

Two Dimensional Random Walk on Percolation Clusters

George M Paily, Shivakumar Jolad, Sanghamitra Neogi*

Department of Physics, Pennsylvania State University, University Park, PA 16803

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Random walk on lattices of different dimensions has been well studied in literature. The behavior of the walk changes drastically when the walk is restrained by constraints like ‘Percolation cluster’. Such walks belong to a different set of Universality class. In this report we illustrate such a behavior by simulating Random walk on a 2D lattice percolation cluster (psuedo infinite). We have been able to observe the asymptotic behavior of the mean square displacement from the origin, the number of returns to the origin and the number of distinct sites visited as a function of time with the help of the simulation. Calculation on these asymptotes gives us fracton dimension \tilde{d} , fractal dimension D and exponent d_w . We show that the Random Walk on a 2D lattice percolation cluster belongs to the universality class with fracton dimension $4/3$. This report is essentially an implementation of the paper [1].

I. INTRODUCTION

Random walk is mathematical formulation of the intuitive idea of taking successive steps, each in a random direction. Problems involving Random walks are ubiquitous in nature. They can be used as good models to understand various phenomenons like Brownian motion, diffusion processes. In polymer physics, random walk helps to describe the ideal chain. It also finds its application in various other fields like economics, brain research etc. The behavior of random walks depends on certain parameters like spatial dimensions of the walk, constraints put on the walk. Unconstrained random walk typically belongs to the universality class where the root mean square distance varies asymptotically as square root of the number of steps taken $\langle R^2 \rangle^{\frac{1}{2}} \rightarrow \sqrt{N}$ and is independent of the dimensions of the walk. On the other hand the number of returns to origin and the number of distinct sites visited in N steps depend on dimensions of the walk. In these higher dimensions Random walks show a fractal structure. One can define dimensions like the fractal dimension, fracton dimension etc on these walks. These dimensions and related exponents define the universality class of the Random Walk. We consider Brownian motion as for example. One of the well known structures in Brownian motion curve has a fractal dimension

of 2 and behaves like a space filling curve. The characteristic of walk also changes when one imposes constraints. One such constraint is to restrict the walk on a ‘Percolation cluster’. In this article we study the Random Walk on an invasion percolation cluster (and estimate the dimension parameters using its asymptotic behavior. We show that this belongs to different universality class with fracton dimension $4/3$.

II. PERCOLATION CLUSTER

A percolation model is a collection of points distributed in space with certain pairs (usually the adjacent neighbors) connected [2]. The distribution of these points can be modeled by assigning a site destruction probability p . A simple linkage can be implemented by linking adjacent occupied sites. Such linked points are said to be connected. All these points of the distribution may be partitioned into *clusters* such that pairs of points in the same cluster are connected but there is no path between points of different clusters. The size of the cluster and number of clusters in a given percolation model depends critically on the site occupation probability. If the percolation model size is such that boundary(surface) effects are negligible, one can effectively treat the model as a system with infinite sites and unbounded volume. In such a system, the size of the cluster may become infinite at some critical density of the linkages. One can then define the *percolation probability* as the probability that a given percolation model has an infinite cluster. The percolation

*Electronic address: gmp161@psu.edu, shivakumar@psu.edu, sun114@psu.edu

probability gives the chance that one can find an infinite sized cluster. This percolation probability $P(p)$ critically depends on p (site destruction probability). When $p > p_c$ (p_c is the critical probability), $P \rightarrow 1$ the system is said to be in the percolating state where one can find clusters of infinite extent. This situation is exactly similar to phase transition from disordered to ordered phase. One can analogously define all critical exponents to this problem. For example, the rms deviation from center $\zeta(p)$, is the analogue of correlation length (both diverge at the transition), the distinct number of sites in the cluster $S(p)$ is the analogue of susceptibility.

III. RANDOM WALK ON PERCOLATION CLUSTER

Percolation cluster at or beyond the transition point provides a constrained arena for Random Walk. The situation of the walker is that of an “ant in the labyrinth”. Here the walker (an ‘ant’) starts at an arbitrary point on the cluster and tries to move randomly to the nearest sites. If the randomly chosen direction of othe next step leads to an occupied site, he moves and the steps increment by 1. If he doesn’t find a site he stay’s there itself and the step is incremented. After long number steps one can analyze the asymptotic behavior of such a walk. The rms displacement, number of returns to the origin and number of distinct sites visited for the walk show some power law dependence on number of steps(not all these exponents are independent).

