STRUCTURAL ACOUSTICS TUTORIAL—PART 1: VIBRATIONS IN STRUCTURES

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Most sounds that you hear throughout the day are radiated by vibrating structures. Walls and windows radiate sound into your house and office building. Windows radiate sound into your automobile, or into other vehicles, like buses, trains, and airplanes. The cones on the speakers of your stereo are vibrating structures that radiate sound into the air around you.

However, these structures are usually not the original sources of the sounds you hear. For example, the walls and windows in your house are driven by acoustic pressure waves caused by passing vehicles, noisy neighbors (often with loud lawn and garden equipment such as leaf blowers), or by the wind through the trees. The pressures impinge on your windows, which in turn vibrate and pass some of the incident sound through to the interior. In airplanes and high-speed trains, tiny pressure waves within turbulence outside the vehicles drive the walls, which then vibrate and radiate sound. There are, of course, many other sources of vibration and the subsequent sound that we hear.

"How do structures vibrate? How do vibration patterns over a structure's surface radiate sound?"

Although often the sounds radiated by vibrating structures are annoying (your neighbor's leaf blower), sometimes they are pleasing, like the sounds radiated by musical instruments. Pianos, violins, guitars, brass instruments, and the air within and around them are complex structural-acoustic systems. The sound from musical instruments (including the human voice) is often reproduced by audio equipment, such as CD players, amplifiers, and speakers. Speakers, with their multiple pulsating pistons mounted on the surfaces of boxes filled with air, are also very complex structural-acoustic systems, and engineers working for speaker companies spend entire careers trying to design systems that reproduce input signals exactly (they haven't succeeded completely yet!).

Those of us who study how structures vibrate and radiate sound usually call ourselves Structural-Acousticians. As I have taught structural acoustics to members of Penn State's Graduate Program in Acoustics (and others in industry and government), I find those taking my courses want answers to the following questions:

- How do structures vibrate?
- How do vibration patterns over a structure's surface radiate sound?
- Conversely, how do sound waves induce vibrations in structures they impinge on?

Some of the inevitable follow-on questions are:

- How can I modify a structure to reduce how much sound it makes (noise control engineering)?
- Or, for designers of musical instruments and loudspeakers, how can I increase how much sound my structure makes, and craft its frequency dependence to be more pleasing to human listeners?

The answers to all of these questions depend on a structure's shape and material properties, which define how fast and strongly different structural waves propagate through it. We will start by studying how structures vibrate in this first part of the article, and then consider in the second part of the article (to appear in a future issue of *Acoustics Today*) how a structural surface's vibration patterns act on surrounding fluids, radiating acoustic sound fields. Also in part two, we will analyze how structures are excited by incident sound fields. I have written mostly about simple structures, like flat plates and cylindrical shells, and use plenty of examples to explain the concepts of vibration and sound radiation.

I have tried to make this tutorial general and interesting to the non-structural acoustician, while also including enough detailed information (yes, equations) for those interested in pursuing structural-acoustics further. I have drawn from the course I teach at Penn State, several outstanding textbooks[1-4] and articles in the field (see the Reference list at the end of the article), and research performed by several members of the Penn State Graduate Program in Acoustics.

Compressional and shear waves in isotropic, homogeneous structures

Structural materials, like metals, plastics, and rubbers, deform in ways far more complicated than air or water. This is because of one simple fact: structural materials can resist shear deformation, and fluids cannot.[5] This means that both dilatational (and compressive) and shear waves can co-exist in structures. By itself, this is not very exciting. However, one more attribute of nearly all practical structures makes the field of structural-acoustics so interesting and complicated: most structures have one or two dimensions that are very small with respect to internal wavelengths. We call these structures plates and beams, and they vibrate flexuraly. Why is this so interesting? Because flexural waves are *dispersive*, meaning that their wave speeds increase with increasing frequency.

Dispersive waves are odd to those not familiar with structural vibrations. Imagine a long plate with two transverse sources at one end which excite flexural waves in the plate. One source drives the plate at a low frequency, while the other vibrates at a high frequency. The sources are turned on at the same time, and somehow the high frequency wave arrives at the other end of the plate faster than the low fre-

quency wave! I will show why this is so later, but first we will study the simpler structural wave types.

The simplest structural waves are those that deform an infinite material longitudinally and transversely. Longitudinal waves, sometimes called compressional waves, expand and contract structures in the same way acoustic waves deform fluids. The wave equation and sound speed for a longitudinal wave traveling in the x direction are

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c_l^2}\frac{\partial^2 w}{\partial t^2} \quad \text{, and} \tag{1a}$$

$$c_l = \sqrt{\frac{B}{\rho}} \quad , \tag{1b}$$

where w is the deformation (also in the x direction), B is the elastic bulk modulus and ρ is the mass density.

The bulk modulus relates the amount of volumetric contraction (per unit volume) to an applied pressure:

$$B_0 = \frac{-p}{\frac{dv}{v}} \quad . \tag{2}$$

Low volumetric changes mean stiffer structures, and faster compressional waves.

So far, we have considered structural waves only in media large in all dimensions with respect to vibrational wavelengths. For audible frequencies, and for most practical structures, one or two geometric dimensions are small with respect to a wavelength. As a longitudinal wave expands or contracts a beam or plate in its direction of propagation, the walls of the structure contract and expand transversely due to the Poisson effect, as shown in Fig. 1. The Poisson's ratio, which relates in and out of plane strain deformations according to:

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{3}$$

determines the amount of the off-axis deformation, which for incompressible materials like rubber approaches the amount of the on-axis deformation (a Poisson's ratio of 0.5).[6]

Longitudinal waves are therefore slower in structures like beams and plates, since the free surfaces of the structural material are exposed to air or fluid. Since the stiffness of most fluids that might surround a beam or plate is smaller than that of the structural material, the free surfaces of the structure act essentially as stress relievers, slowing down the compressional waves. The sound speeds of longitudinal waves in beams and plates are

$$c_l\,(\text{Beams}) = \sqrt{\frac{E}{\rho}} \quad \text{, and} \tag{4a}$$

$$c_l\,(\text{plates}) = \sqrt{\frac{E}{\rho\left(1-\nu^2\right)}} \quad , \tag{4b}$$

where c_l is defined not by the Bulk Modulus, but by the Young's Modulus E, which is related to the volumetric Bulk Modulus according to:

$$E = \frac{B(1+\nu)(1-2\nu)}{(1-\nu)} \quad (E < B) \; . \tag{5}$$

For a typical Poisson's ratio of 0.3, longitudinal wave speeds in plates and beams are 90% and 86% of those in infinite structural media, respectively.

As I mentioned earlier, the key difference between acoustic waves in structural materials and fluid media is a structure's ability to resist shear deformation. This shear stiffness allows pure shear waves to propagate through a structure, with the structure deforming in its transverse direction as the wave propagates in the axial direction (see Fig. 2 below). Shear wave behavior is governed by the same wave equation as longitudinal waves, and acoustic waves in fluid media:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2 w}{\partial t^2} \quad . \tag{6}$$

However, shear waves, which travel at the speed

$$c_s = \sqrt{\frac{G}{\rho}} \quad , \tag{7}$$

are slower than longitudinal waves, since a structure's shear modulus is smaller than its Bulk and Young's Moduli. The shear modulus G is related to E and Poisson's ratio according to:

$$G = \frac{E}{2(1+\nu)} \quad (G < E) \; . \tag{8}$$

Bending waves in beams and plates

Most sound radiated by vibrating structures is caused by bending, or flexural waves traveling through beams, plates,

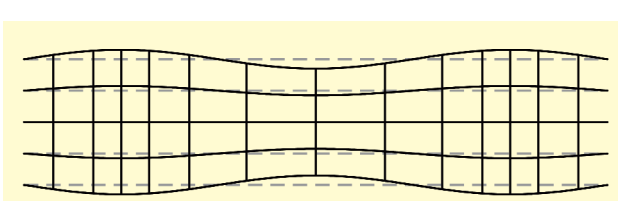

Fig. 1. A longitudinal wave passing through a plate or beam (amplitudes highly exaggerated). As the material expands or contracts along the axis of the plate or beam, the Poisson effect contracts and expands the material in the transverse directions.

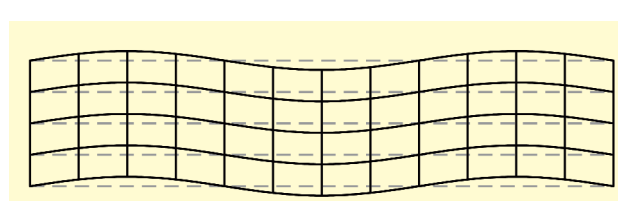

Fig. 2. A shear wave propagating through a plate or beam (amplitudes highly exaggerated). The wave propagates along the plate or beam axis, while deforming the structure transversely.

