MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
HOMEWORK 3 SOLUTIONS

Assigned 9/14, due 9/18 in class.

Problem 1. Working with your group, implement the division algorithm (Chapter 2, Section 3, Theorem 3 in CLO).

Solution. The code is written below.

```
def polydivset(f,g):
    R = f.parent()
    n = len(g)
    p, r, a = f, 0, [R.zero() for i in range(n)]
    while p != 0:
       i, flag = 0, False
       while i < n and not flag:
          if R.monomial_divides( g[i].lt(), p.lt() ) == True:
             q = R(p.lt()/g[i].lt())
             p = p - q*g[i]
             flag == True
          else:
             i = i + 1
       if flag == False:
          r, p = r + p.lt(), p - p.lt()
    return a, r
```

Do Section 3, Exercises 5 and 6 with your division algorithm as well.

In Chapter 2 Section 4, do Exercise 12. Note there is a typo, it should say "or $u \cdot \alpha = u \cdot \beta$ and $\alpha >_\sigma \beta$."

§3 : 1.) Compute the remainder on division of the given polynomial f by the order set F. Use the grlex order, then the lex order in each case.

a.) $f = x^7y^2 + x^3y^2 - y + 1, F = (xy^2 - x, x - y^3)$

Solution. The code is below

```
R.<x, y> = PolynomialRing(QQ,order = 'deglex');
F = [x*y^2 - x, x - y^3];
f = x^7*y^2 + x^3*y^2 - y +1;
polydivset(f,F)
([x^6 + x^2, 0], x^7 + x^3 - y + 1)
```

```
R.<x, y> = PolynomialRing(QQ,order = 'lex');
F = [x*y^2 - x, x - y^3];
f = x^7*y^2 + x^3*y^2 - y +1;
polydivset(f,F)
([x^6 + x^5*y + x^4*y^2 + x^4 + x^3*y + x^2*y^2 +
2*x^2 + 2*x*y + 2*y^2 + 2, x^6 + x^5*y + x^4 + x^3*y +
2*x^2 + 2*x*y + 2], 2*y^3 - y + 1)
```

b.) Repeat part a with the order of the pair F reversed.

Solution. The code is below

```
R.<x, y> = PolynomialRing(QQ,order = 'degrevlex');
F = [x*y^2 - x, x - y^3];
f = x^7*y^2 + x^3*y^2 - y +1;
polydivset(f,F)
([x^6 + x^2, 0], x^7 + x^3 - y + 1)
```

```
R.<x, y> = PolynomialRing(QQ,order = 'invlex');
F = [x*y^2 - x, x - y^3];
f = x^7*y^2 + x^3*y^2 - y +1;
polydivset(f,F)
([x^6 + x^2, 0], -y + x^7 + x^3 + 1)
```

§3 : 2.) Compute the remainder on division:

a.) $f = xy^2z^2 + xy - yz, F = (x - y^2, y - z^3, z^2 - 1)$

Solution. The code is below

```
R.<x, y, z> = PolynomialRing(QQ);
F = [x - y^2, y - z^3, z^2 - 1];
f = x*y^2*z^2 + x*y - y*z;
polydivset(f,F)
([-x*z^2, 0, x^2], x^2 + x*y - y*z)
```

b.) Repeat part a with the order of the set F permuted cyclically.

Solution. The code is below

```
R.<x, y, z> = PolynomialRing(QQ);
F = [y - z^3, z^2 - 1, x - y^2];
f = x*y^2*z^2 + x*y - y*z;
polydivset(f,F)
([0, x*y^2, -x], x^2 + x*y - y*z)
```

```
R.<x, y, z> = PolynomialRing(QQ);
F = [z^2 - 1, x - y^2, y - z^3,];
f = x*y^2*z^2 + x*y - y*z;
polydivset(f,F)
([x*y^2, -x, 0], x^2 + x*y - y*z)
```

§3 : 5.) We will study the division of $f = x^3 - x^2y - x^2z + x$ by $f_1 = x^2y - z$ and $f_2 = xy - 1$.

a.) Compute using grilled order:

$$r_1 = \text{ remainder of } f \text{ on division by } (f_1, f_2)$$
$$r_2 = \text{ remainder of } f \text{ on division by } (f_2, f_1)$$

Your results should be different. Where in the division algorithm did the difference occur? (You may need to do a few steps by hand here.)

Solution. The results are computed below.

```
R.<x, y, z> = PolynomialRing(QQ);
F = [x^2*y - z, x*y - 1];
f = x^3 - x^2*y - x^2*z + x;
polydivset(f,F)
([-1, 0], x^3 - x^2*z + x - z)
```

```
R.<x, y, z> = PolynomialRing(QQ);
F = [x*y - 1, x^2*y - z];
f = x^3 - x^2*y - x^2*z + x;
polydivset(f,F)
([-x, 0], x^3 - x^2*z)
```

b.) Is $r = r_1 - r_2$ in the ideal $\langle f_1, f_2 \rangle$? If so, find an explicit expression $r = Af_1 + Bf_2$. If not, say why not.

