Special Factoring

In Algebra, some products appear very often, and it is a good idea to memorize them. These products are called special products and we call the topic of factoring those products as special factoring. Here is a list of the special products:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

Each of these products can be verified easily by multiplying the expressions on the left to obtain the expressions on the right. Using your knowledge of special products to factor comes in three steps. First, identify the special product. Second, identify the components (what is a and what is b) of this special product. Third, write the answer which will be the left side of the special product.

Now let us look at these identities individually and see how they are used. We will start with the first two identities:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

These are commonly known as the perfect square identities. They are also the hardest ones to identify. Let us attempt to factor the following expression:

$$4x^2 + 12x + 9$$

The first step is to identify this expression as a perfect square. This is done by first recognizing that this is a quadratic trinomial with first and last terms being perfect squares. This makes our expression a candidate to be a perfect square trinomial. This means that it is possible that:

$$4x^2 + 12x + 9 = a^2 + 2ab + b^2$$

If this is true then $4x^2 = a^2$, $12x = 2ab$, and $9 = b^2$. If

$$4x^2 = a^2$$

Then:

$$2x = a$$

And since

$$9 = b^2$$

Then:

$$3 = b$$

And finally, we can verify to see if $12x = 2ab$:

$$2ab = 2(2x)(3) = 12x$$

So $4x^2 + 12x + 9$ is a perfect square trinomial of the form $a^2 + 2ab + b^2$. This means that it can be factored into $(a + b)^2$ and since $a = 2x$ and $b = 3$ then the factorization of $4x^2 + 12x + 9$ is:

$$4x^2 + 12x + 9 = (2x + 3)^2$$

This works similarly for the second perfect square identity. Consider the following expression:

$$9x^2 - 30xy + 25y^2$$

When factoring this quadratic trinomial we once again start by noticing that the first and last terms are perfect squares and so we investigate further to see if this is a perfect square trinomial. If it is, then $9x^2 - 30xy + 25y^2 = a^2 - 2ab + b^2$ and as a result $9x^2 = a^2$, $30xy = 2ab$, and $25y^2 = b^2$. If $9x^2 = a^2$, then $3x = a$ and if $25y^2 = b^2$, then $5y = b$. And finally, we can verify to see if $30xy = 2ab$:

$$2ab = 2(3x)(5y) = 30xy$$

So $\mathbf{9x^2 - 30xy + 25y^2}$ is a perfect square trinomial of the form $\boldsymbol{a^2 - 2ab + b^2}$**.** This means that it can be factored into $\boldsymbol{(a - b)^2}$ and since $\boldsymbol{a = 3x}$ and $\boldsymbol{b = 5y}$ then the factorization of $\mathbf{9x^2 - 30xy + 25y^2}$ is:

$$\mathbf{9x^2 - 30xy + 25y^2 = (3x - 5y)^2}$$

Let us take a look at one more example. Consider the following expression:

$$\boldsymbol{x^2 + 5x + 4}$$

We notice that the first and last terms on this expression are perfect squares. This means that it could be a perfect square. To check we see that $\boldsymbol{a^2 = x^2}$ so $\boldsymbol{a = x}$ and $\boldsymbol{b^2 = 4}$ so $\boldsymbol{b = 2}$. Now as we go to check the middle term we see that:

$$\mathbf{2ab = 2(x)(2) = 4x}$$

Which does not equal the middle term of my original expression which is $5x$. This means that $\boldsymbol{x^2 + 5x + 4}$ is not a perfect square and if it can be factored it should be done some other way.

Check yourself:

In exercises below, factor the quadratic trinomials if they are perfect squares:

1) $x^2 + 10x + 25$

2) $16x^2 - 72xy + 81y^2$

3) $36x^2 + 42x + 49$

Answers:

1) $(x + 5)^2$

2) $(4x - 9y)^2$

3) Not a perfect square

Now let us move on the next special product:

$$\boldsymbol{(a + b)(a - b) = a^2 - b^2}$$

This special product is commonly referred to as a difference of squares. If you see an expression that is a difference of perfect squares it can be factored according to the result of this special product. Let up consider the following expression:

$$4x^2 - y^2$$

We can recognize that this is a difference of two perfect squares and so this expression is of the form $a^2 - b^2$. If:

$$4x^2 - y^2 = a^2 - b^2$$

Then $4x^2 = a^2$ so $2x = a$ and $y^2 = b^2$ so $y = b$. Since $a^2 - b^2 = (a + b)(a - b)$:

$$4x^2 - y^2 = (2x + y)(2x - y)$$

Check yourself:

In exercises below, factor the expressions if they a difference of squares:

4) $x^2 - 9$

5) $121x^2 - 64y^2$

6) $4x^2 + 49$

Answers:

4) $(x + 3)(x - 3)$

5) $(11x + 8y)(11x - 8y)$

6) Not a difference of squares

Now let us move on the final two special products:

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

These special products are commonly referred to as the sum of cubes and difference of cubes. When you recognize an expression as either a sum or difference of cubes you just apply the appropriate formula.

Consider the following expression:

$$8x^3 + y^3$$

Since this expression is a sum of perfect cubes, it is in the form of $a^3 + b^3$. This means that $8x^3 = a^3$ so $2x = a$ and $y^3 = b^3$ so $y = b$. Since $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ so:

$$8x^3 + y^3 = (2x + y)((2x)^2 - 2xy + y^2) = (2x + y)(4x^2 - 2xy + y^2)$$

Check yourself:

In exercises below, factor the expressions if they a sum or difference of cubes:

7) $x^3 - 64$

8) $b^3 + 8c^3$

9) $x^6 + 1$

Answers:

7) $(x - 4)(x^2 + 4x + 16)$

8) $(b + 2c)(b^2 - 2bc + 4c^2)$

9) $(x^2 + 1)(x^4 - x^2 + 1)$

To be able to recognize the sum and difference of cubes expressions it is important to recognize what is a perfect cube so it is strongly suggested that the following list of perfect cubes from one to ten be memorized:

$1^3 = 1$	$6^3 = 216$
$2^3 = 8$	$7^3 = 343$
$3^3 = 27$	$8^3 = 512$
$4^3 = 64$	$9^3 = 729$
$5^3 = 125$	$10^3 = 1000$