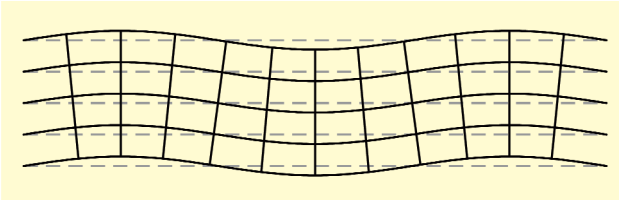

Fig. 3. A flexural, or bending wave propagating through a plate or beam (amplitudes highly exaggerated). As with pure shear, the wave propagates along the plate or beam axis, while deforming the structure transversely. Unlike pure shear, however, a bending wave causes the plate or beam cross sections to rotate about the neutral axis.

and shells, like the example shown in Fig. 3. Bending waves deform a structure transversely, so that they excite acoustic waves in neighboring fluids (we will learn about this phenomenon in part 2 of this tutorial). Although longitudinal and shear wave behavior is simple—similar to that of acoustic waves in air or water—bending waves are far more complicated. In particular, the speed of a bending wave depends not only on the elastic moduli and density of the structural material it travels through, but also on the geometric properties of the beam or plate cross section. Also, bending wave speeds are dispersive, with the curious property of depending on their frequency of oscillation.

I will not derive the bending wave equations for beams and plates in this article, but will show them, along with their corresponding wave speeds. The wave equation and wave speed for flexure in thin[7] beams are

$$\frac{E\,I}{\rho\,A}\,\frac{\partial^4 w}{\partial x^4} = -\frac{\partial^2 w}{\partial t^2}\ \text{, and} \tag{9a}$$

$$c_{B_{Bernoulli-Euler}} = \sqrt[4]{\frac{E\,I}{\rho\,A}\,\omega^2}\ . \tag{9b}$$

Whereas the wave equations for longitudinal and shear (and acoustic) waves are second order, the bending wave equation has a fourth order variation with space. Also, note that the wave speed does not appear explicitly in the flexural wave equation, and that the wave speed depends on frequency, as we learned earlier.

Although the thin beam bending wave equation is more complicated than those for pure longitudinal and shear waves, it is still fairly simple. However, when flexural wavelengths become short with respect to the beam thickness, other terms become important—such as resistance to shear deformation and the rotary mass inertia. Unfortunately, including these effects complicates the wave equation and sound speed considerably, leading to the thick beam[8] wave equation and wave speed:

$$\frac{EI}{\rho\,A}\frac{\partial^4 w}{\partial x^4} - \left(\frac{I}{A}+\frac{EI}{KAG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} = -\frac{\partial^2 w}{\partial t^2} - \frac{I\rho}{KAG}\frac{\partial^4 w}{\partial t^4}\ \text{, and}$$

$$c_B = \sqrt{\frac{\sqrt{\left(\frac{EI}{KAG}-\frac{I}{A}\right)^2\omega^4 + 4\frac{EI}{\rho A}\omega^2} - \omega^2\left(\frac{EI}{KAG}+\frac{I}{A}\right)}{2\left(1-\omega^2\frac{I\rho}{KAG}\right)}}\ . \tag{10b}$$

Two new components appear in the thick beam bending wave equation: a fourth order dependence of motion on both time and space, and a fourth order dependence on time. Some new combinations of factors also appear: *KAG* is the shear factor that is the product of area, shear modulus, and the correction factor *K*, which is the fraction of the beam cross section which supports shear; and *I*/*A*, which represents the rotary inertia.

When shear resistance and rotary inertia are negligible (which is the case for waves with long wavelengths with respect to thickness), the wave equation and wave speed reduce to the simpler forms shown earlier for thin, or Bernoulli-Euler beams. Since long wavelengths imply low frequencies, thin beam theory is sometimes called a low frequency limit of the general, thick beam theory. For very high frequencies, the shear resistance terms become dominant, so that the flexural wave equation simplifies to the shear wave equation, and the bending wave speed approaches the shear wave speed:

$$c_{B_{Shear}} = \sqrt{\frac{K\,G}{\rho}}\ . \tag{11}$$

The only difference between a shear wave in a beam and one in an infinite structural material is the shear correction factor *K*.

Although flexural wave theories for plates are derived in different ways than those for beams, the general thick plate[9] wave equation and wave speed are essentially the same as those for beams, but for a wave propagating in two dimensions:

$$\frac{D}{\rho h}\nabla^4 w - \left(\frac{I'}{h}+\frac{D}{k\,h\,G}\right)\nabla^2\left(\frac{\partial^2 w}{\partial t^2}\right) = -\frac{\partial^2 w}{\partial t^2} - \frac{I'\rho}{k\,h\,G}\frac{\partial^4 w}{\partial t^4}\ \text{, and} \tag{12a}$$

$$c_B = \sqrt{\frac{\sqrt{\left(\frac{D}{KhG}-\frac{I'}{h}\right)^2\omega^4 + 4\frac{D}{\rho h}\omega^2} - \omega^2\left(\frac{D}{KhG}+\frac{I'}{h}\right)}{2\left(1-\omega^2\frac{I'\rho}{KhG}\right)}}\ . \tag{12b}$$

D is a combination of terms called the flexural rigidity:

$$D = I'\frac{E}{\left(1-\nu^2\right)} = \frac{E\,h^3}{12\left(1-\nu^2\right)}\ . \tag{13}$$

As with beams, the low frequency (thin plate) limits of the thick plate equations are simpler, but still dispersive:

$$\frac{D}{\rho h}\nabla^4\,w = -\frac{\partial^2 w}{\partial t^2}\ \text{, where} \tag{14a}$$

$$c_{B_{Thin\ Plate}} = \sqrt[4]{\frac{D}{\rho\,h}\,\omega^2}\ . \tag{14b}$$

And, at high frequencies flexural waves in plates approach pure shear waves, where

$$c_{B_{Shear}} = \sqrt{\frac{KG}{\rho}}\ . \tag{15}$$

For homogenous isotropic plates, the shear correction factor is 5/6 (for beams, *K* depends on the geometry of the cross section).

The wave speeds of thin and thick plates, along with longitudinal and shear wave speeds in a 10 cm thick steel plate are shown in Fig. 4. The low and high frequency limits of the general thick plate wave speed are evident in the plot. So, longitudinal waves are faster than shear waves, which are faster than bending waves. Also, waves in stiffer materials are faster than

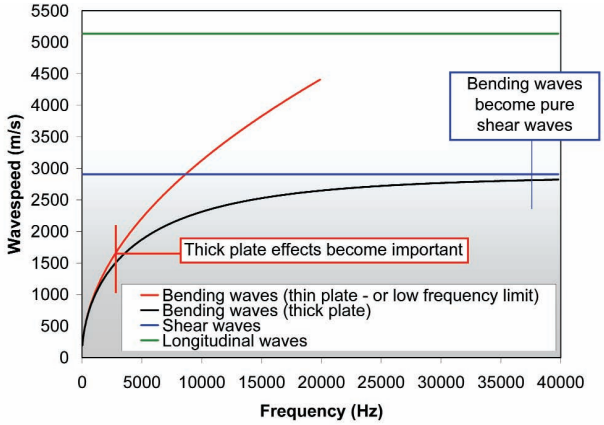

Fig. 4. Wave speeds in a 10 cm thick steel plate; the longitudinal and shear waves are non-dispersive, and the bending waves are dispersive (speed varies with frequency). The thin plate wave speed becomes invalid at high frequencies where rotary inertia and shear resistance become important.

those in flimsy materials. Finally, structural waves are slower in massive materials than they are in lightweight materials.

In this tutorial, we have limited our discussions to homogenous, isotropic beams and plates. Many modern structures are constructed of combinations of materials that are far from homogenous and isotropic, like plates made of honeycomb cores and thin outer metal face sheets; or laminated composites, which are cured assemblies of layers of woven fibers and epoxies. These sorts of structures represent interesting challenges to structural-acousticians, and will be left to future articles.

Modes of vibration

We have learned generally about how waves propagate in structures. Now, we consider waves that reflect from structural boundaries, and how they superpose with waves incident on those boundaries.

