MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
HOMEWORK 1 SOLUTIONS

Assigned 8/24, due 8/28 in class.

Problem 1. State the fundamental theorem of algebra.

Solution 1. Every nonconstant polynomial $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

Problem 2. Consider the variety X cut out of three-dimensional affine space by the vanishing of the polynomials $2x^2 + y^3$ and $4x - y^5$. Use sagemathcloud or some other program to show the dimension is zero, and compute the points that make up X. Does your answer depend on the field?

Solution 2. This problem had a typo. It should have stated *two-dimensional affine space*. We compute this for the intended problem. Note that the dimension increases by 1 if we go to a three dimensional space.

```
R.<x,y>=PolynomialRing(QQ,2);
I = ideal(2*x^2+y^3, 4*x-y^5);
I.dimension()
0
```

```
R.<x,y>=PolynomialRing(RR,2);
I = ideal(2*x^2+y^3, 4*x-y^5);
I.dimension()
0
```

```
R.<x,y>=PolynomialRing(CC,2);
I = ideal(2*x^2+y^3, 4*x-y^5);
I.dimension()
0
```

```
k = GF(2); k
Finite Field of size 2
R.<x,y>=PolynomialRing(k,2);
I = ideal(2*x^2+y^3, 4*x-y^5);
I.dimension()
1
```

We can solve for the roots over $\mathbb{C}$ by hand and discuss which of these roots also exist in $\mathbb{R}$ and $\mathbb{Q}$. If $2x^2 + y^3 = 0$ and $4x - y^5 = 0$, then $x = \frac{y^5}{4}$ and

$$2 \cdot \frac{y^10}{2} + y^3 = 0 \implies y^3 \left(\frac{y^7}{8} + 1 \right) = 0.$$

When the field is $\mathbb{C}$, the variety contains 8 points: $(0,0)$ (with multiplicity 3) and the seven roots of $(\frac{y^7}{8}+1)=0$. Only one of these seven roots is real, namely $(\sqrt[7]{-1/8}, \sqrt[5]{4\sqrt[7]{-1/8}})$. Hence the variety over $\mathbb{R}$ contains two points. When the variety is over $\mathbb{Q}$, there is only one point, $(0,0)$, in the variety.

Over $\mathbb{F}_2$, these polynomials are different. $4 \equiv 0$ and $2 \equiv 0$. Hence, the polynomials are equivalent to $y^5 = 0$ and $y^3 = 0$. Notice that there are no restrictions on x. So over $\mathbb{F}_2$, the variety is 1 dimensional; it is the span of $(1,0)$.

Problem 3. Is every linear subspace of a complex vector space an affine algebraic variety? Why or why not?

Solution 3. Yes. Let W be a subspace if $\mathbb{C}^k$ of dimension ℓ where $1 \leq \ell \leq k$. Let $\{\omega_1, \ldots, \omega_\ell\}$ be a basis for W. Extend this basis to all of $\mathbb{C}^k$: $\{\omega_1, \ldots, \omega_k\}$. Define a linear transformation M such that

$$M\omega_i = \begin{cases} \vec{0} & \text{if } i \leq \ell \\ \omega_i & \text{if } i > \ell \end{cases}.$$

Then W is the null space of the linear transformation M, which corresponds to a set of (linear) polynomials.

Problem 4. How do you think we should define the dimension of an affine algebraic variety? This problem has many right answers, but don't worry about looking up and understanding the correct definitions that are out there (we'll get there). Just think about it and come up with a proposal, no wrong answers here.

Solution 4. Any reasonably conjecture was accepted for this.

§1 : 2a.) Let $\mathbb{F}_2$ be the field of two elements. Consider the polynomial $g(x,y) = x^2y + y^2x \in \mathbb{F}_2[x,y]$. Show that $g(x,y) = 0$ for every $(x,y) \in \mathbb{F}_2^2$ and explain why this does not contradict Proposition 5.

Solution. We can factor g as $g(x,y) = xy(x+y)$. It is clear that if $x = 0$ or $y = 0$, the function evaluates to 0. Consider if $x = y = 1$. Then $x + y = 0$. Hence $g(1,1) = 0$.

This does not violate proposition 5, which requires the field be infinite. $\mathbb{F}_2$ is a finite field.

