): Since $s_q(\bullet)$ is lower semicontinuous in $\bullet$, it follows that $Q(\bullet)$ is also lower semicontinuous by Fatou’s Lemma\(^\dag\)

\(^\dag\)Just to jog your memory, Fatou’s Lemma allows for interchanging integrals and limits under assumptions on the integrand.
Lemma 3 (Fatou’s Lemma) Suppose that there exists a $P$-integrable function $g(\omega)$ such that $f_n(\bullet) \geq g(\bullet)$. Then

$$\liminf_{n \to \infty} \mathbb{E}[f_n] \geq \mathbb{E}\left[\liminf_{n \to \infty} f_n\right].$$

Convexity and closedness of $\text{dom}Q$ is a consequence of the convexity and lower semicontinuity of $Q$. By Prop. 10, $Q(x) < +\infty$ if and only if $h(\xi) - T(\xi)x \in \text{pos} W$ with probability one. But this implies that

$$\text{dom}Q = \{x \in \mathbb{R}^n : Q(x) < +\infty\} = \{x \in \mathbb{R}^n : h(\xi) - T(\xi)x \in \text{pos} W, \text{ with probability one.}\}.$$

(c): In the remainder of the proof, we show the Lipschitz continuity of

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A random variable $Z : \Omega \to \mathbb{R}^d$ is $P$-integrable if $\mathbb{E}[Z]$ is well-defined and finite. Recall that $\mathbb{E}[Z(\omega)]$ is well-defined if it does not happen that $\mathbb{E}[Z(\omega)_+]$ and $\mathbb{E}[(-Z(\omega))_+]$ are $+\infty$ in which case $\mathbb{E}[Z] = \mathbb{E}[Z_+] - \mathbb{E}[(-Z)_+]$. 

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\( \mathcal{Q}(x) \). Consider \( x_1, x_2 \in \text{dom} \mathcal{Q} \). Then, we have

\[ h(\xi) - T(\xi)x_1 \in \text{pos} W, \text{ w.p.1} \quad \text{and} \quad h(\xi) - T(\xi)x_2 \in \text{pos} W, \text{ w.p.1}. \]

By leveraging this claim, if \( \Pi(q) \) is nonempty, we have that

\[ |s_q(h - Tx_1) - s_q(h - Tx_2)| \leq \kappa \|q\| \|T\| \|x_1 - x_2\|. \]

Taking Expectations on both sides, we have that

\[ |s_q(h - Tx_1) - s_q(h - Tx_2)| \leq \kappa \|q\| \|T\| \|x_1 - x_2\|. \]
\[ |\mathcal{Q}(x_1) - \mathcal{Q}(x_2)| \leq \mathbb{E}[|s_q(h - Tx_1) - s_q(h - Tx_2)|] \]
\[ \leq \kappa \mathbb{E}[\|q\|\|T\|\|x_1 - x_2\|], \]

where the first inequality follows from the application of Jensen’s inequality** and the convexity of the norm. The Lipschitz continuity of \( \mathcal{Q}(x) \) on its domain follows.

\[ \blacksquare \]

**Recall that Jensen’s inequality implies that \( f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \), when \( f \) is a convex function and \( \mathbb{E}[X] \) denotes the expectation of a random variable \( X \).
References