MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
HOMEWORK 2 SOLUTIONS

Assigned 8/28, due 9/4 in class.

Problem 1. Find the equation in terms of $x_{HH}, x_{HT}, x_{TH}, x_{TT}$ that implicitizes the parametric representation $x_{HH} = q_H d_H$, $x_{HT} = q_H d_T$, $x_{TH} = q_T d_H$, $x_{TT} = q_T d_T$. Now assume all numbers are nonnegative real, $q_H + q_T = d_H + d_T = 1$, and so $x_{HH} + x_{HT} + x_{TH} + x_{TT} = 1$. Draw the image of the parametric representation in the tetrahedron representing the probability simplex for four outcomes.

Solution 1. First, let us make the problem somewhat easier to read by changing the variables. Let

$$\begin{array}{cccc} x = x_{HH} & y = x_{HT} & z = x_{TH} & w = x_{TT} \\ s = q_H & t = q_T & u = d_H & v = d_T \end{array}$$

Then our parameterization is $x = su, y = sv, z = tu$, and $w = tv$. Notice that each parameter appears in exactly two different equations and is degree one. Therefore, one should be able to "cross multiply" to get the implicitization. That is $xw - yz$.

Remark 0.1. Why did I use the phrase "cross-multiply"? Because this is a *determinantal* variety. Essentially, we are taking an outer product of two coin flips:

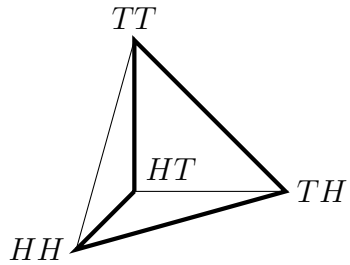
$$\begin{pmatrix} s \\ t \end{pmatrix} \times \begin{pmatrix} u & v \end{pmatrix} = \begin{pmatrix} su & sv \\ tu & tv \end{pmatrix} := \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

The implicitization is defined by the determinant of this matrix.

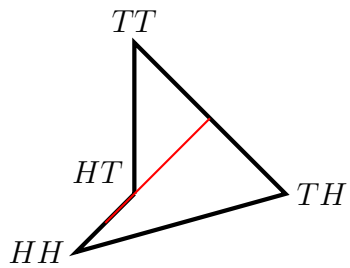
Alternatively, you could have found this using SAGE. Here is the code:

```
T.< x, y, z, w, s, t, u, v > = PolynomialRing(QQ,8);
I=ideal(x-s*u, y - s*v, z - t*u, w - t*v);
I.elimination_ideal([s, t, u, v])
Ideal (y*z - x*w) of Multivariate Polynomial Ring
in x, y, z, w, s, t, u, v over Rational Field
```

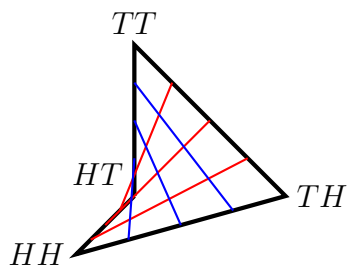
To draw the representation, let us first consider the boundaries. This was discussed in Lecture 2.



The variety is outlined by the highlighted portions depicted above, which contain the extreme cases when $q_* = 1$ or $d_* = 1$. To explain how the interior of this variety can be found, let us consider the case when $s = 1$ and $u = .5$. This is an extreme point on the boundary when $u = .5$. Considering the corresponding extreme point, $t = 1$ and $u = .5$. Transitioning between these two cases is determined by the linear equation $s + t = 1$ because u is held constant.



Following this logic for all the parameters, we get the following picture. Note that this is a smooth surface, but we draw it as a mesh to better illustrate the shape. The red lines correspond to $s + t = 1$ while the blue lines correspond to $u + v = 1$.



For a more detailed discussion on drawing this picture, see `https://www.youtube.com/watch?v=7aNaH-8n-TM`.

Problem 2. Find the equation(s) in terms of x, y, z implicitizing the parametric representation $x = \theta_1^2$, $y = \theta_1\theta_2$, $z = \theta_2^2$. Explain why you think no more equations are needed.

Solution 2. Using the logic from problem 1, it is easy to see that $xz - y^2$ will work as an implicitization.

Remark 0.2. Like before, this too is a determinantal variety.

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \times \begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix} = \begin{pmatrix} \theta_1^2 & \theta_1\theta_2 \\ \theta_1\theta_2 & \theta_2^2 \end{pmatrix} := \begin{pmatrix} x & y \\ y & z \end{pmatrix}.$$

As always, one can use SAGE to solve this problem as well.

```
T.<x, y, z, s, t> = PolynomialRing(QQ,5);
I=ideal(x-s^2, y - s*t, z - t^2);
I.elimination_ideal([s, t])
Ideal (y^2 - x*z) of Multivariate Polynomial Ring
in x, y, z, w, s, t over Rational Field
```

One should not expect more equations because this is an image of a variety is two dimensional (there are two nonredundant parameters), which justifies one equation in the implicitization.