§1 : 5.) In the proof of Proposition 5, we took $f \in k[x_1, \ldots, x_n]$ and wrote it as a polynomial in x_n with coefficients in $k[x_1, \ldots, x_{n-1}]$. To see what this looks like in a specific case, consider the polynomial

$$f(x,y,z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3$$

a. White f as a polynomial in x.

Solution.

$$f(x,y,z) = (y^2z)x^5 - (y^3)x^4 + (z)x^2 + (y+2)x + (y^5 - y^3z - 5z + 3)$$

b. White f as a polynomial in y.

Solution.

$$f(x,y,z) = y^5 - (x^4 + z)y^3 + (x^5z)y^2 + (x)y + (x^2z + 2x - 5z + 3)$$

b. White f as a polynomial in z.

Solution.

$$f(x, y, z) = (x^5y^2 + x^2 - y^3)z + (y^5 - x^4y^3 + xy + 2x - 5z + 3)$$

$\S1:6.)$ Inside $\mathbb{C}^n$ we have the subset $\mathbb{Z}^n$, which consists of all points with integer coordinates.

a. Prove that if $f \in \mathbb{C}[x_1, \ldots, x_n]$ vanishes at every point in $\mathbb{Z}^n$, then f is the zero polynomial.

Solution. Let $n = 1$. Then $f \in k[x]$ where $\deg(f) = m$. Then f has at most m roots. Since $\mathbb{Z}$ is infinite, this is a contradiction. Therefore, f is the zero polynomial.
Now assume this holds true up to $n - 1$. Let $f \in k[x_1, \ldots, x_n]$ be a polynomial that vanishes at all points of $\mathbb{Z}^n$. We can write f as a function of x_n, similar to $\S1\#5$. That is,

$$f = \sum_{i=0}^{N} g_i(x_1, \ldots, x_{n-1})x_n^i.$$

Fix $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$. Then f is a polynomial in one variable with finite degree. Since f vanishes whenever $x_n \in \mathbb{Z}$, we know f is the zero polynomial in $k[x_n]$. Hence the coefficients of f are zero: $g(a_1, \ldots, a_{n-1}) = 0$. Since these are arbitrary elements in $\mathbb{Z}^{n-1}$, we can conclude that each $g_i \in k[x_1, \ldots, x_{n-1}]$ is the zero polynomial. Therefore, f is the zero polynomial in $k[x_1, \ldots, x_n]$.

b. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$, and let M be the largest power of any variable that appears in f. Let $\mathbb{Z}^n_{M+1}$ be the set of points of $\mathbb{Z}^n$, all coordinates of which lie between 1 and $M + 1$. Prove that if f vanishes at all points of $\mathbb{Z}^n_{M+1}$, then f is the zero polynomial.

Solution. This is essentially the same proof as part (a).
Let $n = 1$. Then $f \in k[x]$ where $\deg(f) = M$ has at most M roots. Since $\mathbb{Z}_{M+1}$ has $M + 1$ points, this is a contradiction. Therefore, f is the zero polynomial.
Now assume this holds true up to $n - 1$. Let $f \in k[x_1, \ldots, x_n]$ be a polynomial that vanishes at all points of $\mathbb{Z}^n_{M+1}$. As before, ee can write f as a function of x_n:

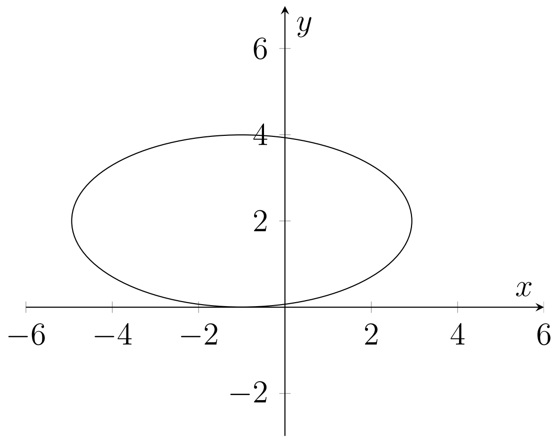
$$f = \sum_{i=0}^{N} g_i(x_1, \ldots, x_{n-1})x_n^i.$$

Fix $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}_{M+1}$. Then f is a polynomial in one variable with finite degree. Since f vanishes whenever $x_n \in \mathbb{Z}_{M+1}$, we know f is the zero polynomial in $k[x_n]$ by the base-case argument. Hence the coefficients of f are zero: $g(a_1, \ldots, a_{n-1}) = 0$. Since these are arbitrary elements in $\mathbb{Z}^{n-1}_{M+1}$, we can conclude that each $g_i \in k[x_1, \ldots, x_{n-1}]$ is the zero polynomial. Therefore, f is the zero polynomial in $k[x_1, \ldots, x_n]$.

