MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
HOMEWORK 3 SOLUTIONS

Assigned 9/14, due 9/18 in class.

Problem 1. Working with your group, implement the division algorithm (Chapter 2, Section 3, Theorem 3 in CLO).

Solution. The code is written below.

def polydivset(f, g):
    R = f.parent()
    n = len(g)
    p, r, a = f, 0, [R.zero() for i in range(n)]
    while p != 0:
        i, flag = 0, False
        while i < n and not flag:
            if R.monomial_divides(g[i].lt(), p.lt()) == True:
                q = R(p.lt()/g[i].lt())
                p = p - q*g[i]
                flag = True
            else:
                i = i + 1
        if flag == False:
            r, p = r + p.lt(), p - p.lt()
    return a, r

Do Section 3, Exercises 5 and 6 with your division algorithm as well.
In Chapter 2 Section 4, do Exercise 12. Note there is a typo, it should say “or \( u \cdot \alpha = u \cdot \beta \) and \( \alpha \gg_\sigma \beta \).”

§3 : 1.) Compute the remainder on division of the given polynomial \( f \) by the order set \( F \). Use the grlex order, then the lex order in each case.
a.) \( f = x^7y^2 + x^3y^2 - y + 1, F = (xy^2 - x, x - y^3) \)

Solution. The code is below

1 R.<x, y> = PolynomialRing(QQ, order = 'deglex');
2 F = [x*y^2 - x, x - y^3];
3 f = x^7*y^2 + x^3*y^2 - y + 1;
4 polydivset(f, F)
5 (\([x^6 + x^2, 0], x^7 + x^3 - y + 1\))
\textbf{Solution.} The code is below
\begin{verbatim}
1 R.<x, y> = PolynomialRing(QQ,order = 'degrevlex');
2 F = [x*y^2 - x, x - y^3];
3 f = x^7*y^2 + x^3*y^2 - y +1;
4 polydivset(f,F)
5 ([x^6 + x^2, 0], x^7 + x^3 - y + 1)
\end{verbatim}

§3 : 2.) Compute the remainder on division:
\textbf{a.)} \( f = xy^2z^2 + xy - yz, F = (x - y^2, y - z^3, z^2 - 1) \)

\textbf{Solution.} The code is below
\begin{verbatim}
1 R.<x, y, z> = PolynomialRing(QQ);
2 F = [x - y^2, y - z^3, z^2 - 1];
3 f = x*y^2*z^2 + x*y - y*z;
4 polydivset(f,F)
5 ([-x*z^2, 0, x^2], x^2 + x*y - y*z)
\end{verbatim}

\textbf{b.)} Repeat part a with the order of the set \( F \) permuted cyclically.

\textbf{Solution.} The code is below
§3 : 5.) We will study the division of \( f = x^3 - x^2y - x^2z + x \) by \( f_1 = x^2y - z \) and \( f_2 = xy - 1 \).

a.) Compute using grilled order:

\[
\begin{align*}
\text{r}_1 &= \text{remainder of } f \text{ on division by } (f_1,f_2) \\
\text{r}_2 &= \text{remainder of } f \text{ on division by } (f_2,f_1)
\end{align*}
\]

Your results should be different. Where in the division algorithm did the difference occur? (You may need to do a few steps by hand here.)

**Solution.** The results are computed below.

\[
\begin{align*}
\text{r}_1 &= \text{remainder of } f \text{ on division by } (f_1,f_2) \\
\text{r}_2 &= \text{remainder of } f \text{ on division by } (f_2,f_1)
\end{align*}
\]

b.) Is \( r = r_1 - r_2 \) in the ideal \( \langle f_1, f_2 \rangle \)? If so, find an explicit expression \( r = Af_1 + Bf_2 \). If not, say why not.
Solution. $r = x - x^2y$. This is clearly in the ideal since
\[ r = 0 \cdot f_1 - x \cdot f_2 \]
c.) Compute the remainder of $r$ on division by $(f_1, f_2)$. Why could you have predicted your answer before doing the division?

Solution. The remainder is 0 since $r$ is in the ideal.
d.) Find another polynomial $g \in \langle f_1, f_2 \rangle$ such that the remainder on division of $g$ by $(f_1, f_2)$ is nonzero.

Solution. Let $g = yz - xy$. Notice that $-y(x^2y - z) + xy(xy - 1) = -x^2y^2 + yz + x^2y^2 - xy = g$, so $g$ is in the ideal.

```
1 R.<x, y, z> = PolynomialRing(QQ);
2 F = [x^2*y - z, x*y - 1];
3 g = y*z + x*y;
4 polydivset(g,F)
5 ([0, -1], y*z - 1)
```

e.) Does the division algorithm give us a solution for the ideal membership problem for the ideal $\langle f_1, f_2 \rangle$? Explain your answer.

