MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY HOMEWORK 3 SOLUTIONS

Assigned 9/4, due 9/11 in class.

Problem 1. Show that given a term order $<$ and an ideal $I \subset k[x_1, \ldots, x_n]$, $in_<(I)$ is an ideal.

Solution 1. It follows directly from the definition since

$$in_<(I) = \langle in_< f \ : \ f \in I \rangle \, .$$

That is, it is the ideal generated from the leading terms of elements in I.

Problem 2. Find the initial ideal and a Gröbner basis of

$$\left\langle x^3y - z^2, wxyz - y^2z^2 \right\rangle \subset k[x, y, z, w].$$

Solution 2. The Gröbner basis can be found using sage.

```
R.<x, y, z, w> = PolynomialRing(QQ,4);
I=ideal(x^3*y - z^2, w*x*y*z- y^2*z^2);
B = I.groebner_basis(); B
[x^4*z^3*w - z^6, y*z^4 - x*z^3*w, x^3*y - z^2,
y^2*z^2 - x*y*z*w]
```

The initial ideal is then the highest order terms of the Gröbner basis. This, too, can be directly calculated in SAGE.

```
J = Ideal(f.lm() for f in B); J
Ideal (x^4*z^3*w, y*z^4, x^3*y, y^2*z^2) of
Multivariate Polynomial Ring in x, y, z, w
over Rational Field
```

As we saw in class, these are the leading terms of the Gröbner basis, $\{ x^4z^3w - z^6, yz^4 - xz^3w, x^3y - z^2, y^2z^2 - xyzw \}$.

Question 0.1. What is the term order being used here? If it were to change, how would that change our calculations?

§5 : 9.) Use the method described in the text to decide whether $x^2 - 4$ is in the ideal

$$\left\langle x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2 \right\rangle .$$

Solution. The method outlined in the book asks us to find the GCD of the set of polynomials and then determine membership. We can simply factor all the generators, identify the common factors and determine if these factors are also factors for $x^2 - 4$.

$x^3 + x^2 - 4x - 4 = (x + 1)(x - 2)(x + 2)$

$x^3 - x^2 - 4x + 4 = (x-1)(x-2)(x+2)$
$x^3 - 2x^2 - x + 2 = (x-2)(x-1)(x+1)$
So the $GCD = x - 2$. It is clear that $(x-2)\big|(x^2-4)$, so $x^2 - 4 \in I$.

Alternatively, we could make the observation that

$$x^2 - 4 = \frac{1}{2}\left(x^3 + x^2 - 4x - 4\right) - \frac{1}{2}\left(x^3 - x^2 - 4x + 4\right)$$

§5 : 10.) Give pseudocode for an algorithm that has input $f, g \in k[x]$ and output $h, A, B \in k[x]$ where $h = \gcd(f, g)$ and $Af + Bg = h$.

Solution. Rather than producing the pseudocode, I've written the SAGE code.

```
def Num10Alg(f, g):
    R = f.parent()
    h, A, B, s, C, D, = f, 1, 0, g, 0, 1
    while s != 0:
       qr = h.quo_rem(s)
       quo = qr[0]
       rem = qr[1]
       a, b, c, d = A, B, C, D
       A, B, C, D = c , d , a - quo*c, b - quo*d
       h = s
       s = rem
    return h, A, B
```

Remark 0.2. If you are familiar with diophantine equations, this process works in the same way. We need two "lines" of division to keep track of:

$$\text{STEP 1:} \quad \begin{cases} 1 \cdot f + 0 \cdot g = f \\ 0 \cdot f + 1 \cdot g = g \end{cases}$$

$$\text{STEP 2:} \quad \begin{cases} 0 \cdot f + 1 \cdot g = g \\ (1) \cdot f + (-q_1) \cdot g = r_1 \end{cases}$$

Notice that the second line is simply $f = q_1 g + r_1$. We can repeat this process.

$$\text{STEP 3:} \quad \begin{cases} 1 \cdot f + (-q_1) \cdot g = r_1 \\ (-q_2) \cdot f + (1 + q_2 q_1) \cdot g = r_2 \end{cases}$$

This second equation can look somewhat confusing, but that is only because it combines two ideas:

$$(-q_2) \cdot f + (1 + q_2 q_1) \cdot g = g + (-q_2)(\underbrace{f - q_1 g}_{r_1}) = r_2$$

Notice that by definition r_2 is the remainder when dividing g and r_1. A moment's thought will convince you that every r_i can be written as an expression of f and g.