Imagine operating a dial that controls frequency, and watching the left and right traveling waves in a finite structure shorten as the dial is turned.[10] As the wavelengths shorten with increasing frequency, they pass through specific frequencies where left and right traveling waves either destructively interfere (anti-resonance), or constructively interfere (resonance). The constructive interference in resonance causes the appearance of 'standing waves' with high vibration amplitudes, where it does not appear that the waves are traveling at all, but that there is a stationary wave that oscillates in place. Remember that in reality, the standing wave is comprised of left and right traveling waves that move at finite speeds.

Modes of beams

We will consider first the simplest flexural resonances—those in a simply supported straight beam of length a. The resonance condition is found from the simply supported boundary conditions, where there is no motion, and no moment resistance by the supports.

The frequencies of resonance for Bernoulli-Euler (thin) beams are found by combining the wave speed equation with the resonance conditions, leading to

$$\omega_m = k_m^2 \sqrt{\frac{EI}{\rho A}} = \frac{m^2\pi^2}{a^2}\sqrt{\frac{EI}{\rho A}}\ , \qquad (16)$$

which correspond to the mode shapes $w_m(x) = \sin(m\pi x/a)$, for $0 < x < a$. There are an infinite number of modes as m increases from 1 to infinity. Note that the resonance frequency is the product of the square of the modal wavenumber[11] ($k_m = m\pi/a$) and the square root of the beam parameters $EI/\rho A$. We will see that for other boundary conditions, the resonance frequencies still depend on $EI/\rho A$, but will change as the wavenumber of the mode shape changes.

The mode shapes for m=1 through 4 are shown in Fig. 5 for a simply supported straight beam. In the mode shapes, dashed lines indicate the locations of maximum amplitude (the anti-nodes). The nodes of the mode shapes are at points of near zero vibration. A useful way of determining the orders of mode shapes measured on structures with nearly simply supported boundary conditions is to count the number of antinodes (try it in the figures).

I often refer to the modes of simply supported structures as the analyst's best friend, since they are easy to incorporate into advanced theories of sound radiation, and into analyses of the flow turbulence acting on structures. It is hard to find something much simpler than a sine wave to integrate!

Unfortunately for the analysts, modes in structures with free (and other) boundary conditions are more complicated than those in ideal simply supported structures, since the free edges impart a near field deformation to the vibration and shapes. For example, for a free beam of length a, the resonance frequencies may be computed only approximately, where

$$\omega_m \cong \frac{\pi^2(2m-1)^2}{4a^2}\sqrt{\frac{EI}{\rho A}}\ , \quad m>3 \qquad (17)$$

For m=2 and 3, the above formula may still be used, but can be in error by greater than 20-30%.

There are no simple formulas for the mode shapes of free beams, but we can refer to Leissa's compendium of plate mode shapes[12] to infer the following:

$$w_m(x) = \frac{\cosh k_m \cos k_m x + \cos k_m \cosh k_m x}{\sqrt{\cosh^2 k_m + \cos^2 k_m}} \quad \text{for even } m \qquad (18)$$

and

$$w_m(x) = \frac{\sinh k_m \sin k_m x + \sin k_m \sinh k_m x}{\sqrt{\sinh^2 k_m - \sin^2 k_m}} \quad \text{for odd } m$$

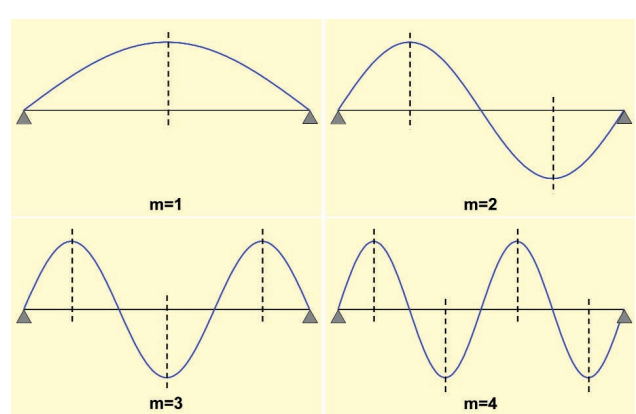

Fig. 5. The first four mode shapes of a simply supported beam. The dashed lines indicate the vibration antinodes, or locations of maximum deformation.

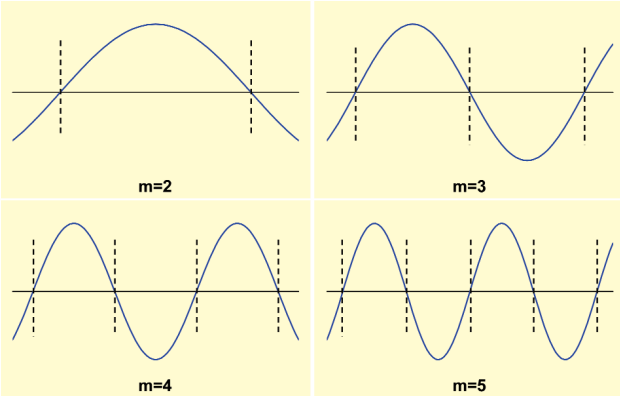

Fig. 6. First four mode shapes of a free beam. The dashed lines indicate the vibration nodes, or locations of zero deformation.

Figure 6 shows sample mode shapes of a straight free beam. Here, dashed lines have been placed at the modal node lines (locations of zero deformation). Whereas counting antinodes can determine mode order for beams with simple supports at their ends, counting nodes determines mode order for free beams.

The only difference between the resonance frequencies for beams with free (or other) boundary conditions and those for simply supported boundary conditions is the wavenumber term k_m. Fortunately, as the mode order m increases and the wavelengths become small with respect to the structural dimensions, the near field deformations around a structure's edges influence the mode shapes and their resonance frequencies less. An exercise to confirm this phenomenon is to plot the mode shapes for high m, and to compute the resonance frequencies of beams with free and simply supported boundary conditions. Notice how as m increases, $\pi^2(2m\text{-}1)^2/(4a^2)$ (the wavenumber term for free beam modes) approaches $m^2\pi^2/a^2$ (the wavenumber term for simply supported beam modes). In Fig. 7, the mode shapes for a simply supported and free beam are shown for large m and k_m, and are nearly identical away from the boundaries.

Modes of plates

Mode shapes in flat plates look like those in beams, but are two-dimensional. We consider again simply supported boundary conditions (the analyst's best friend) at the edges of a thin rectangular plate, where the transverse displacement field of a given mode shape of order (m,n) is:

$$w_{mn}(x,y) = A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\ \ m,n = 1, 2, \ldots \quad , \qquad (19)$$

and the corresponding resonance frequencies of the modes are

$$\omega_{mn} = \left|\bar{k}\right|^2_{mn}\sqrt{\frac{D}{\rho h}} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \quad . \qquad (20)$$

Now, our rigidity/mass term is $D/\rho h$, rather than $EI/\rho A$ for beams, and our wavenumber is now a two-dimensional wavevector,

$$\bar{k}_{mn} = \vec{i}k_m + \vec{j}k_n = \vec{i}\left(\frac{m\pi}{a}\right) + \vec{j}\left(\frac{n\pi}{b}\right) \qquad (21)$$

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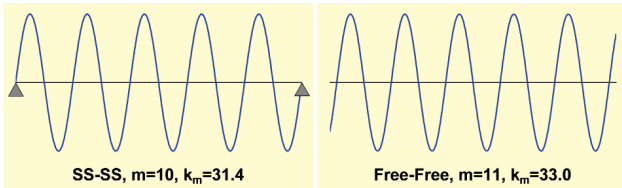

Fig. 7. Simply supported (left) and free (right) beam mode shapes at high mode order.

since bending waves travel in the *xy* plane. The square of the magnitude of the wavevector determines the resonance frequency.

For thick plates, where shear resistance and rotary inertia are important, resonance frequencies cannot be computed with a closed-form equation, since the dependence of wave speed on frequency is so complicated (recall the thick plate wave speed equation). Fortunately, there is an iterative way of computing resonance frequencies, where the magnitude of the wavenumber of a mode shape $|k_{mn}|$ can be compared to the wavenumber of free bending waves (computed using the thick plate equation and recalling that $k_b = \omega / c_b$).

Figure 8 shows how the resonance frequency of a thick plate mode may be computed provided the modal wavenumber k_{mn} is known. First, generate the wavenumber-frequency plot of free bending waves. Next, equate k_{mn} to k_b on the ordinate of the plot to find the corresponding resonance frequency. The plot shows the free wavenumber-frequency curves calculated using both thick and thin plate theory for a 5 mm thick steel plate, and shows how resonance frequencies computed using thin plate theory become inaccurate at high frequencies.

