Outline

- Vectors, Matrices, Norms, Convergence
- Open and Closed Sets
- Special Sets: Subspace, Affine Set, Cone, Convex Set
- Special Convex Sets: Hyperplane, Polyhedral Set, Ball
- Special Cones: Normal Cone, Dual Cone, Tangent Cone
- Some Operations with Sets:
  - Set Closure, Interior, Boundary
  - Convex Hull, Conical Hull, Affine Hull
  - Relative Closure, Interior, Boundary
- Functions
- Special Functions: Linear, Quadratic
Notation: Vectors and Matrices

We consider the space of $n$-dimensional real vectors denoted by $\mathbb{R}^n$

- Vector $x \in \mathbb{R}^n$ is viewed as a column: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- The prime denotes a transpose: $x' = [x_1, \ldots, x_n]$
- For a matrix $A$, we use $a_{ij}$ or $[A]_{ij}$ to denote its $(i, j)$-th entry
- The matrix $A$ is **positive semidefinite** ($A \geq 0$) when
  $$x'Ax \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$
- The matrix $A$ is **positive definite** ($A > 0$) when
  $$x'Ax > 0 \quad \text{for all } x \in \mathbb{R}^n, \; x \neq 0$$
Inner Product and Norms

- For vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \), the inner product is
  \[
  x'y = \sum_{i=1}^{n} x_i y_i
  \]

- **Norm** (length) of a vector:
  - Euclidean norm \( \|x\| = \sqrt{x'x} = \sqrt{\sum_{i=1}^{n} x_i^2} \) (\( l_2 \)-norm)
  - Sum-norm \( \|x\|_1 = \sum_{i=1}^{n} |x_i| \) (\( l_1 \)-norm)
  - Max-norm \( \|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i| \) (\( l_\infty \)-norm)

- Matrix norm induced by (Euclidean) vector norm
  \[
  \|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\| = 1} \frac{\|Ax\|}{\|x\|}
  \]
Norm Properties, Distance, Orthogonal Vectors

- **Properties of a norm**
  - Nonnegativity: \( \|x\| \geq 0 \) for every \( x \in \mathbb{R}^n \)
  - Uniqueness of the zero-value: \( \|x\| = 0 \) if and only if \( x = 0 \)
  - Homogeneity: \( \|\lambda x\| = |\lambda| \|x\| \) for any scalar \( \lambda \) and any \( x \in \mathbb{R}^n \)
  - Triangle relation: \( \|x + y\| \leq \|x\| + \|y\| \) for every \( x \) and \( y \in \mathbb{R}^n \)

- **(Euclidean) Distance** between two vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \)
  \[
  \|x - y\| = \sqrt{n \sum_{i=1}^{n} (x_i - y_i)^2}
  \]

- Vectors \( x \) and \( y \) are **orthogonal** when \( x'y = 0 \)

- For orthogonal vectors \( x \) and \( y \):
  - **Pythagorean Theorem**: \( \|x + y\|^2 = \|x\|^2 + \|y\|^2 \)
  - **Schwartz Inequality**: \( |x'y| \leq \|x\| \|y\| \)
Vector Sequence

Given a vector sequence \( \{x_k\} \subset \mathbb{R}^n \), we say that

- The sequence is **bounded** when for some scalar \( C \), we have
  \[ \|x_k\| \leq C \quad \text{for all } k \]

- The sequence **converges to a vector** \( \tilde{x} \in \mathbb{R}^n \) when
  \[ \lim_{k \to \infty} \|x_k - \tilde{x}\| = 0 \quad (x_k \to \tilde{x}) \]

- Vector \( y \) is an **accumulation point of the sequence** when there is a subsequence \( \{x_{k_i}\} \) such that \( x_{k_i} \to y \) as \( i \to \infty \)

- Example: \( x_k = (1 + (-1)^k a)^k \) with \( 0 < a < 1 \). Accum. points?

- **Bolzano Theorem**
  A bounded sequence \( \{x_k\} \subset \mathbb{R}^n \) has at least one limit point
Set Topology

Typically denoted by capital letters $C$, $X$, $Y$, $Z$, etc.

Let $X$ be a set in $\mathbb{R}^n$.

- A vector $x$ is an accumulation (or a limit) point of the set $X$ when there is a sequence $\{x_k\} \subseteq X$ such that $x_k \to x$.
- The set $X$ is closed when it contains all of its accumulation points.
- The set $X$ is open when its complement set $X^c = \{x \in \mathbb{R}^n \mid x \not\in X\}$ is closed.
  - $\mathbb{R}^n$ and $\emptyset$ are the only sets that are both open and closed.
- The set $X$ is bounded when for some $C$: $\|x\| \leq C$ for all $x \in X$.
- (Th.) The set $X$ is compact when it is both closed and bounded.

Let $\{X_i \mid i \in I\}$ be a family of closed sets $X_i \subseteq \mathbb{R}^n$.