IV. SIMULATION

A. Simulation of 2D Random Walk

Our starting point for simulation was implementing a 2 dimensional random walk and checking to see if it agrees with the expected behavior. The following figures[Fig.1-2] show two typical Random Walks. These figures are typical of Brownian motion. One of the measure of its asymptotic behavior is the rms displacement, which when plotted shows the expected behavior of $\sqrt{(N)}$ as number steps N becomes large[Fig.3].

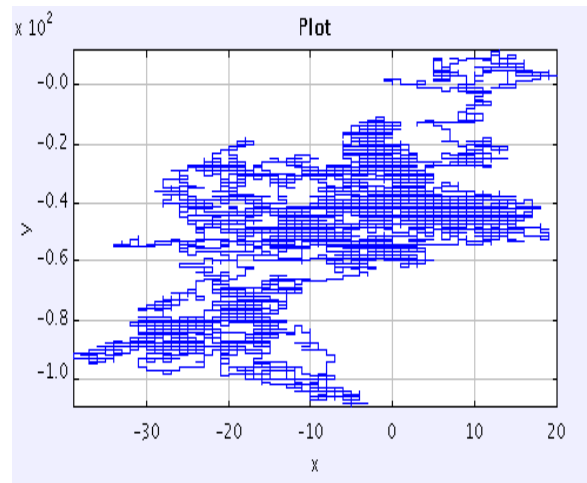


FIG. 1: Random walk in 2D trial 1

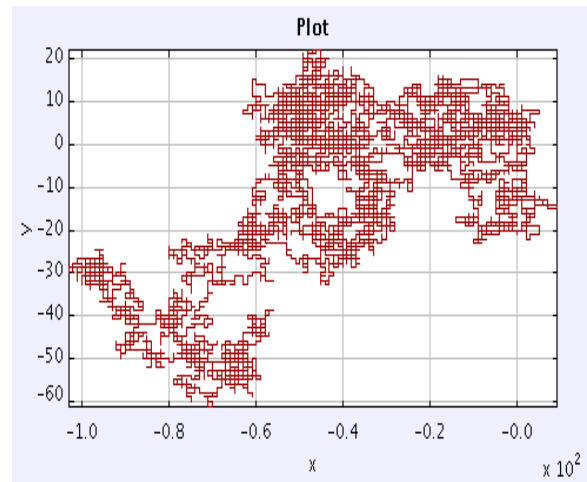


FIG. 2: Random walks in 2D trial 2

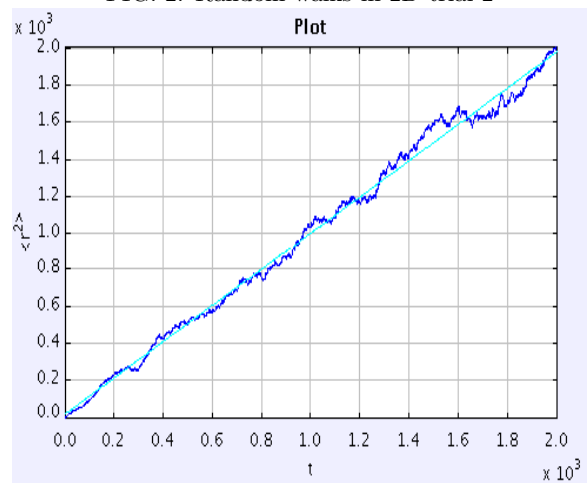
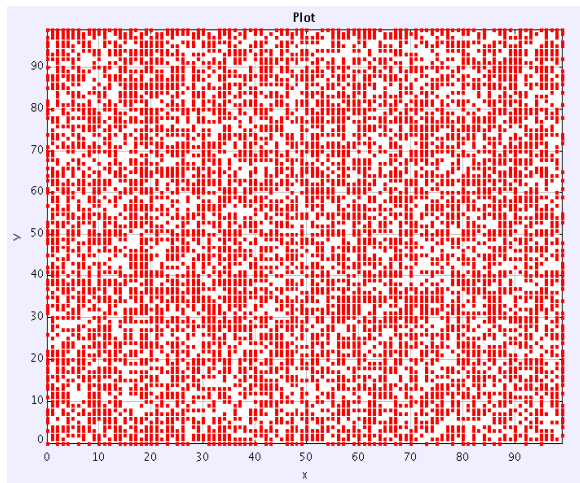
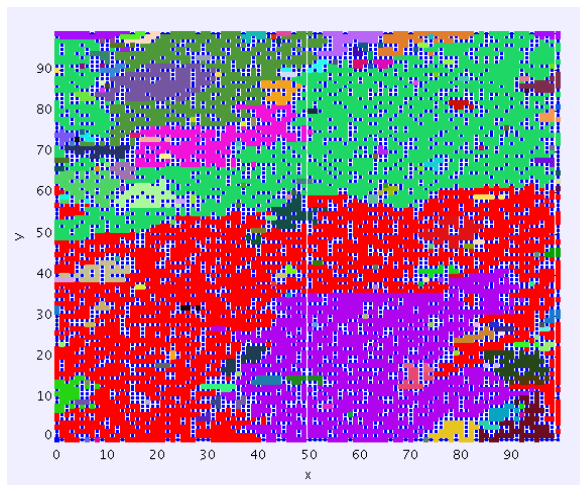
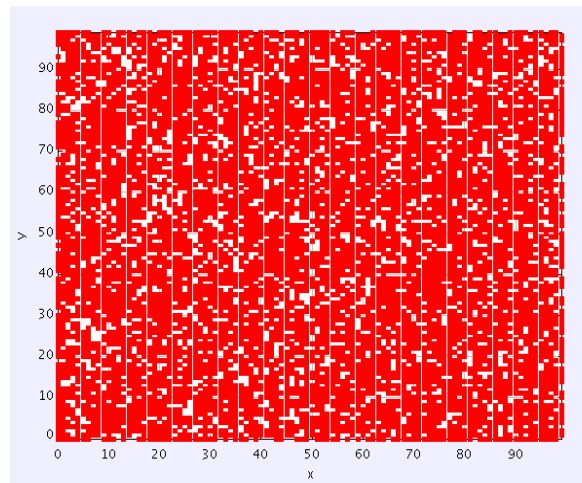
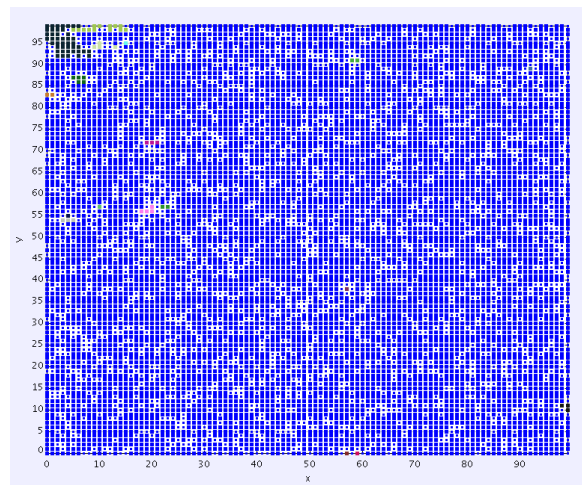


FIG. 3: mean square displacement versus number of steps in 2D RW

FIG. 4: Percolation model when $p < p_c$ FIG. 5: Percolation cluster when $p < p_c$

B. Creating infinite Percolation cluster

The second step is growing a percolation cluster by destroying sites on a 2D lattice. We chose the default value of the lattice to be 125×125 . After some critical probability the cluster size grew to a big size which can be considered as infinite cluster. If N is the number of steps in random Walk and M is the size of cluster (maximum rms distance from origin), one of the requirement for effective infinite cluster random walk is $M \gg N$. The program also identifies individual cluster and plots them in different color. One can show that near the threshold probability the size of clusters formed grow rapidly indicating a transition from disconnected (non conducting) to connected (conducting) phase (disordered to ordered

FIG. 6: Percolation model when $p > p_c$ FIG. 7: Percolation cluster when $p > p_c$

phase).

The following figures [Fig.4-7] show cluster formation and clusters when $p < p_c$ and $p > p_c$. We also wrote a program to count the number of clusters and size of the cluster. At the threshold probability one expects that the size of the largest cluster to suddenly grow and show a discontinuous behavior. We also expect the cluster number to be close to be one when p approaches zero. The following figures [Fig.8-9] plot the size of cluster and numbers of clusters versus the site destruction probability.