The procedure may be used for any combination of boundary conditions, provided the modal wavenumber can be computed. Also, the procedure applies to other shapes, like circles and triangles, provided once again that the modal wavenumber can be computed from the mode shapes. The textbooks by Leissa[12] and Blevins[13] are useful references for the resonance frequencies and mode shapes of many plate and beam configurations.

For free or clamped boundary conditions, the mode shapes of rectangular flat plates are not simple standing sine waves, as shown in the examples in Fig. 9, measured for a 5 mm thick 0.304 m x 0.304 m piece of glass. As with beams, the waves within the plate look like sine functions. However, the free edges vibrate like cosh and sinh functions.

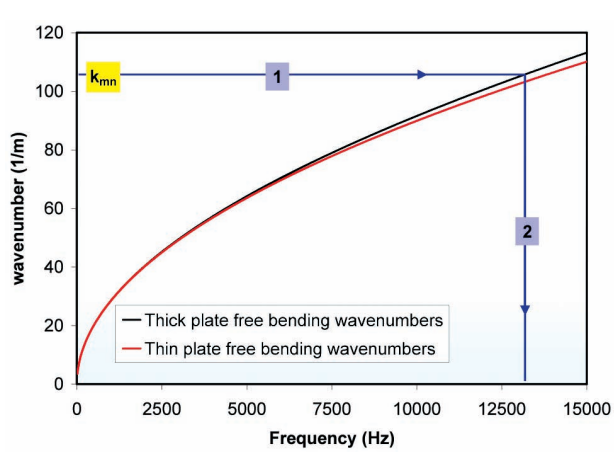

Fig. 8. Procedure for finding the resonance frequency of a mode shape in a thick flat plate. The example is for a steel 5 mm thick plate. Matching the modal wavenumber k_{mn} to the free bending wavenumber locates the corresponding resonance frequency.

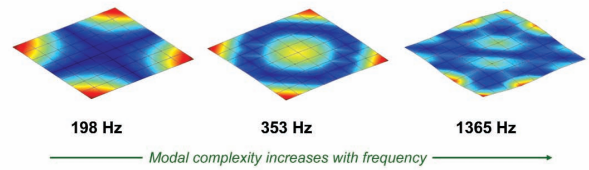

Fig. 9. Mode shapes of a 5 mm thick 0.304 m x 0.304 m glass plate with free boundary conditions, measured with a force hammer and accelerometers. Red colors indicate high relative vibration, and blue colors indicate low relative vibrations. The vibration pattern changes phase across the blue 'lines', with neighboring regions of red/yellow vibrating out of phase with each other.

The mode shapes in Fig. 9 were not computed from an equation. Instead, they were extracted from a series of mobility measurements (which we will discuss next) made using accelerometers and an instrumented force hammer, as shown in Fig. 10. For this test, free boundary conditions were approximated by suspending the plate with soft surgical tubing, which behaves like a very low frequency isolation mount.[14]

Mobility and impedance

A structure's mobility is the amount it vibrates when driven by a fluctuating force. For the measurements shown in Fig. 10, the accelerations at the three accelerometers are measured as each point in a grid mapped over the plate is struck by an instrumented force hammer. The accelerations are integrated over time to compute velocity, which is normalized by the drive force to compute mobility.[15] When the driven and measured points coincide, the mobility is called a drive point mobility; otherwise, the mobilities are called cross-mobilities.

Matrices of drive point and cross-mobilities may be used to extract the resonances of a structure. I will not talk about how this is done here (see the textbooks by Ewins[16] and the article by Avitabile[17] for details on experimental modal analysis), but I will explain how the vibration response of a structure is composed of individual modal contributions, which is a concept key to understanding how structures radiate sound.

For example, consider the mobility at location (x,y) for a rectangular simply supported plate driven by a point drive at (x_o, y_o):

$$\frac{v(x,y)}{F(x_o,y_o)} = \frac{i\omega}{\left(\frac{\rho h a b}{4}\right)} \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\omega_{mn}^2-\omega^2} \times \left[\sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{b}\right]\left[\sin\frac{m\pi x_0}{a}\ \sin\frac{n\pi y_0}{b}\right]\right). \tag{22}$$

The mode shapes of the plate (the sin functions in the *x* and *y* directions) appear as functions of both the drive (shown in magenta) and response (shown in blue) locations. Also, the mass of the plate appears in the denominator of the mobility, where the modal mass (shown in red) of each mode

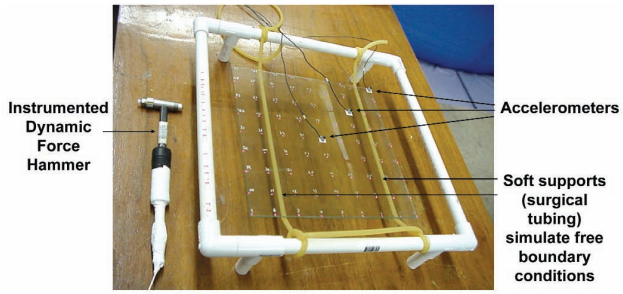

Fig. 10. Instrumentation for measuring the mobilities (and mode shapes) of a plate. A grid of points is struck by the instrumented force hammer while acceleration is measured with the three accelerometers.

shape is 1/4 of the total plate mass.[18] Therefore, mobility is inversely proportional to mass, and lightweight structures generally have high mobilities. Finally, it is clear from the frequency term in the denominator that modes that are excited at frequencies ω near their resonance frequency ω_{mn} will respond quite strongly.

In Fig. 11 the mobility of a 5 mm thick 1 m x 1 m simply supported steel plate is shown, along with the contributions to the mobility by the first few modes of the plate. Clearly, the peaks at the resonances dominate the mobility. Should any tonal source excite the plate at or near one of the resonances, very strong vibrations would ensue. In part II of this article, we will learn about how some mode types radiate sound better than others, and will combine the modal mobilities we see here with modal radiation efficiencies to compute the sound power radiated by individual modes.

The peak responses in Fig. 11 are very high, and in fact, are infinite when computed using the mobility equation I showed earlier. This is because we have not yet considered the energy lost during each cycle of vibration due to damping. As we will see next, damping limits the amplitudes of modes at resonance.

Damping

There have been many articles and textbooks published which describe the damping of structural vibrations, among them those by Nashif, Jones, and Henderson[19] and Beranek.[20] As a structure vibrates, it loses energy in many ways: within the structural material itself, to structures it is connected to, and to neighboring fluids. For now, we will consider internal energy losses; and in part II of this article, will see how sound radiation is also an energy loss and damping mechanism.

Energy lost within a vibrating material depends on its molecular structure and how the material is deformed. The deformation leads to normal and shear strain fields (actually complicated strain tensors). Some materials, like rubbers, are more efficient at converting internal strains into heat, dissipating energy. Other materials, like metals, lose very little of their strain energy as heat. A key concept is that the energy dissipation depends on deformation, or displacement, which is contrary to the energy losses in the fundamental vibrating system usually studied first in basic vibrations courses: the simple harmonic oscillator, or mass-spring-dashpot system.

In a mass-spring-dashpot system, the damper dissipates energy proportional to the oscillating masses velocity, not its displacement. In most structures, however, damping is not due to dashpots, but due to the mechanisms we discussed earlier. The damping we will use here is defined by a loss factor, which causes the structural stiffness to be complex—with its imaginary component related to the loss factor. For a spring, the spring constant simply becomes $k(1+i\eta)$, where k is the spring constant and η is the loss factor (i is the square root of -1). For structural materials, the moduli of elasticity (E, G, and B) become complex in the same manner. For example, the complex Young's Modulus becomes $E(1+i\eta)$.

Complex elastic moduli lead to complex wave speeds (recall the wave speed equations depend on the moduli) and wavenumbers. They also lead to complex resonances, where each mode has a resonance frequency and modal loss factor. The complex resonance frequency ω_{mn} limits the peak amplitudes of the modes in a structure's mobility since the mobility is inversely proportional to $(\omega_{mn}^2-\omega^2)$. The higher the loss factor, the lower the resonant response.