- The intersection $\bigcap_{i \in I} X_i$ is closed.
- When the set $I$ is finite ($I = \{1, \ldots, m\}$) the union $\bigcup_{i \in I} X_i$ is closed.
Special Sets

Let $X$ be a set in $\mathbb{R}^n$. We say that $X$ is:

- **subspace** when $\alpha x + \beta y \in X$ for every $x, y \in X$ and any $\alpha, \beta \in \mathbb{R}$
  - Example: $\{x \mid x'e = 0\}$ where $e \in \mathbb{R}^n$ with $e_i = 1$ for all $i$
  - Is a subspace a closed set? Bounded?

- **affine** when it is a translated subspace:
  
  $X = x + S$ for an $x \in X$ and a subspace $S$

- **cone** when $\lambda x \in X$ for every $\lambda \geq 0$ and every $x \in X$

- **convex** when $\alpha x + (1 - \alpha)y \in X$ for any $x, y \in X$ and $\alpha \in [0, 1]$

[Diagram of a triangle and a line segment illustrating convexity and affine properties]
Convex Sets

Is a subspace convex set? Affine set?

Special convex sets:

- (Euclidean) **Ball** centered at $x_0 \in \mathbb{R}^n$ with a radius $r > 0$

  $$\{ x \in \mathbb{R}^n \mid \|x - x_0\| \leq r \}$$  \hspace{1cm} \text{(closed ball)}

  $$\{ x \in \mathbb{R}^n \mid \|x - x_0\| < r \}$$  \hspace{1cm} \text{(open ball)}

- A **closed ball is convex**; An **open ball is convex**
More Convex Sets

- **Half-space** with normal vector $a \neq 0$: \( \{ x \mid a'x \leq b \} \)
- **Hyperplane** with normal vector $a \neq 0$: \( \{ x \mid a'x = b \} \) (\( b \) is a scalar)

- **Polyhedral set** (given by finitely many linear inequalities):
  \[
  \{ x \mid a'_i x \leq b_i, \ i = 1, \ldots, m \} = \{ x \mid Ax \leq b \}
  \]
  $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$
Dual Cone

Let \( K \subseteq \mathbb{R}^n \) be a cone.

**Def.** A dual cone of \( K \) is the following set

\[
K^* = \{d \in \mathbb{R}^n \mid d'y \geq 0 \quad \text{for all } y \in K\}
\]

- Dual cone appears in formulations of complementarity problems
- Dual cone is always closed and convex (regardless of \( K \))
Normal Cone of a Set

Let $X \subseteq \mathbb{R}^n$ be a nonempty set, and let $\hat{x} \in X$.

**Def.** A **normal cone of the set $X$ at the point** $\hat{x}$ is the following set

$$N(\hat{x}; X) = \{ y \in \mathbb{R}^n \mid y'(x - \hat{x}) \leq 0 \text{ for all } x \in X \}$$

Vectors in this set are called **normal vectors to the set $X$ at $\hat{x}$**

- Normal cone plays an important role in **optimality conditions**
- Normal cone is always **closed and convex** (regardless of $X$)
Tangent Cone of a Set

Let \( X \subseteq \mathbb{R}^n \) be a nonempty set, and let \( \hat{x} \in X \).

**Def.** A tangent cone of the set \( X \) at the point \( \hat{x} \) is the following set

\[
T(\hat{x}; X) = \left\{ y \in \mathbb{R}^n \mid y = \lim_{k \to \infty} \frac{x_k - \hat{x}}{\lambda_k} \quad \text{with} \quad \{x_k\} \subseteq X, \{\lambda_k\} \subseteq \mathbb{R}_{++} \right\}
\]

s. t. \( x_k \to \hat{x}, \lambda_k \to 0 \)

Vectors in this set are called tangent vectors to the set \( X \) at \( \hat{x} \)

**Prop.** The tangent cone \( T(\hat{x}; X) \) is equivalently given by

\[
T(\hat{x}; X) = \left\{ \beta y \in \mathbb{R}^n \mid y = \lim_{k \to \infty} \frac{x_k - \hat{x}}{\|x_k - \hat{x}\|} \quad \text{with} \quad \{x_k\} \subseteq X, \right. \\
\left. \quad x_k \to \hat{x}, x_k \neq \hat{x}, \beta \geq 0 \right\}
\]
Tangent Cone Illustration

Tangent cone $T(\hat{x}; X)$ for any set $X$ and any $\hat{x} \in X$:

- plays an important role in optimality conditions
- **always closed** regardless of the properties of $X$
- when $X$ is convex, then $T(\hat{x}; X)$ is **convex**
Tangent and Normal Cone

Let $X \subset \mathbb{R}^n$ and let $\hat{x} \in X$. Then, we have $-N(\hat{x}; X) \subseteq T(\hat{x}; X)^*$

Prop. If $X$ is convex, then

$$T(\hat{x}; X)^* = -N(\hat{x}; X) \quad \text{for all } \hat{x} \in X$$
Set Interior, Closure and Boundary

Let $X \subseteq \mathbb{R}^n$.