$\S2:1.)$ Sketch the following varieties in $\mathbb{R}^2$.

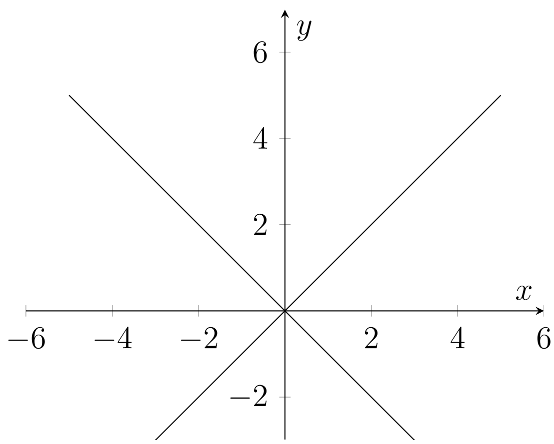
a. $V(x^2 + 4y^2 + 2x - 16y + 1)$

Solution. The solution is depicted below.



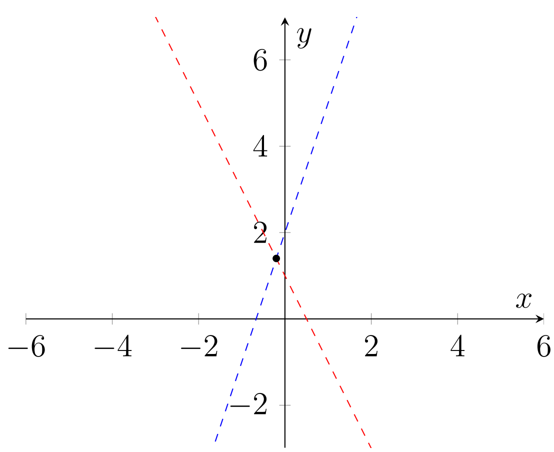
b. $V(x^2 - y^2)$

Solution. The solution is depicted below.



c. $V(2x + y - 1, 3x - y + 2)$

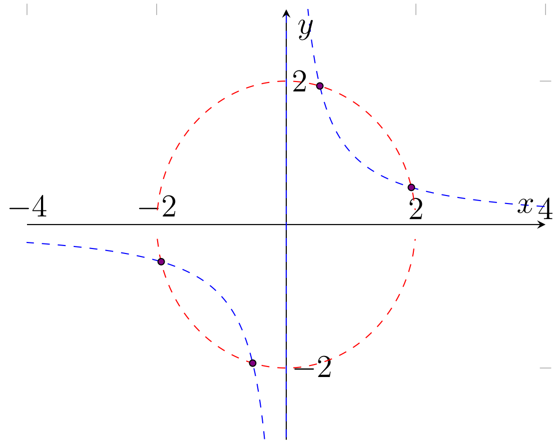
Solution. The solution is depicted below.



§2 : 3.) In the plane $\mathbb{R}^2$, draw a picture to illustrate

$$V(x^2 + y^2 - 4) \cap V(xy - 1) = V(x^2 + y^2 - 4, xy - 1)$$

Solution. The solution is depicted below.



§2 : 6.) Let us show that all finite subsets of k^n are affine varieties.

a. Prove that a single point $(a_1, \ldots, a_n) \in k^n$ is an affine variety.

Solution. Define $\mathscr{F}$ to be the family of polynomials of the form $f_i = x_i - a_i$. It is clear that $(a_1, \ldots, a_n) \in V(\mathscr{F})$. Consider a point $y = (y_1, \ldots, y_n) \in V(\mathscr{F})$. Since $f_i(y) = 0$ for all i, we must have that $y_i = a_i$. Hence, $y = (a_1, \ldots, a_n)$.

b. Prove that every finite subset of k^n is an affine variety.

Solution. By problem 15a, we know all finite unions and intersections of varieties also form a variety. Since a finite set of points can be expressed as a finite union of single-point varieties, we know it is a variety.