The effect of loss factor on mobility is shown in the example in Fig. 12, where the glass plate shown in Fig. 10 was treated with two different damping configurations. The loss factors of the moderately damped plate range from 0.05 to 0.009, and the loss factors of the highly damped plate range from 0.012 to 0.015. For reference, the loss factor of untreated glass is less than 0.01 (about 0.005). The difference between the loss factors can be related approximately to the difference in mobility at resonance, where the square of

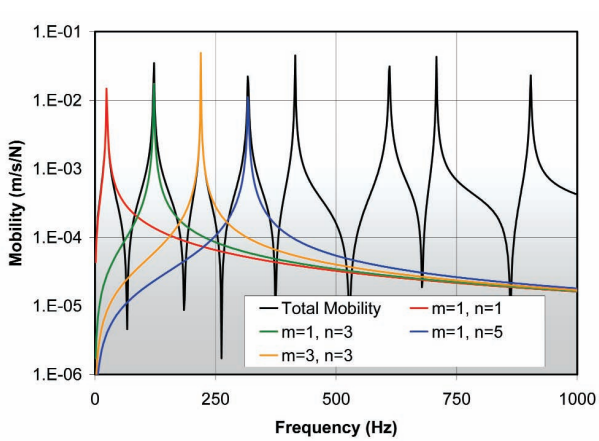

Fig. 11. Magnitude of drive point mobility of a simply supported 5 mm thick 1 m x 1 m rectangular steel plate driven at its center. Contributions of selected mode orders to the mobility are shown at low frequencies.

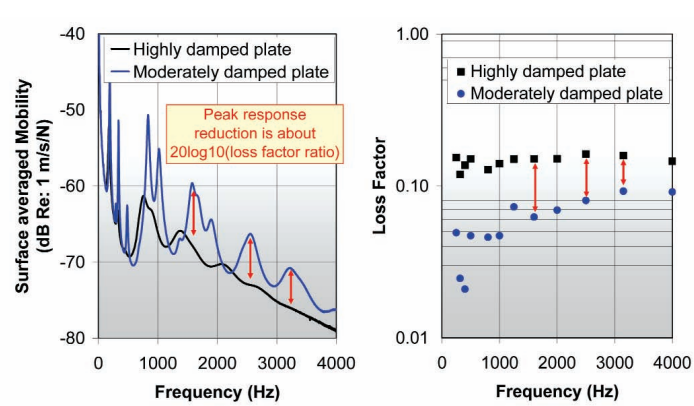

Fig. 12. Example of how increasing damping reduces resonant response. Two glass 5 mm thick 0.304 m x 0.304 m square plates treated with different viscoelastic layers have different loss factors, and different mobility peaks.

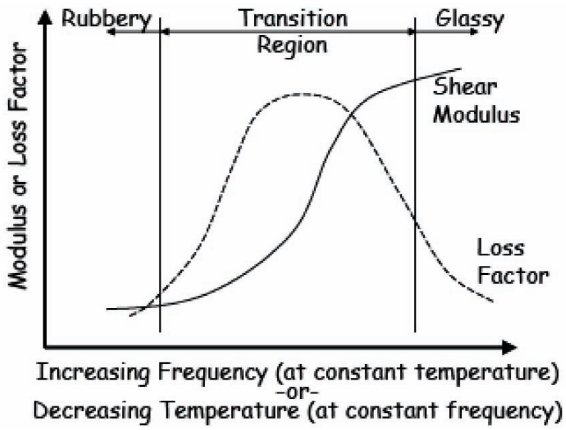

Fig. 13. Typical frequency-temperature dependencies of viscoelastic material properties.

mobility decreases with increasing loss factor.

The structural damping in the example above is caused by embedding thin sheets of viscoelastic material (rubber) between two plates of glass, creating a sandwich structure. The technique, called constrained layer damping (CLD) is well known and often applied by noise control engineers confronted with vibration problems. As the top and bottom structures (which are much stiffer than the viscoelastic material) bend, they strain the rubber, which has a high loss factor, and energy is dissipated. Curiously, thinner sheets of viscoelastomer usually lead to higher loss factors than thick sheets. This is counter-intuitive until the theory behind CLD is well understood. Ungar[21] provides a retrospective on the history of CLD which references many early papers describing its theory and examples of how it has been applied.

The number of viscoelastic materials available to noise and vibration control engineers is daunting, and choosing the right material for a given application requires great care. This is because the elastic moduli and loss factors of all viscoelastomers depend strongly on both frequency and temperature. This is, in fact, why so many different materials are available, as manufacturers tailor their different rubbers to work well in various conditions (hot, warm, cold) and over many frequency ranges.

Figure 13 shows a typical shear modulus and loss factor curve. At low frequencies (or high temperatures), a viscoelastomer is flimsy and rubbery, and not very lossy. At very high frequencies (or very low temperatures), a viscoelastomer is very stiff, or glass-like, and again not very lossy. Within its range of usefulness—mid frequencies and temperatures—a viscoelastomer's elastic modulus varies strongly with frequency and temperature, and where the slope of its modulus is high, it becomes extremely lossy, with loss factors near 1.

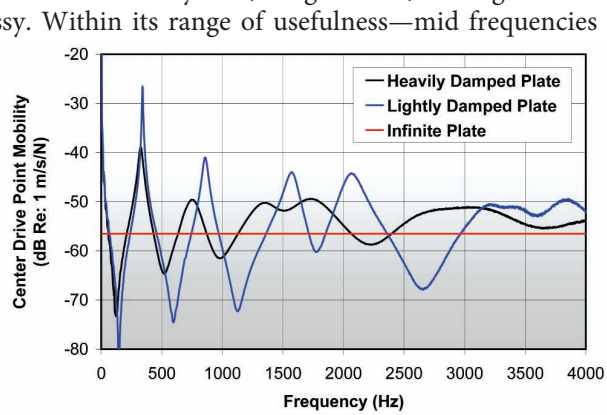

Fig. 14. Magnitudes of drive point mobilities of 5 mm thick 0.304 m x 0.304 m glass plates with light and heavy damping, compared to infinite plate mobility.

The equivalence between the effects of increasing frequency and decreasing temperature in a viscoelastomer is called the time-temperature superposition principle, and laboratories use this principle to measure the frequency and temperature varying dynamic properties of rubbers. Most dynamic property characterization tests measure a rubber's vibration response at a limited set of frequencies, but over a wide range of cold and hot temperatures. The data are shifted in frequency to a common temperature using the time-temperature superposition principle to form so-called 'master' curves of elastic moduli and loss factors. Fortunately, many rubber manufacturers offer these master curves to their customers (some are available on the internet), so that materials well suited to a given application may be chosen easily.

Prospective CLD users should be cautioned though—it is not uncommon for different testing labs to produce different master curves of the dynamic properties of the same viscoelastomer. In my experience, elastic moduli measured by different labs and testing procedures can easily vary by a factor of two. Loss factors, though, are easier to measure, and should be accurate to within 10-20%. The uncertainty in elastic modulus, though, can lead to comparable uncertainties in the damping provided to a CLD-treated structure. Other techniques are available to more accurately characterize the master curves,[22] but they are quite cumbersome, and best suited to applications where high precision is required.

Bending waves in infinite structures

It is sometimes hard to convince acoustics students that understanding how waves behave in infinite structures is worthwhile. However, the simple formulas that describe the mobilities of infinite structures are tremendously useful, since they represent the mean vibration response of complicated, finite structures to force and moment drives. Figure 14 shows an example of our measured glass plate mobilities, along with the drive point mobility of an infinite glass plate, computed using the well-known formula for bending waves in infinite flat plates:[1]

$$Y_{\inf} = \left(v/F\right)_{\inf} = \frac{1}{8\sqrt{D\rho h}} \quad . \tag{23}$$

The infinite plate mobility Y_{inf} is purely real (the general finite plate mobility is complex, although we have only looked at plots of the magnitude of mobility), and for the glass plate clearly approximates the mean mobility.

As structural damping increases, the peaks in a finite structure's mobility become less sharp, until at high damping values and high frequencies, the mobility becomes nearly real, and approaches that of an infinite plate. To physically understand why this is so, consider the waves traveling through a damped plate which are induced by a point drive. As the plate absorbs energy from the vibrations, the wave amplitude decreases as it travels away from the drive. Eventually, the diminished wave strikes a finite boundary, and reflects back toward the source. However, as it travels

back, it continues to lose energy to structural damping. The energy loss occurs for every spatial cycle of the wave, so as frequency increases, the number of spatial cycles, and therefore energy loss, increases. For high structural damping and frequency, the amplitude of the wave that eventually returns to the source is so small it is barely noticeable, and the source mobility resembles that of an infinite structure.

Why is infinite structure theory useful? The mobility equations (we will see more of them later—for beams and shells) can and should be used to check mobility measurements. I have often looked at measured mobility plots and noticed that the levels seemed strange, confirming my suspicions by performing a simple infinite structure mobility calculation. Usually the discrepancy is a neglected gain factor applied to instrumentation, or a forgotten units conversion factor.