Question 0.3. What is the first equation when $r_i = 0$? How does this solve the problem?

§5 : 12.) This exercise will study the one-variable case of the Nullstellensatz problem from §4, which asks for the relation between $I(V(f_1, \ldots, f_s))$ and $\langle f_1, \ldots, f_s \rangle$ when $f_1, \ldots, f_s \in k[x]$. By using GCDs, we can reduce to the case of a single generator. So, in this problem, we will explicitly determine $I(V(f))$ when $f \in k[x]$ is a nonconstant polynomial. Since we are working over the complex numbers, we know by Exercise 1 that f factors completely, i.e.,

$$f = c(x - a_1)^{r_1} \cdots (x - a_\ell)^{r_\ell},$$

where $a_1, \ldots, a_\ell \in \mathbb{C}$ are distinct and $c \in \mathbb{C} - \{0\}$. Define the polynomial

$$f_{\text{red}} = c(x - a_1) \cdots (x - a_\ell).$$

Note that f and f_{red} have the same roots, but their multiplicities may differ. In particular, all roots of f_{red} have multiplicity one. It is common to call f_{red} the reduced or square-free part of f. To explain the latter name, notice that f_{red} is the square-free factor of f of largest degree.

a.) Show that $V(f) = \{a_1, ..., a_l\}$.

Solution. It is clear that $V(f) \supset \{a_1, ..., a_l\}$. Let us consider a point $a \in V(f)$. Then

$$f(a) = c(a - a_1)^{r_1} \cdots (a - a_\ell)^{r_\ell} = 0$$

Hence $f(a)$ is a product of numbers $(a - a_i)^{r_i}$. Since $\mathbb{C}$ is a field, then one of these numbers has to be zero. Hence

$$a = a_i.$$

b.) Show that $I(V(f)) = \langle f_{red} \rangle$.

Solution. $V(f)$ is the union of ℓ varieties of the form $\{a_i\}$. The corresponding ideal is $f_i = x - a_i$. The union of varieties is the product and hence

$$I(V(f)) = \left\langle \prod_{i=1}^{\ell} (x - a_i) \right\rangle \equiv \langle f_{red} \rangle$$

Chapter 2

§1 : 1.) Determine whether the given polynomial is in the given ideal $I \subset k[x]$ using the method of Example 1.

a.) $f(x) = x^2 - 3x + 2, \ I = \langle x - 2 \rangle$

Solution. The method used is simply division.

$$\begin{array}{r|l} & x - 1 \\ \hline x - 2 \,) & x^2 - 3x + 2 \\ & -x^2 + 2x \\ \hline & -x + 2 \\ & x - 2 \\ \hline & 0 \end{array}$$

Since the remainder is 0, f is in the ideal.

b.) $f(x) = x^5 - 4x + 1, \ I = \langle x^3 - x^2 + x \rangle$

Solution. Again, we divide.

$$\begin{array}{r|l} & x^2 + x \\ \hline x^3 - x^2 + x & x^5 \quad - 4x + 1 \\ & -x^5 + x^4 - x^3 \\ \hline & x^4 - x^3 \\ & -x^4 + x^3 - x^2 \\ \hline & -x^2 \end{array}$$

Since the remainder is $-x^2 \neq 0$, f is not in the ideal.

c.) $f(x) = x^2 - 4x + 4,\ I = \langle x^4 - 6x^2 + 12x - 8, 2x^3 - 10x^2 + 16x - 8\rangle$

Solution. We must first find the generating polynomial of I by calculating the GCD (see problem 10).

