Research Statement

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1 Introduction and research agenda

My research is in Mathematical Logic and Foundations of Mathematics. More specifically, I am interested in non-classical logics and the nature of mathematics over foundations other than the classical.

1.1 Intuitionism

In my doctoral dissertation, and in my work since, I have studied *intuitionistic logic*, a non-classical logic first proposed by L. E. J. Brouwer in the early 20th century as an alternative constructive approach to mathematics. The basic tenets of this philosophy can be found in [14] and [15, Chapter 1]. One of the central ideas in this philosophy is that the truth or falsity of a mathematical statement is not independent of our knowledge concerning it - a statement is true if we have a proof of it and false if we can show that assuming the existence of a proof of it leads to a contradiction. Thus certain classical principles like the law of excluded middle, which assert that any arbitrary mathematical statement is either true or false, are unacceptable from an intuitionistic point of view.

The logical analysis of intuitionistic or constructive mathematics began with Arend Heyting's rigorous formalization of intuitionistic propositional calculus, arithmetic, and analysis. Shortly afterward, in 1932, A. N. Kolmogorov proposed a simple and natural, but non-rigorous, semantics for Heyting's intuitionistic propositional logic, called the *calculus of problems* [9]. We have extended a rigorous formulation of this semantics to intuitionistic higher-order logic via sheaves.

1.2 Intuitionistic higher-order logic and sheaf semantics

Higher-order logic is a kind of logic where, in addition to quantifiers over objects, one has quantifiers over pairs of objects, sets of objects/ pairs, and so on. Higher-order logic thus calls for a *many-sorted* or *typed* language. This language, together with the appropriate axioms and rules is rich enough to permit the development of virtually all of mathematics. If the axioms and rules are intuitionistic, we get virtually all of intuitionistic mathematics.

In the later 1930s, Alfred Tarski noticed that for any topological space T, the lattice of open sets of T is a model for intuitionistic propositional calculus. In 1979 Dana Scott, along with M.P. Fourman and J. Hyland, pointed out in a series of papers [4, 13, 5] that the category of sheaves of sets over a topological space T, denoted by $\mathrm{Sh}(T)$ is a model for intuitionistic higher-order logic.

A *sheaf* over a topological space T, is a pair (X, p) where X is a topological space and p is a local homeomorphism of X into T. A sheaf morphism from (X, p) to (Y, q) is then a continuous function $f : X \longrightarrow Y$ such that $p(x) = q(f(x))$ for all $x \in X$[1]. One can then easily define product sheaves,

[1] Sheaves can be defined in various ways. In our paper [2], we used a different but equivalent definition of $\mathrm{Sh}(T)$.

power sheaves, function sheaves, subsheaves, etc. The collection of all sheaves over a topological space T, denoted by $\mathrm{Sh}(T)$, forms a category, and in fact a *topos*.

To interpret intuitionistic higher-order logic in a sheaf model over a topological space T, one then interprets the various sorts as sheaves in $\mathrm{Sh}(T)$, such that the product, function and power sorts are interpreted by the corresponding product, function and power sheaves.

It is well known that the properties of the topological space T determine the properties of $\mathrm{Sh}(T)$, and that in turn impacts the nature of intuitionistic set theory and mathematics in this sheaf model. For example, the axiom of countable choice[2] is true in $\mathrm{Sh}(T)$ for some topological spaces T (such as $T = \mathbb{N}^{\mathbb{N}}$, the Baire space) and not true for some other topological spaces (such as $T = \mathbb{R}$, the real numbers with the usual topology).

The axiom of countable choice has a significant impact on the nature of intuitionistic analysis, since if the axiom of countable choice is true in a sheaf model, then the sheaf representations of the Cauchy reals $\mathbb{R}_C$ and the Dedekind reals $\mathbb{R}_D$ coincide in that sheaf model[3]. See [15, Chapter 5, Volume I; Chapter 15, Volume II] for further discussion on this.

1.3 Degrees of unsolvability and mass problems

Recursion theory, another active area of research in logic and foundations, also started as a constructive approach to mathematics in the 1930's. Alan Turing, in 1936, showed that algorithmically unsolvable problems exist in mathematics, and it then became desirable to compare the unsolvabilities of such problems. This led to Turing's work on *relative computability* and *oracle machines*, which gave rise to the notions of *Turing reducibility* and *Turing degrees*. The set of all Turing degrees, denoted by $\mathcal{D}_{\mathrm{T}}$, forms a partially ordered set. For a detailed discussion, see for example [12, Chapter 9].

Based on the notion of Turing reducibility, Y. T. Medvedev introduced *mass problems* in 1955 [10]. A mass problem is defined as a subset of $\mathbb{N}^{\mathbb{N}}$. Informally, to "solve" a mass problem P means to "find" or "construct" an element of the set $P \subseteq \mathbb{N}^{\mathbb{N}}$. Medevedev also defined a reducibility notion, currently called *strong reducibility* between these mass problems.

Later, in 1963, A. A. Muchnik [11] introduced an alternative notion of reducibility for mass problems. This is called *weak reducibility* or *Muchnik reducibility*.