Infinite structure theory can also be used to make cost-effective back-of-the-envelope estimates. An engineer trying to decide between various structural materials can use the simple equations to conduct tradeoff studies prior to investing time in more rigorous and costly analyses. Finally, infinite structure theory is useful for scaling the response of a structure to that of another geometrically identical structure constructed out of a different material.

In Fig. 15, the measured corner drive point mobilities of an Aluminum and Lexan ribbed panel of nominally identical geometry are compared (the plots are adapted from Reference 23). Note that in the top plot the infinite plate mobility is included. In the bottom plot, the mobility of the Aluminum plate is scaled to that of Lexan using the ratio of the infinite plate mobilities. Also, the frequency of the Aluminum mobility is scaled to that of Lexan using the longitudinal wave speed ratio:

$$\frac{f_{Al}}{f_{Lexan}} = \frac{c_{Al}}{c_{Lexan}} = \frac{\sqrt{E_{Al}/\rho_{Al}}}{\sqrt{E_{Lexan}/\rho_{Lexan}}} \ . \tag{24}$$

The agreement between the two sets of measurements, when scaled, is striking, with the differences in the peak responses due to differences in the structural loss factor.

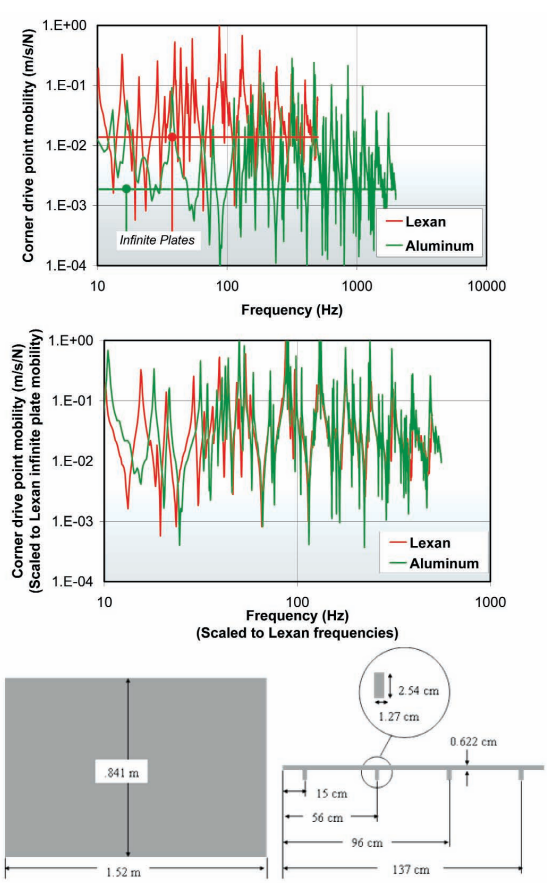

Fig. 15. Drive point mobility measurements of two geometrically identical ribbed panels, one made of Aluminum and the other of Lexan. Top–raw mobility measurements; Middle–mobility and frequency scaled to those of Lexan; Bottom–Geometry of the ribbed panels.

Vibrations in cylindrical shells

In most flat plates, bending and membrane (longitudinal) waves are uncoupled. Take a flat plate and bend it statically, and those waves become coupled. Take the flat plate and wrap it around a circle to form a cylinder, and the coupled bending-membrane waves become continuous around the circle. This is a circular cylindrical shell—a structure that has received almost as much attention over the years as flat plates have.

A typical vibration field in a short cylindrical shell is shown in Fig. 16. In the example, a bending wave wraps around the shell so that radial deformation (and slope) are continuous everywhere. Flexural waves also travel along the axis of the shell, and resemble those in flat plates. The vibration field in a cylindrical shell is decomposed into its circumferential and axial components:

$$w_{mn}(z,\theta,t) = \left[A_{mn}\cos(n\theta) + B_{mn}\sin(n\theta)\right]e^{k_m z}e^{i\omega t} \tag{25}$$

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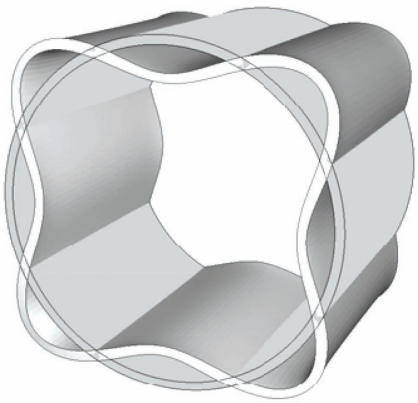

Fig. 16. Example of an n=4 circumferential mode in a cylindrical shell.

where n is the circumferential harmonic for both cosine and sine waves around the circumference.

When bending waves travel around a structure with curvature, they are coupled strongly to in-plane longitudinal waves, which travel at much higher speeds than the bending waves do (recall our discussions of wave speeds earlier). The coupling occurs because curvature causes the transverse displacement to be resisted not only by flexural stiffness, but also by the membrane, or 'hoop' stiffness. Bending waves sped up by membrane stiffness resonate at higher frequencies than they would in an equivalent flat plate.

The theory representing coupled flexural/membrane waves is far too complicated to include here, and I refer you to the compendium by Leissa[24] for the various approaches that have been developed to date. We can, however, look at some of the implications of the phenomena we have discussed so far.

The circumferential harmonics of most interest to structural acousticians are low order, and shown in Fig. 17. Recall that the circumferential motion of a cylindrical shell is comprised of $\cos(n\theta)$ and $\sin(n\theta)$ components, where n is the number of waves around the circumference for a given harmonic. The harmonic that radiates the most sound is n=0, where there is no sinusoidal variation of the wave around the circumference. The n=0 modes of a cylindrical shell are clustered around the ring frequency, which is the frequency at which a membrane wave is continuous around the circumference:

$$f_r = c_l / 2\pi a \tag{26}$$

The ring frequency is sometimes expressed in dimensionless form, normalized to the radius a and longitudinal wave speed: $\Omega_r = 2\pi f_r a/c_l$. n=0 modes are sometimes called

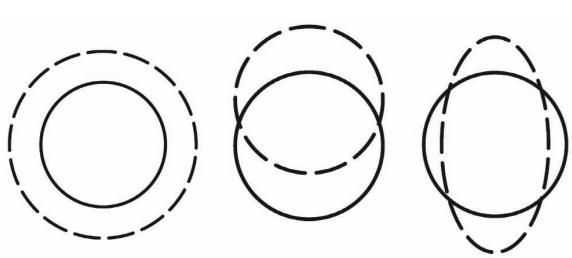

Fig. 17. Circumferential deformation of low-order modal harmonics: breathing, or n=0 circumferential motion (left); beam, or n=1 circumferential motion (center); and ovaling, or n=2 circumferential motion (right).

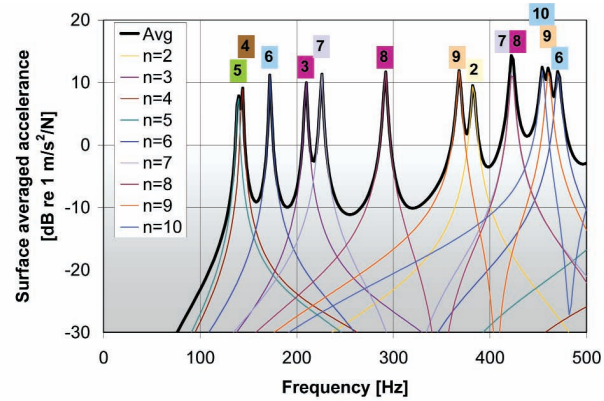

Fig. 18. Circumferential harmonics of the surface-averaged accelerance of a short 3 mm thick, 1.2 m long, 0.8 m diameter steel cylinder.

'breathing' modes, and as you might imagine, radiate sound very efficiently.

n=1 motion is a single wave around the circumference, and it appears that the cylinder cross section vibrates as a rigid body about the undeformed shape. n=1 motion generally occurs in long cylinders, and can be represented more simply using beam theory, with the area and inertia of the cylinder cross-section input to the beam equations. For long cylinders (usually called pipes), beam theory can be used to model vibrations up to the frequency where the higher order harmonics cut on. In short cylinders, though, the beamlike n=1 modes do not appear until high frequencies.

When harmonics above n=1 appear in a cylinder's vibration, they are called 'lobar' modes. The first lobar mode is n=2, where the cross section deforms as an oval. Ovaling is often the first mode to cut on in cylinders of medium length.

