What Is a Tensor?

As we have seen in class, there are a lot of different ways to understand a tensor. The most fundamental way is to think of a tensor as a vector in a space constructed from taking the tensor product.

Let V, W be vector spaces. The **tensor product** $V \otimes W$ is the set of elements

$$\sum_{i=1}^{n} c_i v_i \otimes w_i$$

such that the following holds:

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
(3) $(cv) \otimes w = c(v \otimes w) = v \otimes (cw)$

for all $v_i \in V$ and $w_i \in W$.

Given two vectors v, w, we can treat $v \otimes w$ as the **Kronecker product**:

$$(v_1, v_2, v_3) \otimes (w_1, w_2) = (v_1 w_1, v_1 w_2, v_2 w_1, v_2 w_2, v_3 w_1, v_3 w_2)$$

Exercise 1. Show that the vector η below is an element of $\mathbb{R}^2 \otimes \mathbb{R}^3$ but that there does not exist vectors $v \in \mathbb{R}^2, w \in \mathbb{R}^3$ such that $v \otimes w = \eta$.

$$\eta = (0, 1, 1, 2, 0, 1)$$

Solution 1. Suppose the contrary. Because the first entry is zero and $\mathbb{R}$ is a field, then either $v_1 = 0$ or $w_1 = 0$. The second entry indicates that $v_1 \neq 0$ and the fourth entry indicates $w_1 \neq 0$, which is a contradiction.

Let $\{e_1, e_2, e_3\}$ be a basis for V_1 and $\{u_1, u_2\}$ be a basis for V_2. Then a basis for $V_1 \otimes V_2$ is

$$\{e_1 \otimes u_1, e_1 \otimes u_2, e_2 \otimes u_1, e_2 \otimes u_2, e_3 \otimes u_1, e_3 \otimes u_2, \}$$

Exercise 2. Consider the claim mentioned above. Let $V_1 = \mathbb{R}^3$, $V_2 = \mathbb{R}^2$ and let both e_i and u_i be standard basis vectors. Show that the basis

$$\{e_1 \otimes u_1, e_1 \otimes u_2, e_2 \otimes u_1, e_2 \otimes u_2, e_3 \otimes u_1, e_3 \otimes u_2, \}$$

is the standard basis for the space $\mathbb{R}^6$.

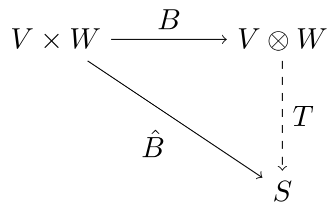
Solution 2. Write out the Kronecker product for each one:

- $e_1 \otimes u_1 = (1, 0, 0, 0, 0, 0)$
- $e_1 \otimes u_2 = (0, 1, 0, 0, 0, 0)$
- $e_2 \otimes u_1 = (0, 0, 1, 0, 0, 0)$
- $e_2 \otimes u_2 = (0, 0, 0, 1, 0, 0)$
- $e_3 \otimes u_1 = (0, 0, 0, 0, 1, 0)$
- $e_3 \otimes u_2 = (0, 0, 0, 0, 0, 1)$

Remark 0.1. Given two vector spaces over the same field, k^m, k^n, their tensor product $k^m \otimes k^n \cong k^{mn}$. Similarly, $k^m \otimes k^n \cong k^n \otimes k^m$. A lot of sloppiness in the treatment of tensors arises from this fact.

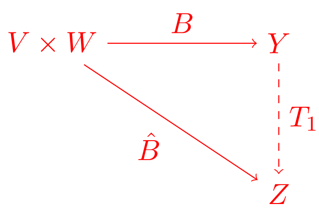
Remark 0.2. In many resources, the preferred definition of a tensor product is one based on the "universal property." A universal property is exactly what it sounds like–a property that must hold true regardless of the setting. With the rise of category theory in the 20th century, defining mathematical concepts using its universal property has become the norm, especially in geometry. Such definitions avoid a choice of representation (e.g. a choice of basis) and, as a result, are often heralded as more elegant. While I agree with this view, I think such definitions can appear too mysterious to one just learning the subject. Despite the question of its helpfulness, I present the definition here:

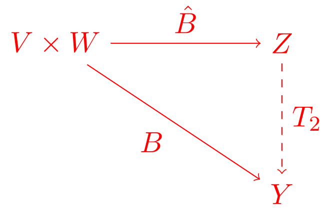
The **tensor product** of vector spaces V and W is a vector space $V \otimes W$ equipped with a fixed surjective bilinear map $B : V \times W \to V \otimes W$ such that for any vector space S and bilinear map $\hat{B} : V \times W \to S$, there is a unique linear map $T : V \otimes W \to S$ such that $\hat{B} = T \circ B$. This is depicted as follows:



Exercise 3. Use the universal property to show that if the tensor product exists, it must be unique.