Mass problems that are mutually strongly (weakly) reducible are called *strongly (weakly) equivalent*. The equivalence classes under the relation of *strong (weak) equivalence* $\equiv_{\mathrm{s}}$ ($\equiv_{\mathrm{w}}$) are then called the *strong (weak) degrees* or *Medvedev (Muchnik) degrees*. The set of strong (weak) degrees form a lattice, denoted by $\mathcal{D}_{\mathrm{s}}$ ($\mathcal{D}_{\mathrm{w}}$).

The mass problems were introduced by Medvedev in an attempt to rigorously define the notion of a problem in Kolmogorov's calculus of problems. Medvedev and Muchnik showed in their papers, [10] and [11], respectively, that $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ are models of intuitionistic propositional logic.

1.4 The Muchnik topos

In my Ph.D. dissertation [1], we studied sheaf models over general topological spaces. In the process a number of results on this topic were given better and cleaner presentations. We then specialized to sheaves over poset spaces with the corresponding topologies of upwardly closed sets,

[2] It is well established that the full axiom of choice is non-constructive and hence is not true in any model of intuitionistic logic that is also not a model of classical logic (see [3]), however the axiom of countable choice is true in some, but not all, models of intuitionistic logic.

[3] The Cauchy and the Dedekind reals are not, in general, identical sets in an intuitionistic setting.

or the Alexandrov topologies. These special sheaf models can also be seen as *Kripke models* for intuitionistic higher-order logic, and are thus important models to study in their own right.

We proved that a certain generalized version of the axiom of countable choice is true in a sheaf model over a poset, and hence in such a model, the sheaf representations of the Cauchy reals and the Dedekind reals are identical. We also proved that Markov's principle, another principle that is not true in all models of intuitionistic logic, is true in sheaf models over poset spaces.

We then specialized further to the sheaf model $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$, where $\mathcal{D}_{\mathrm{T}}$ is the poset of Turing degrees with its Alexandrov topology. This is what we call the *Muchnik topos*, as it is the direct extension of Muchnik's $\mathcal{D}_{\mathrm{w}}$ interpretation of intuitionistic propositional calculus. Thus the Muchnik topos, while being a special topological and Kripke model for intuitionistic higher-order logic, is historically significant as a continuation of the work of Kolmogorov and Muchnik connecting constructivism with computability.

1.5 The Muchnik reals

As discussed above, the Dedekind and the Cauchy real numbers are identical in a sheaf model over any poset space, so these are identical in the Muchnik topos. We have however discovered, within the Muchnik topos, a new sheaf representation of intuitionistic real numbers, the *Muchnik reals*, which are different from the Cauchy and the Dedekind reals.

The Muchnik reals can be described as the sheaf (X, p) over $\mathcal{D}_{\mathrm{T}}$ where $X = \{(\mathrm{a}, r) \in \mathcal{D}_{\mathrm{T}} \times \mathbb{R} \mid \mathrm{a} \geq \deg_{\mathrm{T}}(r)\}$ and $p((\mathrm{a}, r)) = \mathrm{a}$[4]. Informally, a Muchnik real is a real number that "comes into existence" only when we have enough Turing oracle power to compute it.

Within the Muchnik topos, we have also obtained a *choice principle* for the Muchnik reals. Full details on these can be found in [1], and my forthcoming paper with Stephen Simpson [2].

2 Current line of research

The following subsections describe the current foci of my research.

2.1 The Muchnik topos and the Muchnik reals

I am currently working on extending my work on the Muchnik topos to understand the nature of intuitionistic mathematics as interpreted in this new model. One of the means of doing that is testing the truth of various principles that are classically valid but constructively questionable, or classically invalid but constructively admissible. Some of these principles, which are valid on a classical reading of the logical operators, but in a constructive setting may or may not be true, are listed below.

1. **WLEM** (The weak law of excluded middle): For any proposition A, $\neg A$ or $\neg\neg A$ is true. Although the law of excluded middle is unacceptable from a constructive point of view, the weaker version of it is true in some models of intuitionistic logic. We have shown in [1, Section 4.2] that WLEM is not true in all sheaf models, but is true in a sheaf model over a directed poset[5]. Since the poset of Turing degrees $\mathcal{D}_{\mathrm{T}}$ is a directed poset, WLEM is true in $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$.

2. **MP** (Markov's principle): This can be expressed as $(\forall x\,(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x\, A(x)) \implies \exists x\, A(x)$, where $A(x)$ is an arithmetical formula. MP is also not true in all sheaf models, but

[4] For an alternative definition of the Muchnik reals, see [2].

[5] A poset $(K, \leq)$ is said to be directed if for any $\alpha, \beta \in K$, there is a $\gamma \in K$ such that $\alpha, \beta \leq \gamma$.

we have shown in [1, Section 4.3] that MP is true in a sheaf model over any poset space, and hence MP is true in the Muchnik topos.

3. **WKL** (Weak König's lemma): Every infinite decidable tree has an infinite branch.

4. **LPO** (The limited principle of omniscience): For any binary sequence α, either $\alpha(n) = 0$ for all n or there exists n such that $\alpha(n) = 1$.

