MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
MIDTERM SOLUTIONS

Problem 1. What is the definition of...

- an ideal in a polynomial ring?

  Solution. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and $I \subset R$. I is called an ideal of R if
  (1) $0 \in I$
  (2) If $a, b \in I$, then $a + b \in I$.
  (3) If $a \in I$ and $b \in k[x_1, \ldots, x_n]$, then $a \cdot b \in I$.

- an affine variety?

  Solution. A set V in the affine space k^n is called an (affine) variety it is the zero locus of a set of polynomials $\{f_1, \ldots, f_s\}$ for $f_i \in k[x_1, \ldots, x_n]$.

- a Gröbner basis?

  Solution. Let I be an ideal of a polynomial rink $k[x_1, \ldots, x_n]$. A Gröbner basis $\{g_1, \ldots, g_s\}$ of I satisfies the following:
  (1) $I = \langle g_1, \ldots, g_s \rangle$
  (2) $in_<(I) = \langle in_<(g_1), \ldots, in_<(g_s) \rangle$

- the Zariski closure of a set in affine space?

  Solution. Let $A \subset k^n$. The Zariski closure of A is the smallest algebraic variety V that contains A.

- a prime ideal in a polynomial ring?

  Solution. Let $R = k[x_1, \ldots, x_n]$ and let $I \subsetneq R$ be an ideal. We say I is prime if and only if

  $$ab \in I \iff a \in I \text{ or } b \in I$$

Problem 2. Let $I = \langle x^2, xy - z \rangle$ be an ideal in $\mathbb{C}[x, y, z]$.

(1) Is $(x^2, xy - z)$ a Gröbner basis for I with respect to the graded lexicographic order with $x > y > z$?

  Solution. No, this is not a Gröbner basis.

(2) Why or why not? If not, provide such a Gröbner basis.

  Solution. Consider $y(x^2) - x(xy - z) = xz$ but $xz \notin \langle x^2, xy \rangle$. So the basis does not generate the initial ideal.
  Let $f = x^2$ and $g = xy - z$. Form the S-polynomial $S(f, g) = y(x^2) - x(xy - z) = xz$. The result xz is not divisible by the leading term of f (x^2) or g (xy), so add $h = xz$ to

the list. Now $S(f,g)$ reduces to zero, and $S(f,h) = zf - xh = 0$. This leaves $S(g,h) = z(xy-z) - y(xz) = -z^2$, so add $\ell = z^2$ to the basis. Checking again, all now reduce to zero, so (f, g, h, ℓ) is our basis.

(3) What is the dimension of $V(I)$?

Solution. The dimension is one.

Since $x^2 \in I$, $V(I)$ is contained in the $yz-$plane. Furthermore, we have $z = xy$. Hence, the solution set is all points of the form $(0, y, 0)$. So it is one-dimensional.

(4) What is the radical of I?

Solution. From the previous part, we know the radical is just $\langle x, z\rangle$. Alternatively, we can compute this directly.

$$\begin{aligned} V(x^2, xy - z) &= V(x^2) \cap V(xy - z) \\ &= [V(x) \cup V(x)] \cap V(xy - z) \\ &= V(x) \cap V(z) = V(x, z) \end{aligned}$$

Problem 3. Consider the polynomial parameterization $x = a^2b, y = ab^2, z = ab$.

(1) Find an equation in $\mathbb{C}[x, y, z]$ satisfied by every point in the image (an equation in the implicitization ideal)

Solution. $z^3 - xy$

(2) Is this the only equation?

Solution. Yes. It is parametrized by two parameters, a, b and it lives in a three dimensional space. We expect it to be a hypersurface.

(3) Why or why not?

Solution. We know the closure of the image is irreducible, so we just need to check that the dimension is 2; then it must be a hypersurface, defined by one equation. This can be done e.g. by computing the Jacobian of the parameterization map, or by checking that x, y is a standard set.

Problem 4. Dickson's Lemma says that given a possibly infinite set of monomials generating a monomial ideal J, we can choose finitely many from the given set that also generate J. Use this and your knowledge of the division algorithm to prove Hilbert's Basis Theorem for a nonzero ideal I in a multivariate polynomial ring.

Solution. Proof is on page 76-77 of CLO.

Problem 5. **(5 pts)** Let $I \subset \mathbb{C}[x, y, z]$ be given by $I = \langle -x^2y + y^2, -x^3y + yz\rangle$.

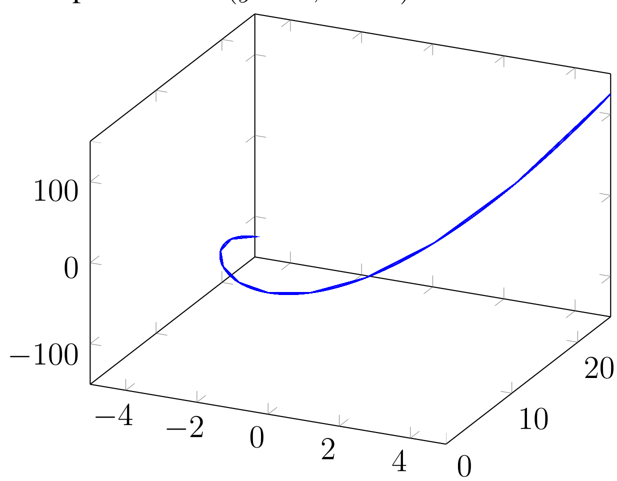
(1) Decompose $V(I)$ into irreducible components.

Solution. This decomposes into $V(y-x^2, z-x^3)$ the twisted cubic and $V(y)$ a coordinate plane.

$$\begin{aligned} V(-x^2y+y^2, -x^3y+yz) &= V(y(y-x^2), y(z-x^3)) \\ &= V(y(y-x^2)) \cap V(y(z-x^3)) \\ &= [V(y) \cup V(y-x^2)] \cap [V(y) \cup V(z-x^3)] \\ &= V(y) \cup [V(y-x^2) \cap V(z-x^3)] \\ &= V(y) \cup V(y-x^2, z-x^3) \end{aligned}$$

(2) Describe the components

Solution. $V(y)$ is the $xz-$plane and $V(y-x^2, z-x^3)$ is the twisted cubic (depicted below).



(3) Show that the components are irreducible

Solution. It is clear that $V(y)$ is irreducible since $\langle y \rangle$ is a prime ideal (this follows from Proposition 6 on page 198).

The twisted cubic, $V(y-x^2, z-x^3)$, can be parametrized in the following way.

$$\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$$

Since this is a rational parametrization, by Proposition 6 (pg 200), this is an irreducible variety.

(4) What is the dimension of each component, and of $V(I)$?

Solution. As a result the dimension of $V(I)$ is the dimension of the larger component, the plane, which is two. The dimension of the twisted cubic is one.