Solution 3. Let Y and Z both be tensor products of the vector spaces V, W. Then, we can observe the following two diagrams hold:





This means $\hat{B} = T_1 \circ B$, $B = T_2 \circ \hat{B}$, and, hence, $\hat{B} = T_1 \circ T_2 \circ \hat{B}$ and $B = T_2 \circ T_1 \circ B$. Thus, T_1 and T_2 are invertible maps and must be isomorphisms.

0.1 Tensors as Maps. Let us now pay close attention to the subscripts of the vectors. Then the tensor product written previously

$$(v_1, v_2, v_3) \otimes (w_1, w_2) = (v_1w_1, v_1w_2, v_2w_1, v_2w_2, v_3w_1, v_3w_2)$$

could be reorganized as a matrix in the obvious way:

$$\begin{pmatrix} v_1w_1 & v_1w_2 \\ v_2w_1 & v_2w_2 \\ v_3w_1 & v_3w_2 \end{pmatrix}$$

Notice that this is a linear map whose domain is $\mathbb{R}^2$ and whose range is $\mathbb{R}^3$. In this way, elements in $\mathbb{R}^3 \otimes \mathbb{R}^2 \cong \mathscr{L}(\mathbb{R}^2, \mathbb{R}^3)$.

Exercise 4. Explain why this matrix must be rank 1. Does that mean all of $\mathbb{R}^3 \otimes \mathbb{R}^2$ is congruent to only rank 1 matrices?

Solution 4. Observe that the columns are multiples of each other. Elements of $\mathbb{R}^3 \otimes \mathbb{R}^2$ are all finite sums of rank 1 tensors, which means matrices of higher rank are constructible.

Remark 0.3. "Rank" is a dangerous word in algebra, especially when talking about tensors. Above, we are considering the *matrix rank*. In physics, the *rank* is often used to denote the *order* of the tensor, like a vector is "rank 1," a matrix is "rank 2," etc. So when you search for "tensor rank" on the internet, you'll likely find this definition. In algebraic geometry, the word "order" is used to denote this. In algebraic geometry, the *tensor rank*, or simply the *rank*, is the minimal number of (matrix) rank 1 tensors needed to construct a tensor. If a tensor M can be constructed by 3 and no fewer elements of the form $v \otimes w$, then M has (tensor) rank 3.

Unfortunately, the terminology gets even worse with the introduction of the *generic rank*, sometimes denoted as "grank." We may discuss this later in the course.

There is a very nifty theorem from linear algebra called the Riesz Representation Theorem. It states that one can identify any $\vec{x}$ (from a finite dimensional space[1]) with a map $\varphi_{\vec{x}}(\cdot) : V \to k$ via the inner product:

$$\varphi_{\vec{x}}(\cdot) = \langle \vec{x}, \cdot \rangle .$$

Moreover, every linear functional is of this form. In this sense, we can identify the linear operator represented by

$$\begin{pmatrix} v_1w_1 & v_1w_2 \\ v_2w_1 & v_2w_2 \\ v_3w_1 & v_3w_2 \end{pmatrix}$$

with the map

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \langle w, \cdot \rangle$$

Exercise 5. Let $v = (2, 1, 3)$ and $w = (-1, 10)$. Pick two vectors and demonstrate the above claim over $\mathbb{R}$.