An example of the surface averaged mobility of a medium length cylindrical shell is shown in Fig. 18. As with the flat plate mobility, the contributions to the mobility by the first several shell modes are shown in the figure, except this time, the mode orders do not increase with increasing frequency. In fact, the first shell mode to cut on is the n=5 mode, followed by the n=4, 6, and 3 harmonics. This behavior is typical of cylindrical shells, where lower order harmonics often cut on at frequencies above those of higher order harmonics. In this example, the breathing and beam modes have not yet cut on for frequencies up to 500 Hz (they do cut on eventually, though).

Since for pipes the n=1 beamlike modes dominate low frequency response, pipe mobilities can be approximated with infinite beam mobilities until the lobar modes cut on. A common approximation for the lobar mode cut on frequency is $\Omega > 0.77\ h/a$, where h is the thickness (recall Ω is $2\pi fa/c_l$). Another infinite mobility equation is used for shells above the lobar mode cut on frequency; and for dimensionless frequencies above 0.6, the cylindrical shell mobility approaches that of a flat plate without curvature. This is because as the flexural wavelengths become short with respect to the radius of curvature of the shell, the shell effectively becomes a flat plate and the flexural motion is uncoupled from the membrane motion.

So, the mobility Y of an infinite pipe (and the mean mobility of a finite one) may be computed as:

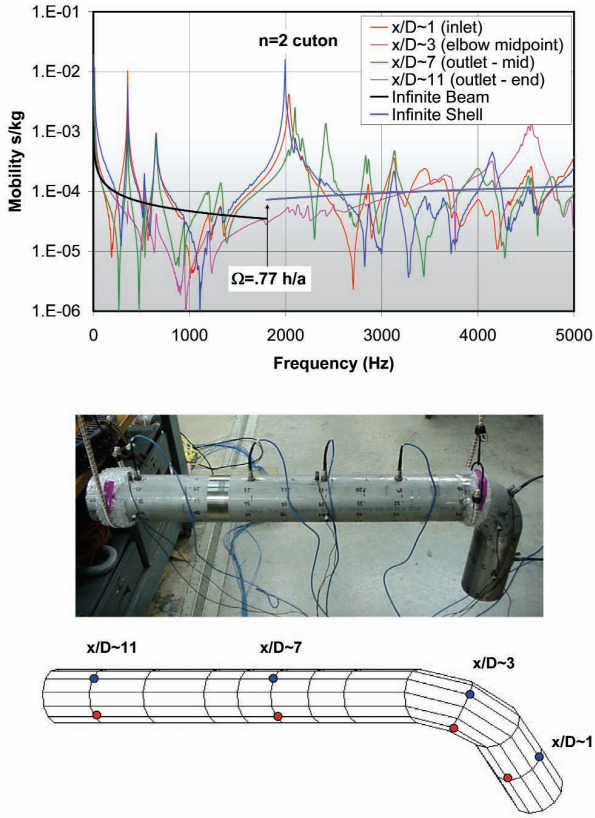

Fig. 19. Mobilities of a 7.62 mm diameter Schedule 40 steel pipe with elbow measured at different locations along the length compared to infinite beam and infinite shell mobilities. n=2 circumferential motion cuts on at 2 kHz, and the pipe transitions from beam-like to shell-like vibration.

$$\mathrm{Re}\{Y\} = \left[4\pi a \rho_s h \sqrt{\frac{\Omega c_l^2}{\sqrt{2}}} \right]^{-1}, \text{ for } \Omega < 0.77\frac{h}{a}, \tag{27a}$$

$$\mathrm{Re}\{Y\} = \frac{0.66}{2.3 c_l \rho_s h^2}\sqrt{\Omega}, \text{ for } 0.77\frac{h}{a} < \Omega < 0.6, \tag{27b}$$

$$\text{and } Y = \frac{1}{8\sqrt{D\rho h}}, \text{ for } \Omega > 0.6. \tag{27c}$$

An example of the measured mobilities in a pipe (this one with an elbow) is shown in Fig. 19 (adapted from Doty[25]). Mobilities measured at several locations are compared to the infinite beam and shell mobilities, once again demonstrating the usefulness of infinite structure theory.

Modeling vibrations—Finite elements

Although infinite structure theory is useful for estimating structural vibration and the mean effects of changes to parameters like thickness, Young's Modulus, and density, sometimes more exact knowledge of the modes and mobilities of complex structures is required. In these situations, analysts often turn to the most popular structural modeling method available—finite element analysis (FEA).

Finite elements are used to subdivide a structure into small increments, each of which behaves according to assumed local functions, usually linearly or quadratically. The textbook by Zienkiewicz[26] is generally recognized as the most authoritative summary of finite element theory, and can provide more detail than I do here. For simple structures, finite element models can be constructed quickly, like the ones of the glass plate shown in Fig. 20. The modes for the glass plate finite element model, constructed using plate elements, were computed using a widely available commercial code, and compare well to those measured with accelerometers and force hammers. The mode shapes match almost exactly, which is not surprising for a simple plate, but the resonance frequencies differ slightly. Discrepancies between the resonance frequencies of FE and actual mode shapes are common, and are usually caused by mismatches between the modeled and actual geometries and material properties. Often, the properties of FE models are updated to better reflect reality when measured data are available.

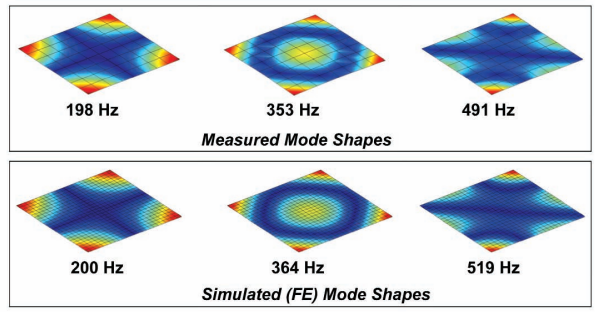

Fig. 20. Measured and simulated (FE) mode shapes of a free square glass plate.

For plate and beam finite elements, nearly all commercial software includes rotary inertia and shear resistance by default (these are important for thick beams and plates, remember?). Also, a limitless variety of cross-sections and inhomogeneities along the beam/plate lengths can be modeled simply with finite elements. For thick-walled structures, solid continuum elements are used, and may also be used to model thin sheets of viscoelastic damping material[12].

The main usefulness of FEA is its ability to simulate the response of complicated structures. Examples of a propeller, modeled with solid continuum elements, and a rib-stiffened metal equipment enclosure, modeled with plate and beam elements, are shown in Fig. 21. These sorts of models simply cannot be constructed readily with analytic or infinite structure methods. Mode shapes are shown for both examples, and mobilities for drives at any location or orientation may be computed directly, or as a summation of the mode shapes, just as we learned for flat plates earlier.

In the propeller mode example, the blades vibrate torsionally, and behave like plates cantilevered from the hub. The relative phasing between the individual blade vibrations is based on the circumferential harmonic (n) of the mode shape, where the hub behaves like the cylindrical shells we considered earlier—with a $\cos(n\theta)$ and $\sin(n\theta)$ dependence on θ.

The motion in the modes of the equipment cabinet model is mostly localized to the panels between the frame. In the example, three of the panels on the right side of the cabinet vibrate like fundamental plate modes with clamped edge conditions, but with different relative phasing: the top panel vibrates out of phase with the bottom panel. Sometimes, the vibrations of panels in framed structures, like airplane fuselages and ship hulls, may be approximated with simple plate theory.

The FE models shown here have been constructed with enough elements to resolve spatially the mode shapes in the

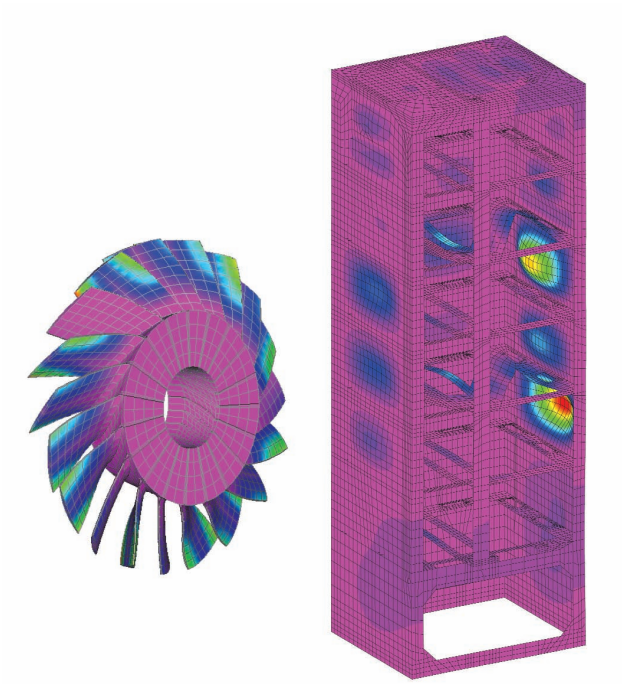

Fig. 21. Finite element models and sample mode shapes for a marine propeller (left) and a metal equipment cabinet (right).

examples. At very high frequencies, when the structural wavelengths shorten to the point where they are similar to the element sizes, the FE model becomes inaccurate. A good rule of thumb is to use at least six, and preferably eight elements to model a structural wavelength. The wavelengths for a given analysis frequency can be estimated using the wave speed formulas shown earlier in this article.

