Homework 1
Due 01/24/2017

Exercise 1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, \ x_2 \geq 0\}$.

(a) Is $X$ a cone? Answer yes or no, and justify your answer.

(b) Find the dual cone of $X$?

(c) Find the normal cone $N(\hat{x}; X)$ and the tangent cone $T(\hat{x}; X)$ to the set $X$ at each of the following choices for $\hat{x}$:
   
   (i) $\hat{x} = (0, 0)$,
   
   (ii) $\hat{x} = (0, 1)$.

Exercise 2. Let $X = (0, 0) \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, \ x_2 > 0\}$.

(a) Is $X$ a cone? Answer yes or no, and justify your answer.

(b) Find the dual cone of $X$?

(c) Find the normal cone $N(\hat{x}; X)$ and the tangent cone $T(\hat{x}; X)$ to the set $X$ at $\hat{x} = (0, 0)$.

Exercise 3. In this exercise, we use the notion of a (lower) level set of a function, defined as follows.

Definition 1. The (lower) level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}.$$

When the function is convex, we refer to this set simply by a level set.

Consider now a convex function $f : \mathbb{R}^n \to \mathbb{R}$. Let $\hat{x}$ be a given vector in $\mathbb{R}^n$, and consider the level set $X$ associated with $\hat{x}$, i.e.,

$$X = \{x \in \mathbb{R}^n \mid f(x) \leq f(\hat{x})\}.$$
Prove the following:

(a) Show that the set $X$ is convex.

(b) Assuming that $f$ is continuously differentiable over $\mathbb{R}^n$, show that the tangent cone $T(\hat{x}; X)$ and the normal cone $N(\hat{x}; X)$ of the level set $X$ at the point $\hat{x}$ are given by

$$T(\hat{x}; X) = \{d \mid \nabla f(\hat{x})'d \leq 0\},$$

$$N(\hat{x}; X) = \{\lambda \nabla f(\hat{x}) \mid \lambda \geq 0\}.$$

*Hint:* You may find it useful to exploit convexity of $f$ and the first-order Taylor expansion of $f$ at $\hat{x}$.

**Exercise 4.** Prove the following:

(a) Suppose $g(x) = f(Ax)$ where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to (-\infty, +\infty]$ and $f : \mathbb{R}^m \to (-\infty, +\infty]$. If $f$ is convex, then $g$ is convex.

(b) Suppose for $i \in I$ (an arbitrary index set), $f_i : \mathbb{R}^n \to (-\infty, +\infty]$ and consider the function $g : \mathbb{R}^n \to (-\infty, +\infty]$ defined as

$$g(x) = \sup_{i \in I} f_i(x).$$

Suppose the $f_i$ are convex for all $i$. Then $g$ is a convex function. (hint: use the notion of epigraphs)

**Exercise 5.** Consider the halfspace defined by $H = \{x \in \mathbb{R}^n : a'x + \alpha \geq 0\}$, where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Formulate and solve the optimization problem of finding the point in $H$ that has the smallest Euclidean norm.

**Exercise 6 (Extra credit).** Let $X_1$ and $X_2$ be two sets with and let $\hat{x} \in X_1 \cap X_2$. Prove that the following relations hold for the tangent cones and the normal cones of $X_1$, $X_2$ and $X_1 \cap X_2$ at the point $\hat{x}$:

$$T(\hat{x}; X_1 \cap X_2) \subseteq T(\hat{x}; X_1) \cap T(\hat{x}; X_2),$$

$$N(\hat{x}; X_1) \cap N(\hat{x}; X_2) \subseteq N(\hat{x}; X_1 \cap X_2).$$