Solution 5. The inner product over $\mathbb{R}$ is the dot product. If v and w are as above, then the matrix is

$$\begin{pmatrix} -2 & 20 \\ -1 & 10 \\ -3 & 30 \end{pmatrix}.$$

We pick our two vectors to be the standard basis vectors. Then

$$\begin{pmatrix} -2 & 20 \\ -1 & 10 \\ -3 & 30 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} (w \cdot e_1)$$

and

$$\begin{pmatrix} -2 & 20 \\ -1 & 10 \\ -3 & 30 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} (w \cdot e_2)$$

This observation is just a rephrasing of $V_1 \otimes V_2 \cong \mathscr{L}(V_2, V_1)$. In particular, it is that $V_1 \otimes V_2 \cong V_1 \otimes V_2^*$.

Exercise 6. Prove that when both V_1 and V_2 are finite dimensional, $V_1 \otimes V_2 \cong V_1^* \otimes V_2$.

Solution 6. It is clear that $V_1 \otimes V_2 \cong V_2 \otimes V_1$ since we can take each basis element $e_j \otimes u_i$ of $V_1 \otimes V_2$ and map it to the basis element $u_i \otimes e_j$ of $V_2 \otimes V_1$. By the above discussion $V_2 \otimes V_1 \cong V_2 \otimes V_1^*$. If we employ the same logic with the bases, we get $V_1^* \otimes V_2 \cong V_2 \otimes V_1^*$.

[1]This theorem applies to infinite dimensions as well; however, $V \cong V^*$ is guaranteed to hold when V is finite dimensional. Given the scope of this class, we restrict to this case.

0.2 Composing Tensors. Tensors can be seen as vectors, as multi-dimensional arrays (e.g. matrices), and as linear maps. Let us focus in for a moment on how we can compose such maps.

Let W be a vector space of dimension n, let $\{\beta_1, \ldots \beta_n\}$ be an orthonormal basis of W, and let $\{\varphi_{\beta_1}, \ldots, \varphi_{\beta_n}\}$ be a corresponding basis of W^* such that

$$\varphi_{\beta_i}(\beta_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Exercise 7. Let $W = \mathbb{R}^3$ and let $w_i = e_i$, the standard basis. Find φ_{w_i}. Show that it is a basis for W^*.

Solution 7. Let $w = (w_1, w_2, w_3)$. Then

$$\varphi_{e_1}(w) = e_1 \cdot w = w_1.$$

Similar results hold for e_2 and e_3. Consider the linear functional f. By the Riesz representation theorem, there exists a vector $x \in W$ such that

$$\begin{aligned} f(\cdot) &= \langle x, \cdot \rangle = \langle a_1 e_1 + a_2 e_2 + a_3 e_3, \cdot \rangle \\ &= a_1 \langle e_1, \cdot \rangle + a_2 \langle e_2, \cdot \rangle + a_3 \langle e_3, \cdot \rangle \\ &= a_1 \varphi_{\beta_1}(\cdot) + a_2 \varphi_{\beta_2}(\cdot) + a_3 \varphi_{\beta_3}(\cdot) \end{aligned}$$

Based on the previous discussion, we can take any element in the space $V \otimes U \otimes W$ and think of it as a map $T : W \to V \otimes U$

$$T(\cdot) = \sum_i^n a_i (v_i \otimes u_i) \varphi_{w_i}(\cdot)$$

Similarly, we can think of another element of $W \otimes X \otimes Y$ as a map $S : X \otimes Y \to W$

$$S(\cdot) = \sum_i^n b_i w_i \varphi_{x_i \otimes y_i}(\cdot)$$

Hence, using the maps S and T, we can construct a map $R = T \circ S$ that can be identified as an element of $V \otimes U \otimes X \otimes Y$:

$$\begin{aligned} T \circ S &= \sum_i^n a_i (v_i \otimes u_i) \varphi_{w_i}(\cdot) \circ \sum_i^n b_i w_i \varphi_{x_i \otimes y_i}(\cdot) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i b_j (v_i \otimes u_i) \varphi_{w_i}(w_j) \varphi_{x_j \otimes y_j}(\cdot) \\ &= \sum_{i=1}^n \sum_{j=1}^n c_{ij} (v_i \otimes u_i) \varphi_{x_j \otimes y_j}(\cdot) \end{aligned}$$

Alternatively, we can write these two elements as matrices in $(V \otimes U) \otimes W$ and $W \otimes (X \otimes Y)$ by treating $(V \otimes U)$ and $(X \otimes Y)$ as their corresponding vector spaces. This is a "flattening" of the tensor. When we multiply these two matrices, we can a third matrix that can be identified with the correct vector in $V \otimes U \otimes X \otimes Y$.