§3 : 4.) Consider the parametric representation

$$x = \frac{t}{1+t}, \quad y = 1 - \frac{1}{t^2}.$$

(a) Find the equation of the affine variety determined by the above parametric equations.

Solution.

$$x = \frac{t}{1+t} \implies x + xt = t \implies x = t(1-x) \implies t = \frac{x}{1-x}$$

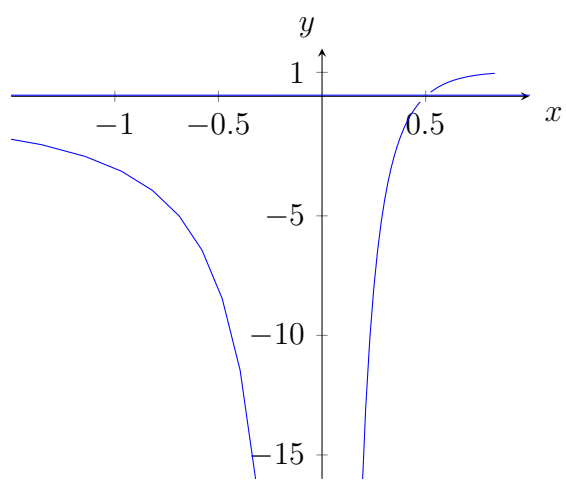
$$y = 1 - \frac{1}{\left(\frac{x}{1-x}\right)^2} \implies x^2 y = x^2 - (1-x)^2 \implies x^2(1-y) = (1-x)^2$$

Hence, the implicitization is

$$0 = x^2 y - 2x + 1$$

(b) Show that the above equations parametrize all points of the variety found in part (a) except for the point $(1, 1)$.

Solution. Suppose $x = 1$. Then $1 + t = t$, which is impossible. Similarly, if $y = 1$, then $0 = -1/t^2$, which is also impossible. Hence $(1, 1)$ is not in the parameterization. It is, however, in the variety.



Let $x_0, y_0 \in V$ such that there does not exist a corresponding t in the parameterization. Since the parameterization has $x = \frac{t}{1+t}$ and $y = 1 - \frac{1}{t^2}$, then either $x_0 = 1$ or $y_0 \geq 1$. If $x_0 = 1$, then $y_0 = 1$ and this is the point discussed above. If $y_0 > 1$, then

$$x = \frac{1 \pm \sqrt{1 - y_0}}{y_0},$$

which implies that x_0 is imaginary. Thus, the parameterization agrees with the implicit equation on all numbers in $\mathbb{R}^2$ except $(1, 1)$.

§3 : 11.) In this problem, we will derive the parameterization

$$x = t(u^2 - t^2),$$
$$y = u,$$
$$z = u^2 - t^2,$$

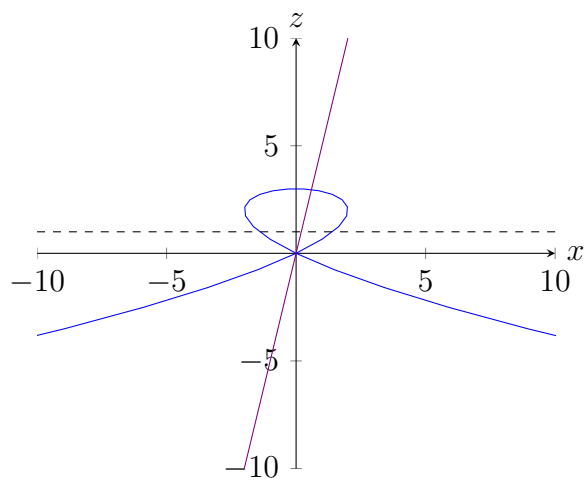
of the surface $x^2 - y^2z^2 + z^3 = 0$ considered in the text.

(a) Adapt the formula in part (d) of Exercise 8 to show that the curve $x^2 = cz^2 - z^3$ is parametrized by

$$z = c - t^2,$$
$$x = t(c - t^2).$$

Solution. In part (d) of Exercise 8, the parameterization for $y^2 = cx^2 - x^3$ is given as $x = c - t^2$ and $y = t(c - t^2)$. If we simply relabel the variables, swapping x for z and y for x, we get the same result.