§2 : 8.) Consider the set

$$X = \{(x, x) \mid x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2,$$

which is the straight line $x = y$ with the point $(1, 1)$ removed. To show that X is not an affine variety, suppose that $X = V(f_1, \ldots, f_s)$. Then each f_i vanishes on X, and if we can show that f_i also vanishes at $(1, 1)$, we will get the desired contradiction. Thus, here we prove if $f \in \mathbb{R}[x, y]$ vanishes on X, then $f(1, 1) = 0$.

Solution. Let $g(t) = f(t, t)$. Then $g \in \mathbb{R}[t]$. Notice that for all $t \neq 1$, $g(t) = 0$. By proposition 5, $g(t)$ is the zero polynomial. Hence $f(1, 1) = g(1) = 0$.

§2 : 9.) Let $R = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane. Prove that R is not an affine variety.

Solution. Suppose it is. Let R denote the upper half plane and let $f(x, y) = y - x^2$. Because the intersection of two varieties forms a variety by Lemma 2, consider

$$R \cap V(f) = \{y = x^2 \mid x \neq 0\}.$$

Define $g(t) = f(t, t^2)$. Notice that $g(t) \in \mathbb{R}[t]$ and that $g(t) = 0$ for all values of $t \neq 0$. By proposition 5, g is the zero function and $f(0, 0) = g(0) = 0$. This implies that $(0, 0) \in R$, which is a contradiction. Hence R is not a variety.

§2 : 15.) In Lemma 2, we showed that if V and W are affine varieties, then so are their union $V \cup W$ and intersection $V \cap W$.

a. Prove that finite unions and intersections of affine varieties are again varieties.

Solution. Lemma two provides the base case for induction. Suppose both intersections and unions hold for set of $n-1$. Let $\mathscr{V}$ be a finite set of varieties $\{V_1, \ldots, V_n\}$. Notice

$$\bigcap_{i=1}^{n} V_i = \left(\bigcap_{i=1}^{n} V_i\right) \cap V_n$$

and similarly

$$\bigcap_{i=1}^{n} V_i = \left(\bigcup_{i=1}^{n} V_i\right) \cup V_n.$$

Therefore, the base case argument proves the inductive step.

b. Give an example to show that an infinite union of affine varieties need not be an affine variety.

Solution. Consider the set X from §2#8. With the point $(1,1)$ removed, the set is NOT a variety. It can be constructed, however, as an infinite union of single-point varieties:

$$X = \bigcup_{t \in \mathbb{R}, t \neq 1} V(x-t, y-t).$$

c. Give an example to show that the set-theoretic difference $V - W$ of two affine varieties need not be an affine variety.

Solution. Consider the set X from §2#8. We can see that $\bar{X} := V(x-y) = \{(x,x) \mid x \in \mathbb{R}\}$ forms a variety and that $V(x-1, y-1) = \{(1,1)\}$ is a variety. Their set-theoretic difference, as we have seen, does not form a variety.

d. Let $V \subset k^n$ and $W \subset k^m$ be two affine varieties and let

$$V \times W = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in k^{m+n} : (x_1, \ldots, x_n) \in V, (y_1, \ldots, y_m) \in W\}$$

be their cartesian product. Prove that $V \times W$ is an affine variety in k^{n+m}.

Solution. Let $\mathscr{F} = \{f_1, \ldots, f_s\}$ and $\mathscr{G} = \{g_1, \ldots, g_r\}$ be such that $V = V(\mathscr{F})$ and $W = V(\mathscr{G})$. Notice that $f_i \in k[x_1, \ldots, x_n] \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ for $1 \leq i \leq s$. Similarly $g_j \in k[y_1, \ldots, y_m] \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ for $1 \leq j \leq r$. Thus, $V(\mathscr{F}, \mathscr{G})$ is a variety in k^{m+n} and it is clear from the definition that $V \times W \subset V(\mathscr{F}, \mathscr{G})$. Let $a = (a_1, \ldots, a_n, b_1, \ldots, b_m) \in V(\mathscr{F}, \mathscr{G})$. Since

* a must be in the zero locus of both $\mathscr{F}$ and $\mathscr{G}$,
* $f_i(a) = f_i(a_1, \ldots, a_n)$ for all $1 \leq i \leq s$, and
* $g_j(a) = g_j(b_1, \ldots, b_m)$ for all $1 \leq j \leq r$,

it follows that $a \in V \times W$. Hence $V \times W = V(\mathscr{F}, \mathscr{G})$. Therefore, it is a variety.