Solution. $r = x - x^2y$. This is clearly in the ideal since

$$r = 0 \cdot f_1 - x \cdot f_2$$

c.) Compute the remainder of r on division by (f_1, f_2).Why could you have predicted your answer before doing the division?

Solution. The remainder is 0 since r is in the ideal.

d.) Find another polynomial $g \in \langle f_1, f_2 \rangle$ such that the remainder on division of g by (f_1, f_2) is nonzero.

Solution. Let $g = yz - xy$. Notice that $-y(x^2y - z) + xy(xy - 1) = -x^2y^2 + yz + x^2y^2 - xy = g$, so g is in the ideal.

```
R.<x, y, z> = PolynomialRing(QQ);
F = [x^2*y - z, x*y - 1];
g = y*z + x*y;
polydivset(g,F)
([0, -1], y*z - 1)
```

```
R.<x, y, z> = PolynomialRing(QQ);
F = [x^2*y - z, x*y - 1];
g = y*z - 1;
polydivset(g,F)
([0, 0], y*z - 1)
```

e.) Does the division algorithm give us a solution for the ideal membership problem for the ideal $\langle f_1, f_2 \rangle$? Explain your answer.

Solution. No, it does not on its own since elements that are in the ideal may not yield a zero remainder.

§3 : 6.) Using the grlex order, find an element g of $\langle f_1, f_2 \rangle = \langle 2xy^2 - x, 3x^2y - y - 1 \rangle \subset \mathbb{R}[x, y]$ whose remainder on division by (f_1, f_2) is nonzero.

Solution. Consider

$$g = -\frac{3}{2}x(2xy^2 - x) + y(3x^2y - y - 1) = -3x^2y^2 + \frac{3x^2}{2} + 3x^2y^2 - y^2 - y$$

```
R.<x, y, z> = PolynomialRing(QQ);
F = [2*x*y^2 - x, 3*x^2*y - y - 1];
g = 3x^2/2 - y^2 - y;
polydivset(g,F)
([0, 0], 3/2*x^2 - y^2 - y)
```

§4 : 12.) Another important weight order is constructed as follows. Let $u = (u_1, \ldots, u_n)$ be in $\mathbb{Z}^n_{\geq 0}$, and fix a monomial order $>_\sigma$ (such as $>_{lex}$ or $>_{grevlex}$) on $\mathbb{Z}^n_{\geq 0}$. Then for $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, define $\alpha >_{u,\sigma} \beta$ if and only if

$$u \cdot \alpha > u \cdot \beta \text{ or } u \cdot \alpha = u \cdot \beta \text{ and } \alpha >_\sigma \beta.$$

We call $>_{u,\sigma}$ the weight order determined by u and $>_\sigma$.

a.) Use Corollary 6 to prove that $>_{u,\sigma}$ is a monomial order.

Solution. Since the dot product yields elements in $\mathbb{Z}$, which is totally ordered by $>$ and $>_\sigma$ is a total order by definition, it follows that $>_{u,\sigma}$ is a total order.
Suppose $u \cdot \alpha > u \cdot \beta$. Then consider

$$u \cdot (\alpha + \gamma) = u \cdot \alpha + u \cdot \gamma > u \cdot \beta + u \cdot \gamma = u(\beta + \gamma).$$

Alternatively, suppose $\alpha >_\sigma \beta$. By definition, $>_\sigma$ satisfies this property.

b.) Find $u \in \mathbb{Z}^n_{\geq 0}$ so that weight order $>_{u,lex}$ is the order $>_{grlex}$.

Solution. If $u = (1, 1, \ldots, 1)$, then we get the graded lexicographic ordering.

Remark 0.1. Projection onto the diagonal results in the ℓ_1 norm, which is equivalent to ordering by degrees.

c.) In the definition of $>_{u,\sigma}$, the order $>_\sigma$ is used to break ties, and it turns out that ties will always occur in this case. More precisely, prove that given $u \in \mathbb{Z}^n_{\geq 0}$, there are $\alpha \neq \beta$ in $\mathbb{Z}^n_{\geq 0}$ such that $u \cdot \alpha = u \cdot \beta$.

Solution. Let $u = (u_1, u_2, \ldots, u_n)$. Define $\alpha = (u_2, 0, \ldots, 0)$ and $\beta = (0, u_1, \ldots, 0)$. Then $u \cdot \alpha = u_1 u_2 = u \cdot \beta$.

d.) A useful example of a weight order is the elimination order introduced by Bayer and Stillman (1987b). Fix an integer $1 \leq i \leq n$ and let $u = (1, ..., 1, 0, ..., 0)$, where there are i 1s and $n - i$ 0s. Then the ith elimination order $>_i$ is the weight order $>_{u,\ grevlex}$. Prove that $>_i$ has the following property: if x^α is a monomial in which one of $x_1, ..., x_i$ appears, then $x^\alpha >_i x^\beta$ for any monomial involving only $x_{i+1}, ..., x_n$.

Solution. Let α and β be as described. Then $u \cdot \alpha > 0 = u \cdot \beta$.