The equations used to represent a finite element model are assembled into linear matrices. The matrices, although large and complicated, look similar to simple mass-spring-damper lumped parameter model equations assuming time-harmonic frequency dependence $e^{i\omega t}$:

$$\left\{ \begin{array}{l} -\omega^2 \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & & \\ \vdots & & \ddots & \\ m_{N1} & & & m_{NN} \end{bmatrix} \\ +i\omega \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & & \\ \vdots & & \ddots & \\ b_{N1} & & & b_{NN} \end{bmatrix} \\ +(1+i\eta) \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & & \\ \vdots & & \ddots & \\ k_{N1} & & & k_{NN} \end{bmatrix} \end{array} \right\} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{Bmatrix} . \qquad (28)$$

Recall from basic vibrations theory that the time-harmonic response of a simple spring-mass-damper system is:

$$\left(-\omega^2 m + i\omega b + k\right) d = F \quad . \qquad (29)$$

The matrices in the finite element system of equations represent the element masses (m_{ij}), damping (b_{ij}), and stiffnesses (k_{ij}), and contain many, many terms; equal to the number of degrees of freedom (DOF) in the model (N). The more elements that are used, the larger the matrices become. The system shown in equation (28) is usually written more compactly in matrix form as:

$$\left[-\omega^2 \mathbf{M} + i\omega \mathbf{B} + (1+i\eta)\mathbf{K}\right]\mathbf{d} = \mathbf{F} \quad . \qquad (30)$$

A finite element computer program will assemble the mass, stiffness, and damping matrices (**M**, **B**, and **K**) based on the element geometries and material properties, and solve for the vibration response (the displacement vector **d**) based on the loads applied in the force vector **F**. A separate solution is required for each analysis frequency since the assembled matrix is frequency-dependent. Finite element programs can also extract the eigenvalues (normal modes) of the system when no loads are applied. It is easy to see that models that include many elements will require long computational times, and significant storage space due to the size of the matrices. In spite of these computational requirements, FE modeling is extremely popular, and used routinely to simulate the vibrations in large models of aircraft, automobiles, ships, and submarines. Finite element models are sometimes also used to simulate acoustic regions, although another numerical method—Boundary Element (BE) analysis—is used more commonly for that purpose. We will learn about BE modeling in part 2 of this tutorial, and also see how BE models of fluids may be coupled to FE models of structures.

Summary

In part 1 of this article on structural acoustics, I have presented basic vibration theories for beams, plates and shells, shown the speeds at which different waves travel through solids, how they can reinforce each other in finite structures to form modes, and how those modes define a structure's mobility. Structural damping limits the vibration peaks at modal resonance frequencies, and at high frequencies can cause the mobility of a finite structure to converge to that of an infinite structure.

In part 2, we will see how these structural vibrations interact with neighboring acoustic media to radiate sound. Conversely, we will also consider how acoustic waves incident on structures cause structural vibration, and subsequent sound re-radiation. As part of our discussion, we will learn about boundary element numerical modeling methods, which are used widely to compute fluid-structure interaction of complex structures in air and water.

Acknowledgments

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3 F. J. Fahy, *Sound and Structural Vibration: Radiation, Transmission, and Response* (Academic Press, 1987).

4 M. C. Junger and D. Feit, *Sound, Structures, and Their Interaction* (Acoustical Society of America, 1993).

5 Note we are considering only acoustic waves in our discussion here, since fluid dynamicists will take great issue with ignoring a viscous fluid's ability to resist shear.

6 Note that there are two off-axis directions, so that a maximum Poisson's ratio of 0.5 implies the deformation in both off-axes sums to the deformation in the main axis.

7 Thin beam theory is sometimes called Bernoulli-Euler theory.

8 Thick beam theory is usually attributed to Timoshenko.

9 Thick plate theory is usually attributed to Mindlin.

10 Exhibits like this for horizontal acoustic resonance tubes are common in science museums, where the frequency of sound within the tube is adjusted until acoustic resonance of the air column occurs and the sound waves excite a shallow pool of water in the tube leading to strong water pulsations at the peaks of the sound waves.

11 The wave number of a free wave is the radial frequency, ω, divided by its sound speed. The wavenumber of a mode shape is the number of radians over the spatial vibration pattern divided by the length of the pattern.

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14 With these boundary conditions, low-frequency resonances exist where the plate vibrates as a rigid body attached to soft springs. These resonances do not affect those of the flexural modes in the plate.

15 Mobility is also called admittance.

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18 This is only true for simply supported, homogeneous plates. However, the modal mass of any mode shape is always a fraction of the total static mass, and for high order mode shapes of plates with other boundary conditions, the 1/4 factor is a reasonable approximation. For beam-like flexural modes, the modal mass is about half of the static mass.

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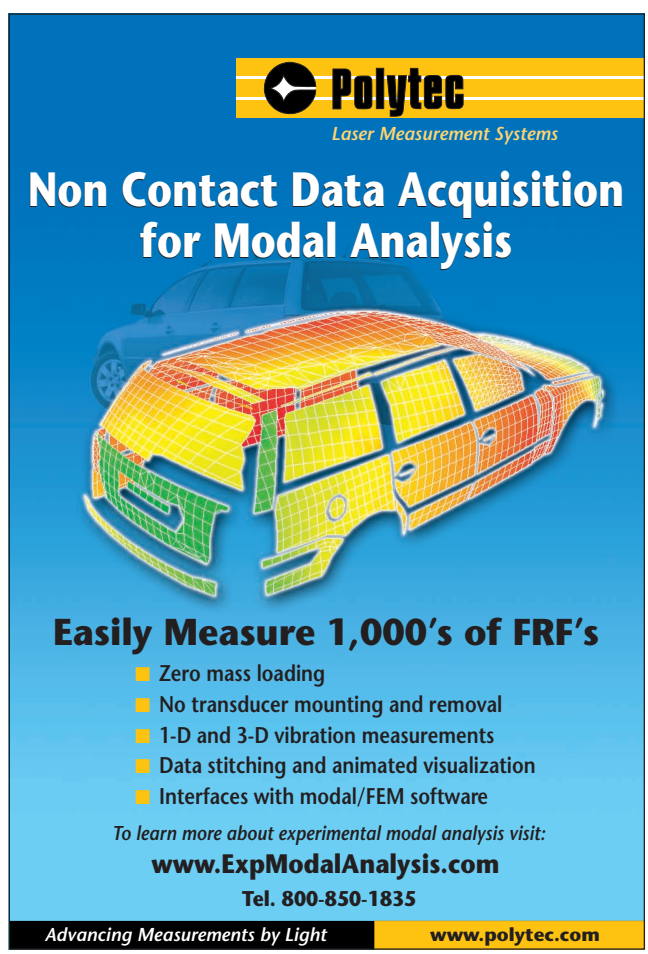
Steve Hambric and his daughter, Lily

Stephen A. Hambric is head of the Structural Acoustics Department at the Applied Research Lab at Penn State Univ. and Associate Professor in the Graduate Program in Acoustics. Prior to joining Penn State in 1996, Dr. Hambric worked for nine years in the Computational Mechanics Office at the Naval Surface Warfare Center, Carderock Division. Dr. Hambric has directed many numerical and experimental flow and structural acoustics research and development programs for the Navy, U.S. industry, and the U.S. Nuclear Regulatory Commission. He has authored over 60 conference and journal articles and advised many graduate students at Penn State. He teaches courses in Structural Acoustics, and Writing for Acousticians on campus at Penn State, and also to off-campus students working in industry and government. He currently serves on the board of directors of the Institute for Noise Control Engineering (INCE), on the Executive Committee of the American Society of Mechanical Engineers (ASME) Noise Control and Acoustics Division, and as an associate editor of ASME's *Journal of Vibration and Acoustics.*