5. **WLPO** (The weak limited principle of omniscience): For any binary sequence α, either $\alpha(n) = 0$ for all n or it is not the case that $\alpha(n) = 0$ for all n.

6. **LLPO** (The lesser limited principle of omniscience): For any binary sequence α with at most one non-zero term, either $\alpha(n) = 0$ for all even n or $\alpha(n) = 0$ for all odd n.

7. The Bolzano-Weierstrass theorem: This well-known theorem of classical real analysis says that every bounded infinite set of real numbers has a point of accumulation.

Note that, we currently have two sets of intuitionistic real numbers within the Muchnik topos, the Cauchy/ Dedekind reals and the Muchnik reals. So a principle, like the Bolzano-Weierstrass theorem, about the real numbers naturally splits into two questions.

Some examples of principles that are incompatible with classical logic are the various intuitionistic forms of Church's thesis, expressing the idea that "all operations in mathematics are algorithmic" and continuity principles for *choice sequences*.

2.2 Realizability interpretations of intuitionistic logic

S. C. Kleene in 1945 [7] proposed an interpretation of intuitionistic number theory that connected intuitionism and the theory of recursive functions. This interpretation that he called *recursive realizability* started off with the conjecture that "if a closed formula of the form $\forall x\, \exists y\, \varphi(x, y)$ is provable in intuitionistic number theory, then there must be a general recursive function $F : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, the formula $\varphi(\overline{n}, \overline{F(n)})$, where $\overline{n}$ and $\overline{F(n)}$ are the numerals denoting the natural numbers n and $F(n)$", is true. Kleene's realizability was inspired as much by Brouwer's intuitionism as by Hilbert's idea that quantified statements be treated finitistically as expressing "incomplete communications". So this connection between constructivism and computability had a genesis that was different from the Kolmogorov/ Muchnik interpretation.

Since its introduction, various realizability interpretations have been devised not only for elementary arithmetic, but for higher-order arithmetic, type theory, set theory, and higher-order logic.

I am currently studying these realizability interpretations. Besides exploring another set of interpretations of intuitionistic logic, a purpose of this study is to compare the Muchnik topos with the realizability interpretations.

3 Future Plans

3.1 Sheaf models like the Muchnik topos

As evident from the above discussions, there are multiple connections between intuitionism and computability theory that date back to the earliest years of these two subjects. The Muchnik topos and the various realizability interpretations illustrate some of these connections. Further connections, similar to the Muchnik topos can be established via the various other degree structures, two of which are described below, that naturally arise in computability theory.

A set of natural numbers is said to be *arithmetical*, if it can be defined by a formula in the language of Peano arithmetic. A set X of natural numbers is said to be *arithmetical in a set* Y of natural numbers, or equivalently, X is *arithmetically reducible* to Y, if X is definable in the language of Peano arithmetic extended by a predicate for membership in Y. This relation is reflexive and transitive and thus the arithmetical equivalence relation defined by: X is *arithmetically equivalent* to Y iff X is arithmetical in Y and Y is arithmetical in X, is an equivalence relation. The equivalence classes under this relation are called *arithmetical degrees*, which form a poset.

A set of natural numbers is called *hyperarithmetical* if it is definable by a formula of second-order arithmetic with only universal set quantifiers and also by a formula of second-order arithmetic with only existential set quantifiers. Then *hyperarithmetical equivalence* and *hyperdegrees* can be defined in the same way as arithmetical equivalence and arithmetical degrees. The hyperdegrees also form a poset.

We plan to study the sheaf models over the posets of the arithmetical degrees and the hyperarithmetical degrees. These degree structures are closely related to the Turing degrees, and hence these sheaf models will provide further connections between constructivism and computability theory. The axiom of countable choice will be true in these models and hence the Cauchy and Dedekind reals will coincide in these sheaf models, by virtue of the fact that these are sheaf models over poset spaces. We then plan to ask the same questions about the Cauchy/ Dedekind reals that we are presently asking of the Muchnik topos.

Apart from the above two degree structures, there are other reducibility notions in computability theory, such as truth-table reducibility, weak truth-table reducibility, LR-reducibility, and so on. These notions also yield degree structures that are posets. Hence sheaf models over these would also provide connections between constructivism and computability theory. We plan to draw comparisons between these models and also with the realizability interpretations.

4 Importance and significance

A general reason to pursue this work is to serve the purpose of the study of logic to clarify foundational principles, a part of which is understanding which combinations of such do and do not imply others. With a variety of models, which are related to each other, one can attempt to separate these foundational principles.

A more specific purpose is to understand constructive logic and mathematics, the underlying philosophy of which has been shrouded in mysticism and subject to multiple interpretations since its inception.

Lastly, intuitionism and computability both started off as constructive schools of thought and have roots in the same set of basic ideas, but grew to become quite different, so much so that computability theory is now largely regarded as a branch of the classical study of foundations of mathematics. Hence investigating the connections between the two is, in a way, studying the divide between classical and constructive mathematics.

5 Summary

My research agenda clearly reflects my interest in non-classical logics. I am also interested in connections between logic and algebra, and applications of logic to computer science.

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