Remark 0.3. The parameterization given is derived in an interesting way. We consider the line $x = 1$, and study all lines intersecting $x = 1$ and the origin. Each of these lines intersects the curve at one point. Here, the variables are swapped. So we draw a line at $z = 1$ and consider the intersection of $(0, 0)$ with the line $(t, 1)$. This line is depicted below in violet.



It is clear that this construction gives a parameterization of the entire curve because it uniquely identifies a point for every value of t. How does one deduce this relationship algebraically from this construction? For every t, we find the line $x = tz$. We can substitute this back into the original equation and solve for z.

(b) Now replace the c in part (a) by y^2, and explain how this leads to the above parametrization of $x^2 - y^2z^2 + z^3 = 0$.

Solution. If we simply write y^2 in place of c, we can algebraically get the function $x^2 - y^2z^2 + z^3 = 0$ and the equations $z = y^2 - t^2$ and $x = t(y^2 - t^2)$. This, however, means we are using y as a parameter as well as a variable in the space (i.e.

the third equation is $y = y$). To separate these two concepts, we need only introduce a parameter that is equal to y: u. This gives us

$$\begin{cases} x = t(u^2 - t^2) \\ y = u \\ z = u^2 - t^2 \end{cases}$$

(c) Explain why this parametrization covers the entire surface $V(x^2 - y^2z^2 + z^3)$.

Solution. We can construct the same argument by first fixing an arbitrary u and letting $y = u$. Then the problem reduces to precisely the situation described in part 1. We consider the point $(t, u, 1)$ and the unique line through $(0, u, 0)$. This line intersects all the points in $x^2 - u^2z^2 + z^3$. And this gives us $y = u$ and $x = tz$. We can then solve by plugging back into the equation:

$$x^2 - y^2z^2 + z^3 = 0 \implies (tz)^2 - u^2z^2 + z^3 = 0 \implies z = u^2 - t^2$$

§4 : 3.) Prove the following equalities of ideals in $\mathbb{Q}[x, y]$.

(a) $\langle x + y, x - y\rangle = \langle x, y\rangle$

Solution. It is clear that $x \pm y \in \langle x, y\rangle$. Notice

$$\frac{1}{2}(x + y) + \frac{1}{2}(x - y) = x$$

and

$$\frac{1}{2}(x + y) - \frac{1}{2}(x - y) = y.$$

Hence the ideals are equal.

(b) $\langle x + xy, y + xy, x^2, y^2\rangle = \langle x, y\rangle$

Solution. Again, it is clear that $\langle x + xy, y + xy, x^2, y^2\rangle \subset \langle x, y\rangle$. For ease, we refer to $\langle x + xy, y + xy, x^2, y^2\rangle$ as I. Let us show the reverse containment. Observe that

$$x + xy - (y + xy) = x - y \text{ and } \frac{-1}{2}(x - y)^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = xy.$$

Thus $xy \in I$ so $(x + xy) - (xy) = x \in I$ and similarly $y \in I$.

(c) $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3\rangle = \langle x^2 - 4, y^2 - 1\rangle$

Solution. First, we show that $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3\rangle \subset \langle x^2 - 4, y^2 - 1\rangle$:

$$(x^2 - 4) - (y^2 - 1) = x^2 - y^2 - 3 \quad \text{and} \quad 2(x^2 - 4) + 3(y^2 - 1) = 2x^2 + 3y^2 - 11.$$

Now we show the reverse containment:

$$\frac{1}{5}\left[(2x^2 + 3y^2 - 11) + 3(x^2 - y^2 - 3)\right] = x^2 - 4 \quad \text{and} \quad \frac{1}{3}\left[(2x^2 + 3y^2 - 11) - 3(x^2 - 4)\right] = y^2 - 1$$

§4 : 6.) The word "basis" is used in various ways in mathematics. In this exercise, we will see that "a basis of an ideal," as defined in this section, is quite different from "a basis of a subspace," which is studied in linear algebra.

(a) First, consider the ideal $I = \langle x\rangle \subset k[x]$. As an ideal, I has a basis consisting of the one element x. But I can also be regarded as a subspace of $k[x]$, which is a vector space over k. Prove that any vector space basis of I over k is infinite.

Solution. Elements in $\langle x \rangle$ can be written as a linear combination of elements from the set $\mathscr{B} = \{x, x^2, x^3, \ldots, x^n, \ldots\}$. We know this set space $k[x]$. We also know that no one element can be written as a linear combination of another using elements over k. Thus, $\mathscr{B}$ is an infinite basis for I over k.

(b) In linear algebra, a basis must span and be linearly independent over k, whereas for an ideal, a basis is concerned only with spanning—there is no mention of any sort of independence. The reason is that once we allow polynomial coefficients, no independence is possible. To see this, consider the ideal $\langle x, y \rangle \subset k[x, y]$. Show that zero can be written as a linear combination of y and x with nonzero polynomial coefficients.