Exercise 8. Let $W = \mathbb{R}^3$ and let $U = V = X = Y = \mathbb{R}^2$. Let $(1, 2, 1, 3, 1, 0, -1, -1, 1, -1, 0, 0)$ be a vector in $V \otimes U \otimes W$ and let $(-1, 0, -1, 2, 1, 1, 1, -1, 1, -1, 0, 2)$ be a vector in $W \otimes X \otimes Y$. Find the corresponding tensor in $V \otimes U \otimes X \otimes Y$ that arises from the composition.

Solution 8.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 2 \\ -2 & 1 & -2 & 5 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

As we've seen in class, the spaces over which we contract (like W) are typically parameter spaces.

0.3 Partial Traces. In linear algebra, the trace is a map from $\mathscr{L}(V) \to k$. It is considered interesting for a variety of reasons including

- it is invariant under a change of basis,
- it is linear, and
- it is related to the inner product.

In the language of tensors, square matrices are elements in the tensor product of isomorphic spaces, e.g. $V \otimes V$, since these are isomorphic to $\mathscr{L}(V)$.

Consider an arbitrary element in $V \otimes V$:

$$\sum_{i=1}^{n} c_i (v \otimes w).$$

Then the trace arises from

$$\text{tr}\left[\sum_{i=1}^{n} c_i (v \otimes w)\right] = \sum_{i=1}^{n} c_i \text{tr}\left[(v \otimes w)\right].$$

Exercise 9. What is the trace of an outerproduct?

Solution 9. It is the inner product!

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} (w_1, w_2, \ldots, w_n) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & \ldots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \ldots & v_2 w_n \\ \ldots & \ldots & \ddots & \ldots \\ v_n w_1 & v_n w_2 & \ldots & v_n w_n \end{pmatrix}$$

Hence, the trace is $v_1 w_1 + \ldots v_n w_n$.

When working with higher order tensors, the analog of the trace known as the **partial trace**. The traditional trace maps matrices to scalars while the partial trace reduces the order by 2.

Exercise 10. What must be true of a tensor's order for all possible partial traces to exist?

Solution 10. The order must be of the form 2^n. Do not confuse the order with the dimension. For example, if I am considering $A \in \mathscr{L}(V \otimes V)$ where $\dim V = 3$, the order of A is 4 (since it is in a space isomorphic to $V \otimes V \otimes V \otimes V$), but it is a vector in a space isomorphism to k^{81}.

Partial traces only make sense for tensors which are linear operators, e.g. elements of $\mathscr{L}(V_1 \otimes V_2 \otimes \ldots \otimes V_n)$.

To get a handle on partial traces, it is best to think of tensors as multi-dimensional arrays instead of vectors. In other words, we want to be mindful of the indices and organize the entries accordingly.

Consider the space $V \otimes W$, where W may have a different dimension. Elements in $\mathscr{L}(V \otimes W) \cong V \otimes W \otimes V^* \otimes W^*$ have four indices, each corresponding to a particular space. Let these indices be i, j, k, ℓ, corresponding to V, W, V^*, W^*. Since we work over finite dimensions, remember that spaces are congruent to their duals. Let A be a tensor in this space. If we wish to trace out A with respect to *only* W, we will "match" the indices and sum. Therefore,

$$\text{tr}[A] = \hat{a}_{ik} = \sum_{j=1}^{n} a_{ijkj}$$

Thus,

$$A = \begin{pmatrix} a_{1111} & a_{1112} & \dots & a_{11nn} \\ a_{1211} & a_{1212} & \dots & a_{12nn} \\ \dots & \dots & \ddots & \dots \\ a_{nn11} & a_{nn12} & \dots & a_{nnnn} \end{pmatrix} = \begin{pmatrix} \sum a_{1i1i} & \sum a_{1i2i} & \dots & \sum a_{1ini} \\ \sum a_{2i1i} & \sum a_{2i2i} & \dots & \sum a_{2ini} \\ \dots & \dots & \ddots & \dots \\ \sum a_{ni1i} & \sum a_{ni2i} & \dots & \sum a_{nini} \end{pmatrix}$$

Exercise 11. Let $A \in \mathscr{L}(V \otimes V)$ where $\dim(V) = 2$. What are the two distinct partial traces of A?