C. Imposing Random walk on Percolation Cluster

The next step is to impose the random walk on the percolation cluster. The initial point of the RW is cho-

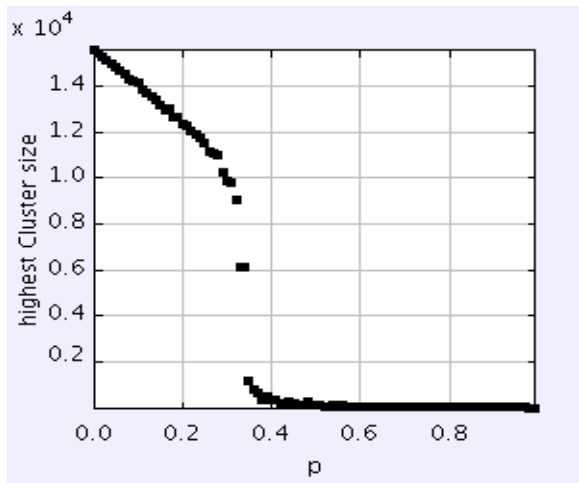


FIG. 8: Percolation cluster versus p (Site destruction probability))

sen such that it lies on one of the cluster. Physically one expects that there is high probability that it lands on infinite cluster (if it exists). The lattice size is chosen to be much bigger than the average size of the random walk. In our case on a 125×125 lattice we had a 1000 steps random walk. We ran around 4000 such walks and did an ensemble average of them. The time of execution depends mainly on the lattice size and site occupation probability. Small increase in lattice width lead to large increases in the execution time. The dynamic display of clusters with different colors and the display of individual random walks take a lot of execution time. Hence we did separate programs- (1) to illustrate cluster formation and to dynamically display 2D random walk. (2) to do calculations (ensemble averages) on the random walks (3) to perform calculations on Percolation cluster- like cluster size and cluster number. The results and calculations are shown in the following sections.

V. ASYMPTOTIC BEHAVIOR: THEORETICAL PREDICTIONS AND SIMULATION RESULTS

A. RMS displacement

After large number of steps t the mean square displacement $\langle R^2 \rangle$ of the RW on Percolation behaves as

$$\langle R^2 \rangle = t^{\frac{2}{d_w}} \quad (1)$$

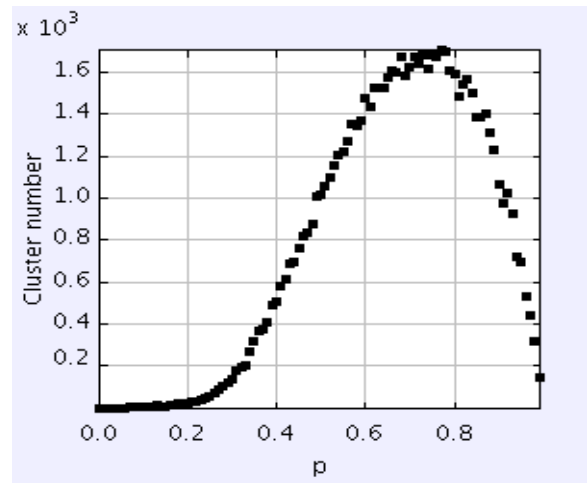


FIG. 9: Percolation cluster number versus p (Site destruction probability))

Here the exponent d_w is related to fractal dimension D of the cluster and fracton \tilde{d} dimension of the walk by

$$d_w = 2D/\tilde{d} \quad (2)$$

Alexander and Orbach conjectured the value of \tilde{d} as $4/3$ [3]. Various simulations have supported this. Our results for the best estimate can be read by the following figure[Fig.10]. It was generated for 125×125 lattice with 1000 step walks and averaged over 4000 runs. We evaluated the cluster at the expected transition point $p = 0.38$. The following figure shows the plot of our best linear fit for $\log\langle R^2 \rangle$. It gives $d_w = 2.801$ which is very close to the expected value of 2.844!

B. Number of distinct sites Visited

The number of distinct sites visited S_t after time (steps) t varies asymptotically as

$$\langle S_t \rangle = t^{\frac{\tilde{d}}{2}} \quad (3)$$

The following figure illustrates the asymptotic behavior along with a line fit. Our simulations gave slope of 0.5919 which is close to the expected value of 0.66, but still there is significant difference which we are not able to explain at this time.