$$\begin{aligned} x^4 - 6x^2 + 12x - 8 &= \big(2x^3 - 10x^2 + 16x - 8\big) \cdot \big(\tfrac{1}{2}x + \tfrac{5}{2}\big) + \big(11x^2 - 24x + 12\big) \\ 2x^3 - 10x^2 + 16x - 8 &= \big(11x^2 - 24x + 12\big) \cdot \big(\tfrac{2}{11}x - \tfrac{62}{121}\big) + \big(\tfrac{184}{121}x - \tfrac{224}{121}\big) \\ 11x^2 - 24x + 12 &= \big(\tfrac{184}{121}x - \tfrac{224}{121}\big) \cdot \big(\tfrac{1331}{184}x - \tfrac{7381}{1058}\big) + -\tfrac{484}{529} \\ \tfrac{184}{121}x - \tfrac{224}{121} &= -\tfrac{484}{529} \cdot \big(-\tfrac{24334}{14641}x + \tfrac{29624}{14641}\big) + 0 \end{aligned}$$

Hence, the GCD is a constant and, therefore, f is automatically in the ideal.

d.) $f(x) = x^3 - 1,\ I = \langle x^9 - 1, x^5 + x^3 - x^2 - 1\rangle$.

Solution. As with the previous problem, we need to find the GCD of the two generators for I.

$$\begin{aligned} x^9 - 1 &= \big(x^5 + x^3 - x^2 - 1\big) \cdot \big(x^4 - x^2 + x + 1\big) + \big(-x^4 + x\big) \\ x^5 + x^3 - x^2 - 1 &= \big(-x^4 + x\big) \cdot -x + \big(x^3 - 1\big) \\ -x^4 + x &= \big(x^3 - 1\big) \cdot -x + 0 \end{aligned}$$

Now, we need only check that the generator of I divides f, which is clearly does. Therefore, $f \in I$.

§1 : 3.) Find implicit equations for the affine varieties parametrized as follows.

a.) In $\mathbb{R}^3$ or $\mathbb{C}^3$

$$\begin{cases} x_1 = t - 5 \\ x_2 = 2t + 1 \\ x_3 = t + 6 \end{cases}$$

Solution. We can find this using substitution or Gaussian elimination.
Via substitution, we get

$$\begin{cases} x_2 = 2(x_1 + 5) + 1 \\ x_3 = x_1 + 11 \end{cases}$$

b.) In $\mathbb{R}^4$ or $\mathbb{C}^4$

$$\begin{cases} x_1 = 2t - 5u \\ x_2 = t + 2u \\ x_3 = -t + u \\ x_4 = t + 3u \end{cases}$$

Solution. We can find this using substitution or Gaussian elimination.

$$\begin{pmatrix} 2 & -5 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -5/2 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & 1/11 & 0 & 0 & -2/11 \\ 0 & 0 & 1 & 0 & 11/4 & 3/4 \\ 0 & 0 & 0 & 1 & 1/4 & -3/4 \end{pmatrix}$$

The implicit equations are $4x_1 + 11x_3 + 3x_4 = 0$ and $4x_2 + x_3 - 3x_4 = 0$

c.) In $\mathbb{R}^3$ or $\mathbb{C}^3$

$$\begin{cases} x = t \\ y = t^4 \\ z = t^7 \end{cases}$$

Solution. We can use substitution to get $y - x^4 = 0$ and $z - x^7 = 0$.

§2 : 1.) Rewrite each of the following polynomials, ordering the terms using the lex order, the grlex order, and the grevlex order, giving $LM(f)$, $LT(f)$, and mdeg(f) in each case.

a.) $f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$.

Solution. Below is the itemized list of answers for each order.

* Lex order ($x > y > z$):
  (1) $f(x, y, z) = x^3 + x^2 + 2x + 3y - z^2 + z$
  (2) $LM(f) = x^3$
  (3) $LT(f) = x^3$
  (4) mdeg$(f) = (3, 0, 0)$
* grlex order ($x > y > z$):
  (1) $f(x, y, z) = x^3 + x^2 - z^2 + 2x + 3y + z$
  (2) $LM(f) = x^3$
  (3) $LT(f) = x^3$
  (4) mdeg$(f) = (3, 0, 0)$
* grevlex order ($x > y > z$):
  (1) $f(x, y, z) = x^3 + x^2 - z^2 + 2x + 3y + z$
  (2) $LM(f) = x^3$
  (3) $LT(f) = x^3$
  (4) mdeg$(f) = (3, 0, 0)$

b.) $f(x, y, z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$.