Solution. No, it does not on its own since elements that are in the ideal may not yield a zero remainder.

§3 : 6.) Using the grlex order, find an element $g$ of $\langle f_1, f_2 \rangle = \langle 2xy^2 - x, 3x^2y - y - 1 \rangle \subset \mathbb{R}[x,y]$ whose remainder on division by $(f_1, f_2)$ is nonzero.

Solution. Consider
\[ g = -\frac{3}{2}x(2xy^2 - x) + y(3x^2y - y - 1) = -3x^2y^2 + \frac{3x^2}{2} + 3x^2y^2 - y^2 - y \]

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1 R.<x, y, z> = PolynomialRing(QQ);
2 F = [2*x*y^2 - x, 3*x^2*y - y - 1];
3 g = 3*x^2/2 - y^2 - y;
4 polydivset(g,F)
5 ([0, 0], 3/2*x^2 - y^2 - y)
```
Another important weight order is constructed as follows. Let \( u = (u_1, \ldots, u_n) \) be in \( \mathbb{Z}_\geq 0^n \), and fix a monomial order \( >_\sigma \) (such as \( >_{\text{lex}} \) or \( >_{\text{grevlex}} \)) on \( \mathbb{Z}_\geq 0^n \). Then for \( \alpha, \beta \in \mathbb{Z}_\geq 0^n \), define \( \alpha >_{u, \sigma} \beta \) if and only if
\[
u \cdot \alpha > u \cdot \beta \quad \text{or} \quad u \cdot \alpha = u \cdot \beta \quad \text{and} \quad \alpha > \sigma \beta.
\]
We call \( >_{u, \sigma} \) the weight order determined by \( u \) and \( >_\sigma \).

a.) Use Corollary 6 to prove that \( >_{u, \sigma} \) is a monomial order.

**Solution.** Since the dot product yields elements in \( \mathbb{Z} \), which is totally ordered by \( > \) and \( >_\sigma \) is a total order by definition, it follows that \( >_{u, \sigma} \) is a total order.

Suppose \( u \cdot \alpha > u \cdot \beta \). Then consider
\[
u \cdot (\alpha + \gamma) = u \cdot \alpha + u \cdot \gamma > u \cdot \beta + u \cdot \gamma = u (\beta + \gamma).
\]
Alternatively, suppose \( \alpha > \sigma \beta \). By definition, \( >_\sigma \) satisfies this property.

b.) Find \( u \in \mathbb{Z}_\geq 0^n \) so that weight order \( >_{u, \text{lex}} \) is the order \( >_{\text{grevlex}} \).

**Solution.** If \( u = (1, 1, \ldots, 1) \), then we get the graded lexicographic ordering.

**Remark 0.1.** Projection onto the diagonal results in the \( \ell_1 \) norm, which is equivalent to ordering by degrees.

c.) In the definition of \( >_{u, \sigma} \), the order \( >_\sigma \) is used to break ties, and it turns out that ties will always occur in this case. More precisely, prove that given \( u \in \mathbb{Z}_\geq 0^n \), there are \( \alpha \neq \beta \) in \( \mathbb{Z}_\geq 0^n \) such that \( u \cdot \alpha = u \cdot \beta \).

**Solution.** Let \( u = (u_1, u_2, \ldots, u_n) \). Define \( \alpha = (u_2, 0, \ldots, 0) \) and \( \beta = (0, u_1, \ldots, 0) \). Then \( u \cdot \alpha = u_1 u_2 = u \cdot \beta \).

d.) A useful example of a weight order is the elimination order introduced by Bayer and Stillman (1987b). Fix an integer \( 1 \leq i \leq n \) and let \( u = (1, \ldots, 1, 0, \ldots, 0) \), where there are \( i \) 1s and \( n - i \) 0s. Then the \( i \)th elimination order \( >_i \) is the weight order \( >_{u, \text{grevlex}} \).

Prove that \( >_i \) has the following property: if \( x^\alpha \) is a monomial in which one of \( x_1, \ldots, x_i \) appears, then \( x^\alpha > x^\beta \) for any monomial involving only \( x_{i+1}, \ldots, x_n \).

**Solution.** Let \( \alpha \) and \( \beta \) be as described. Then \( u \cdot \alpha > 0 = u \cdot \beta \).