MATH 497 INTRODUCTION TO APPLIED ALGEBRAIC GEOMETRY
HOMEWORK 5

Assigned 9/18, due 9/25 in class.

Reading this week is pages 115-136 and 169-182 or so in your book, Cox, Little, and O'Shea's *Ideals, Varieties, and Algorithms*. We will probably skip singular points and go to Chapter 4 (Nullstellensatz, decomposition into irreducibles) next. With Chapter 4, we've covered most of the background, except projective space and projective varieties (Chapter 8), which we might just learn as we use it. Hence we will be focusing more on applications soon, and on material less well covered by existing books.

Problem 1. *Write up your solutions (the defining equations, and how you got them) to the four coin problems from class and recitation: 3 or 4 coins, with and without the two-pocket situation.*

Problem 2. *Now think of the table of probabilities as a matrix or linear map from the vector space spanned by the outcomes for some of the coins to that spanned by the outcomes of the other. What is the rank of this matrix in each of the four cases above? Explain how this gives you a concise description of the defining equations.*

Problem 3. *Now suppose we have a two-person, two-pocket, four-coin model (so 16 coins in the pocket and two which-pocket coins). We are going to combine their probability distributions by multiplying and then rescaling until they sum to one. So before rescaling, with* $i, j, k, l \in \{H, T\}$, *we have*

$$\begin{aligned} x_{ijk\ell} &= p_{L,1}s_{L,i,1}s_{L,j,2}s_{L,k,3}s_{L,\ell,4} + p_{R,1}s_{R,i,1}s_{R,j,2}s_{R,k,3}s_{R,\ell,4} \\ y_{ijk\ell} &= q_{L,1}t_{L,i,1}t_{L,j,2}t_{L,k,3}t_{L,\ell,4} + q_{L,R,1}t_{R,i,1}t_{R,j,2}t_{R,k,3}t_{R,\ell,4}. \end{aligned}$$

And we define $z_{ijk\ell} = x_{ijk\ell}y_{ijk\ell}$.

(1) *Argue that we can drop the parameters associated with pockets (equivalently just set these coins to be fair). Thus we can write the model with 32 parameters as*

$$z_{ijk\ell} = (s_{L,i,1}s_{L,j,2}s_{L,k,3}s_{L,\ell,4}+s_{R,i,1}s_{R,j,2}s_{R,k,3}s_{R,\ell,4})(t_{L,i,1}t_{L,j,2}t_{L,k,3}t_{L,\ell,4}+t_{R,i,1}t_{R,j,2}t_{R,k,3}t_{R,\ell,4})$$

or changing the names of the variables to simplify the notation,

$$z_{ijk\ell} = (a_ib_jc_kd_\ell + e_if_jg_kh_\ell)(m_in_jo_kp_\ell + q_ir_js_kt_\ell)$$

where $ijk\ell \in \{H, T\}^4$, *so there are 32 parameters, two for each coin, and one letter per coin.*

(2) *Argue that the implicitization in terms of the* $z_{ijk\ell}$ *is a hypersurface in* $\mathbb{P}^{15}$ *(defined by one equation). Equivalently, with the appropriate assumptions about probabilities summing to one, you can show it is a hypersurface in the probability simplex on 16 events.*
(3) *Can you find the polynomial that defines this hypersurface?*

In Chapter 3, Section 1, do Exercises 1, 5, and 6.
In Chapter 3 Section 2, do Exercises 1 and 3.