Solution.

$$(x)y + (-y)x = 0$$

(c) More generally, suppose that $f_1, \ldots, f_s$ is the basis of an ideal $I \subset k[x_1, \ldots, x_n]$. If $s \geq 2$ and $f_i \neq 0$ for all i, then show that for any i and j, zero can be written as a linear combination of f_i and f_j with nonzero polynomial coefficients.

Solution.

$$(f_j)f_j + (-f_j)f_i = 0$$

(d) A consequence of the lack of independence is that when we write an element $f \in \langle f_1, \ldots, f_s \rangle$ as $f = \sum_{i=1}^{s} h_i f_i$, the coefficients h_i are not unique. As an example, consider $f = x^2 + xy + y^2 \in \langle x, y \rangle$. Express f as a linear combination of x and y in two different ways. (Even though the h_is are not unique, one can measure their lack of uniqueness. This leads to the interesting topic of syzygies.)

Solution.

$$(x)x + (x + y)y = x^2 + xy + y^2 = (x + y)x + (y)y$$

(e) A basis $f_1, \ldots, f_s$ of an ideal I is said to be minimal if no proper subset of $f_1, \ldots, f_s$ is a basis of I. For example, x, x^2 is a basis of an ideal, but not a minimal basis since x generates the same ideal. Unfortunately, an ideal can have minimal bases consisting of different numbers of elements. To see this, show that x and $x + x^2$, x^2 are minimal bases of the same ideal of $k[x]$. Explain how this contrasts with the situation in linear algebra.

Solution. It is clear that $\langle x \rangle = \langle x + x^2, x^2 \rangle$. It is trivially true that the set $\{x\}$ is minimal. Consider the set $\{x + x^2, x^2\}$. Proper subsets of this basis are $\{x + x^2\}$ and $\{x^2\}$. Notice that $x \notin \langle x^2 \rangle$, otherwise $x = f \cdot x^2$, which implies $x^2 \big| x$, a contradiction. A similar argument can be made for the other $\langle x + x^2 \rangle$.

§3 : 7.) Show that $I(V(x^n, y^m)) = \langle x, y \rangle$ for any positive integers n and m.

Solution. We assume we are working over $k[x, y]$ (otherwise, we can simply embed our answer into a higher dimension). Since $V(x^n, y^m) = V(x^n) \cap V(y^m)$, $V(x^n) = \{(0, y) \mid y \in k\} = V(x)$ and $V(y^m) = \{(x, 0) \mid y \in k\} = V(y)$, we can concluse that $V(x^n, y^m) = V(x) \cap V(y) = V(x, y)$ and $I(V(x^n, y^)) = I(V(x, y))$. Therefore, $\langle x, y \rangle \subset I(V(x^n, y^m))$.

For the other containment, it is clear that $(0, 0) \in V(x^n, y^m)$. Hence any polynomial $f(x, y) \in I(V(x^n, y^m))$ cannot contain a nonzero constant term. Therefore, $I(V(x^n, y^m)) \subset \langle x, y \rangle$.

§3 : 8.) The ideal $I(V)$ of a variety has a special property not shared by all ideals. Specifically, we define an ideal I to be radical if whenever a power f^m of a polynomial f is in I, then f itself is in I. More succinctly, i is radical when $f \in I$ if and only if $f^m \in I$ for some positive integer m.

(a) Prove that $I(V)$ is always a radical ideal.

Solution. Let $f^m \in I(V)$. Then for $v \in V$, $[f(v)]^m = 0$ and, as a result, $f(v) = 0$. Hence $f \in I(V)$.

Remark 0.4. Here, we are evoking the fact that f is a polynomial over a field. Hence $f(v)$ is a field element. Suppose $f(v) \neq 0$ but $[f(v)]^m = 0$ for some $m > 1$, where m is the smallest such integer to yield this result. Then $f(v) \cdot [f(v)]^{m-1} = 0$, making $f(v)$ a zero divisor. Fields, however, do not contain zero divisors.

(b) Prove that $\langle x^2, y^2 \rangle$ is not a radical ideal. This implies that $\langle x^2, y^2 \rangle \neq I(V)$ for any variety $V \subset k^2$.

Solution. Suppose not. Without loss of generality, we may assume $x \in \langle x^2, y^2 \rangle$. This implies that $x = f(x, y)x^2 + g(x, y)y^2$ for polynomials $f, g \in k[x, y]$. This, however, is impossible since the multidegree on the right-hand side is 2 or larger, but on the left-hand side is 1.