Solution 11. Our matrix A is

$$A = \begin{pmatrix} a_{1111} & a_{1112} & a_{1121} & a_{1122} \\ a_{1211} & a_{1212} & a_{1221} & a_{1222} \\ a_{2111} & a_{2112} & a_{2121} & a_{2122} \\ a_{2211} & a_{2212} & a_{2221} & a_{2222} \end{pmatrix}$$

Then we have two partial traces:

$$\rho_1(A) = \begin{pmatrix} a_{1111} + a_{2121} & a_{1112} + a_{2122} \\ a_{1211} + a_{2221} & a_{1212} + a_{2222} \end{pmatrix}$$

$$\rho_2(A) = \begin{pmatrix} a_{1111} + a_{1212} & a_{1121} + a_{1222} \\ a_{2111} + a_{2212} & a_{2121} + a_{2222} \end{pmatrix}$$

Exercise 12. Consider the two operators you got from the previous exercise. Find their traces. Are they the same?

Solution 12. Both have partial traces

$$a_{1111} + a_{1212} + a_{2121} + a_{2222}$$

Depicting Tensors: Tensor Networks

Imagine for a moment that you wanted to represent a tensor using a picture, but you wanted to do it in such a way that a lay person could easily pick up the rules without knowing any linear algebra. How might you do it?

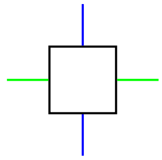
In particular, we'd like to capture the following:

(1) space in which the tensor lives (i.e. how many vector spaces must I tensor product to get the ambient space of the tensor),
(2) and how the tensors can be composed (from now on we will say **contracted** instead).

The answer is tensor networks. To make the diagrams completely explicit, I will color the wires based on the vector spaces they represent.

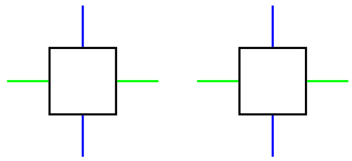
A tensor is represented by a box (in special cases, they can be circles and triangles). Each wire represents a vector space; the tensor product of the vector spaces corresponding to each wire is the ambient space where the tensor resides. Because $V \otimes U \cong U \otimes V$ (why?), the order in which we "read" the diagram is unimportant so long as we are consistent.

Therefore, an element $A \in V \otimes V \otimes V \otimes V$ is represented by

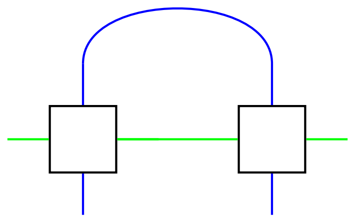


Imagine that these wires are bendable and that two tensors can be composed/contracted if and only if they both have wires of the same color.

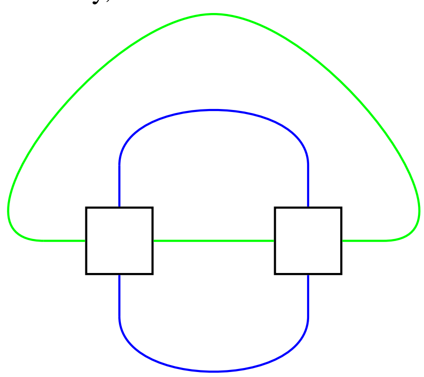
If I draw the *same* picture multiple times, it does not necessarily mean that I have three copies of the same tensor. Instead, it means I have two tensors coming from the same space.



I can compose these two tensors in a lot of different ways. The number of wires left over from the composition tells me the space in which these two tensor lives. For example,



is an element of $V \otimes V \otimes V \otimes V$. Similarly,



Is a scalar.

Exercise 13. Prove that the above picture is a trace of the composition.

Solution 13. The above diagram is the composition of two, rank 4 tensors living in the same space. Call these tensors T, U.

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} t_{ijkl}u_{ijkl}.$$

Define a new variable $v_{ijklijkl} = t_{ijkl}u_{ijkl}$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} v_{ijklijkl}.$$