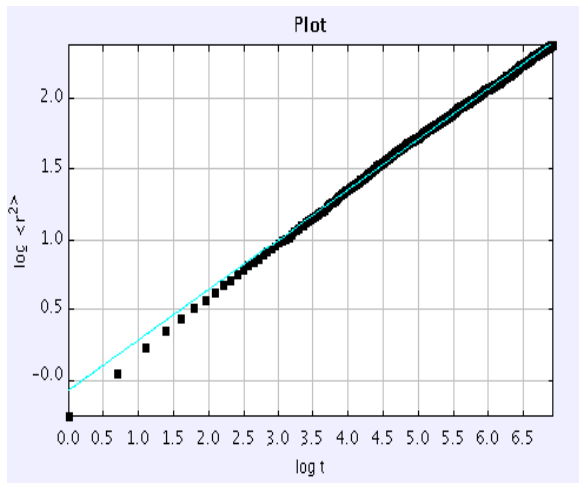


FIG. 10: $\log(\text{Mean square displacement})/2$ v/s $\log(\text{number of steps})$

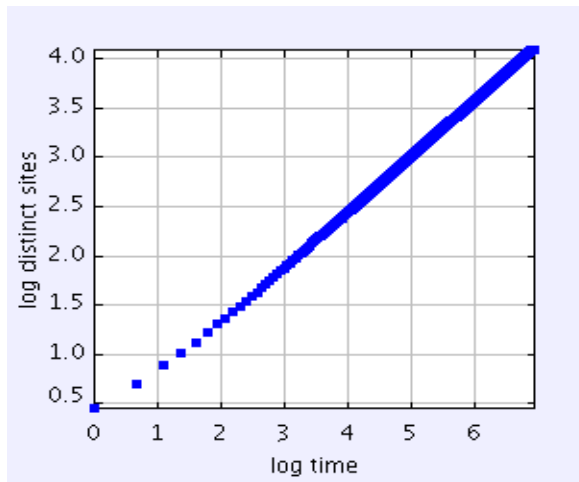


FIG. 11: $\log(\text{Number of distinct points})$ v/s $\log(\text{number of steps})$

C. return to Origin

Probability of returns to origin after steps t , $P_0(t)$ decreases asymptotically as

$$\langle P_0(t) \rangle = t^{-\frac{\tilde{d}}{2}} \quad (4)$$

we had to fit to a parabolic fit(2nd order) to this data so as to account for slight curvature seen. But we notice the linear term was dominant and has value of 0.6452 which gave \tilde{d} of 1.29 which is very close to the expected value of 1.33.

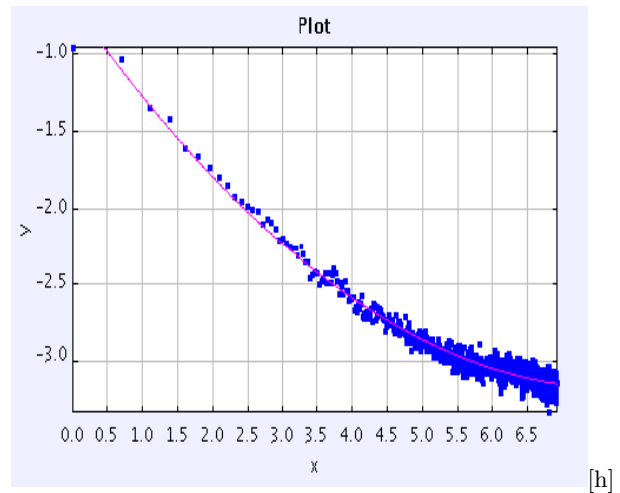


FIG. 12: $\log(\text{Number walks returning to origin})$ v/s $\log(\text{number of steps})$

VI. CONCLUSION

In summary we have shown that our simulation of Random walks on Percolation clusters behaved nearly as expected. We could calculate the various exponents from this simulation and established that the fracton and fractal dimension of such Random walks are different from the unconstrained walks in 2 Dimension. Such constarined walks have wide applicability. This new universality class model can be used to describe various phenomena ranging from mobile networks to conduction in gels.

VII. ACKNOWLEDGMENT

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