Solution. Below is the itemized list of answers for each order.

* Lex order ($x > y > z$):
  (1) $f(x, y, z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$

(2) $LM(f) = x^5yz^4$
(3) $LT(f) = -3x^5yz^4$
(4) mdeg$(f) = (5, 1, 4)$

$*$ grlex order ($x > y > z$):
(1) $f(x, y, z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$
(2) $LM(f) = x^5yz^4$
(3) $LT(f) = -3x^5yz^4$
(4) mdeg$(f) = (5, 1, 4)$

$*$ grevlex order ($x > y > z$):
(1) $f(x, y, z) = 2x^2y^8 - 3x^5yz^4 - xy^4 + xyz^3$
(2) $LM(f) = x^2y^8$
(3) $LT(f) = 2x^2y^8$
(4) mdeg$(f) = (2, 8, 0)$

§2 : 5.) Show that grevlex is a monomial order according to Definition 1.

Solution. Definition 1 states

A **monomial ordering** $>$ on $k[x_1, \dots, x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq}0$, or equivalently, an relation on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\leq 0}$, saying:
(1) $>$ is a total (or linear) ordering on $\mathbb{Z}^n_{\geq 0}$.
(2) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
(3) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq}0$.

Let us go through each item.

(1) Let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ are elements in $\mathbb{Z}^n_{\geq 0}$. Consider

$$(a_1 - b_1, \dots, a_n - b_n).$$

The (ℓ_1) norm of α, β map the vectors into $\mathbb{R}$, which is totally ordered. Let us consider the case when $|\alpha| \neq |\beta|$. For each entry, either $a_i - b_i > 0$, $a_i - b_i < 0$, $a_i - b_i = 0$. Let k be the right-most nonzero result for $a_k - b_k$. If $a_k - b_k < 0$ then $\alpha > \beta$. If $a_k - b_k > 0$ then $\beta > \alpha$. If no such k exists, then it is clear that $\alpha = \beta$. Hence, grevlex is a total ordering.

(2) Without loss of generality, assume $\alpha \geq \beta$ and consider $\gamma = (c_1, \dots, c_n)$. First, suppose $|\alpha| > |\beta|$.

$$\sum a_i \geq \sum b_i \implies \sum a_i + \sum c_i > \sum b_i + \sum c_i \implies |\alpha + \gamma| > |\beta + \gamma|.$$

Now suppose $|\alpha| = |\beta|$. Then for some k, $a_k - b_k < 0$ where $a_{k+n} = b_{k+n}$ for $n > 0$. We have the result since $(a_k + c_k) - (b_k + c_k) = a_k - b_k < 0$.

(3) Let S be a set of elements, of the form $\alpha = (a_1, \dots, a_n)$, indexed in such a way so that

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots.$$

If there exists a smallest element, then we are done. So suppose not. Since $\mathbb{Z}_{\geq 0}$ is well ordered, there must exist some k such that $\deg(\alpha_k) = \deg(\alpha_{k+1}) = \dots$. Thus, for these elements, we are only using the neglex ordering. Let $\ell = \deg(\alpha_k)$. There are only finitely many ways to express an integer ℓ as a sum of n positive integers. Therefore, the decreasing list must terminate.

§2 : 11.) Let $>$ be a monomial order on $k[x_1, \dots, x_n]$.

a.) Let $f \in k[x_1, \ldots, x_n]$ and m be a monomial. Show that $LT(m \cdot f) = m \cdot LT(f)$.