- A vector $\hat{x} \in X$ is an **interior point of** $X$ if there exist a ball $B(\hat{x}, r)$ such that $B(\hat{x}, r) \subset X$.
- The **interior of the set** $X$ is the set of all interior points of $X$; it is denoted by $\text{int}(X)$.
- A vector $\hat{x} \in X$ is a **accumulation point of** $X$ if there exist a sequence $\{x_k\} \subset X$ converging to $\hat{x}$.
- The **closure of the set** $X$ is the set of all accumulation points of $X$; it is denoted by $\text{cl}(X)$.
- The **boundary of the set** $X$ is the intersection of the closure set of $X$ and the closure set of its complement ($\mathbb{R}^n \setminus X$), i.e.,

$$\text{bd}(X) = \text{cl}(X) \cap \text{cl}(\mathbb{R}^n \setminus X)$$

**Example:** $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \ 0 < x_2 < 1\}$
Convex and Conical Hulls

A convex combination of vectors $x_1, \ldots, x_m$ is a vector of the form

$$\alpha_1 x_1 + \ldots + \alpha_m x_m \quad \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{m} \alpha_i = 1$$

The convex hull of a set $X$ is the set of all convex combinations of the vectors in $X$, denoted $\text{conv}(X)$.

A conical combination of vectors $x_1, \ldots, x_m$ is a vector of the form

$$\lambda_1 x_1 + \ldots + \lambda_m x_m \quad \text{with } \lambda_i \geq 0 \text{ for all } i$$

The conical hull of a set $X$ is the set of all conical combinations of the vectors in $X$, denoted by $\text{cone}(X)$. 
Affine Hull

An *affine combination* of vectors $x_1, \ldots, x_m$ is a vector of the form

$$t_1 x_1 + \ldots + t_m x_m$$

with $\sum_{i=1}^{m} t_i = 1$, $t_i \in \mathbb{R}$ for all $i$

The *affine hull* of a set $X$ is the set of all affine combinations of the vectors in $X$, denoted $\text{aff}(X)$

The *dimension* of a set $X$ is the dimension of the affine hull of $X$

$$\dim(X) = \dim(\text{aff}(X))$$
Relative Interior, Closure, and Boundary

Let $X \subseteq \mathbb{R}^n$.

- A vector $\hat{x} \in X$ is a relative interior point of $X$ if there exist a ball $B(\hat{x}, r)$ such that $B(\hat{x}, r) \cap \text{aff}(X) \subset X$

- The relative interior of the set $X$ is the set of all interior points of $X$; it is denoted by $\text{rint}(X)$

- The relative boundary of the set $X$ is the intersection of the affine hull $\text{aff}(X)$ and the boundary $\text{bd}(X)$

Example: $X = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \ x_2 = 1 \right\}$

$\text{aff}(X) =$?
Mappings

Let $X \subseteq \mathbb{R}^n$.

- We write $F : X \to \mathbb{R}^m$ to denote a mapping $F$ from the set $X$ to $\mathbb{R}^m$.
- $F$ is a \textbf{function} when $F : X \to \mathbb{R} \ (m = 1)$.
- We refer to $X$ as a \textbf{domain} of the mapping $F$.
- The \textbf{image of $X$ under mapping} $F$ is the following set:

$$F(X) = \{ y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in X \}$$

- The \textbf{inverse image of $Y \subseteq \mathbb{R}^m$ under mapping} $F$ is the following set:

$$F^{-1}(Y) = \{ x \in X \mid F(x) \in Y \}$$
Special Mappings and Functions

A mapping $F : X \to \mathbb{R}^m$ with $X \subseteq \mathbb{R}^n$ is

- **affine** when for all $x \in X$,

$$F(x) = Ax + b \quad \text{for some matrix } A \text{ and a vector } b \in \mathbb{R}^m$$

*Example:* Space translation (by a given $x_0 \in \mathbb{R}^n$) is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$: $F(x) = x + x_0$ for all $x \in \mathbb{R}^n$

- **linear** when for all $x \in X$, we have $F(x) = Ax$ for a matrix $A$

*Example:* For a coordinate subspace $S = \{x \in \mathbb{R}^n \mid x_i = 0, \text{ for } i \in I\}$, where $I$ is a subset of coordinate indices ($I \subseteq \{1, \ldots, n\}$, the projection on $S$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$F(x) = Px \quad \text{with } [P]_{ii} = 1 \text{ for } i \notin I \text{ and } P_{ij} = 0 \text{ otherwise}$$
A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is

- **quadratic** when for all $x$
  $$f(x) = x'Qx + a'x + b$$
  for a matrix $Q$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

- **affine** when for all $x$
  $$f(x) = a'x + b$$
  for a vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

- **linear** when for all $x$
  $$f(x) = a'x$$
  for a vector $a \in \mathbb{R}^n$
References for this lecture

The material for this lecture:

• (B) Bertsekas D.P. *Nonlinear Programming*
  * See Appendix there for topology in \( \mathbb{R}^n \) and linear algebra

• (BNO) Bertsekas, Nedić, Ozdaglar *Convex Analysis and Optimization*
  Chapter 1 for convex sets and functions

• (FP) Facchinei and Pang *Finite Dimensional ..., Vol I*
  * Chapter 1 for Normal Cone, Dual Cone, and Tangent Cone

• **NOTE!!!**
  * Normal cone definition used in (FP) is different than that of (B,BNO)