Solution. Let γ be the exponential vector of the monomial m. Let g_1 be the leading term of f and $g_2, g_3, \ldots$ denote the remaining terms. Since $\alpha > \beta \implies \alpha+\gamma > \beta+\gamma$, $mg_1 > mg_i$ for all $i \in \{2, \ldots, n\}$.

b.) Let $f, g \in k[x_1, \ldots, x_n]$. Is $LT(f \cdot g)$ necessarily the same as $LT(f) \cdot LT(g)$?

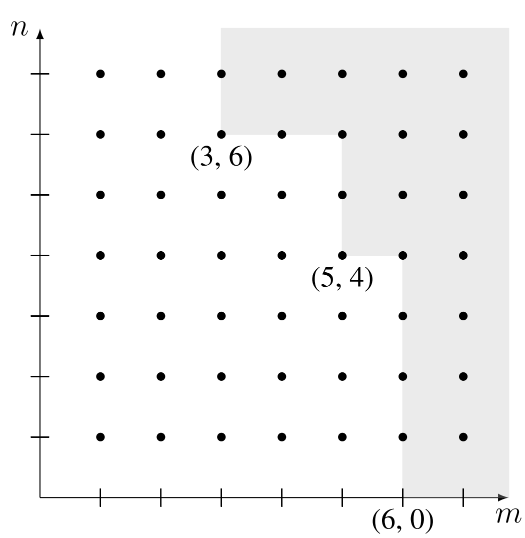
Solution. Let $f = f_1 + f_2 + \ldots$ and let $g = g_1 + g_2 + \ldots$ where f_i and g_i are monomials indexed by order. For each i, $LT(g_i f) = g_i f_1$ by part a. This yields $f_1 g$. It's leading term is $f_1 g_1$.

	f_1	f_2	f_3	$\ldots$
g_1	$g_1 f_1$	$g_1 f_2$	$g_1 f_3$	$\ldots$
g_2	$g_2 f_1$	$g_2 f_2$	$g_2 f_3$	$\ldots$
g_3	$g_3 f_1$	$g_3 f_2$	$g_3 f_3$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

c.) Let $f_i, g_i \in k[x_1, \ldots, x_n]$. Is $LM(\sum f_i g_i)$ necessarily equal to $LM(f_i) \cdot LM(g_i)$ for some i?

Solution. Unfortunately, we can have some cancellation of terms. For example, consider $f_1 = (x-1), g_1 = x, f_2 = (1-x), g_2 = x$. Then $f_1 g_1 + f_2 g_2 = 0$.

§4 : 4.) Let $I \subset k[x, y]$ be the monomial ideal spanned over k by the monomials x^β corresponding to β in the shaded region below:



a.) Use the method given in the proof of Theorem 5 to find an ideal basis for I.

Solution. $I = \langle x^6, x^5 y^4, x^3 y^6 \rangle$. Hence $\alpha(1) = 6, \alpha(2) = 5$ and $\alpha(3) = 3$. Let $J \subset k[x]$ be $J = \langle x^6, x^5, x^3 \rangle = \langle x^3 \rangle$. In other words, J is the projection of I onto $k[x]$.

The largest power of y is 6. Let $0 \leq k \leq 5$ and define J_k to be generated by the monomial ideals x^β such that $x^\beta y^k \in I$. Hence

$$J = \left\langle x^3 \right\rangle$$
$$J_0 = \left\langle x^6 \right\rangle$$
$$J_1 = \left\langle x^6 \right\rangle$$
$$J_2 = \left\langle x^6 \right\rangle$$
$$J_3 = \left\langle x^6 \right\rangle$$
$$J_4 = \left\langle x^5 \right\rangle$$
$$J_5 = \left\langle x^5 \right\rangle$$

Then $I = \langle x^3y^6, x^6, x^6y, x^6y^2, x^6y^3, x^5y^4, x^5y^5 \rangle$

b.) Is your basis as small as possible, or can some βs be deleted from your basis, yielding a smaller set that generates the same ideal?

Solution. Since $I = \langle x^6, x^5y^4, x^3y^6 \rangle$, we know this is not the smallest basis possible. The terms $x^6y, x^6y^2, x^6y^3, x^5y^5$ can be deleted.