<u>Simplifying Radical Expressions</u>

Recall that taking a root of a number is opposite of raising a number to a power. So if: $\sqrt[n]{\boldsymbol{a}} = \boldsymbol{b}$ then, $\boldsymbol{b^n} = \boldsymbol{a}$. A logical consequence of this is that $\left(\sqrt[n]{\boldsymbol{a}}\right)^{\boldsymbol{n}} = \boldsymbol{a}$. By convention, $\sqrt{\boldsymbol{x}} = \sqrt[2]{\boldsymbol{x}}$. Let us consider the following expression:

$$\sqrt{\mathbf{16}} = \boldsymbol{x}$$

By using the definition of roots, we can say that the statement above implies that:

$$\mathbf{16} = \boldsymbol{x}^{\mathbf{2}} \text{ or } \mathbf{16} = \boldsymbol{x} \cdot \boldsymbol{x}$$

So to figure out the value of $\boldsymbol{x}$, we just need to know what number has to be multiplied by itself to equal **16**. From our knowledge of the multiplication table we know that:

$$\mathbf{4 \cdot 4 = 16} \text{ so } \boldsymbol{x} = \mathbf{4}$$

This means that:

$$\sqrt{\mathbf{16}} = \mathbf{4}$$

Let us consider another example:

$$\sqrt[3]{\mathbf{8}} = \boldsymbol{x}$$

Just like in example above $\boldsymbol{x}$ is such a number that:

$$\mathbf{8} = \boldsymbol{x} \cdot \boldsymbol{x} \cdot \boldsymbol{x}$$

So to figure out the value of $\boldsymbol{x}$, we just need to know what number has to be multiplied by itself three times to equal **8**. From our knowledge of the multiplication table we know that:

$$\mathbf{2 \cdot 2 \cdot 2 = 8} \text{ so } \boldsymbol{x} = \mathbf{2}$$

This means that:

$$\sqrt[3]{\mathbf{8}} = \mathbf{2}$$

Check yourself:

In exercises below, evaluate the following radical expressions:

1) $\sqrt{81}$

2) $\sqrt[3]{8}$

3) $\sqrt[6]{1}$

Answers:

1) 9

2) 2

3) 1

There is a connection between roots and fractional exponents. Consider the following expression:

$$\left(a^{\frac{1}{n}}\right)^{n}$$

By using the rules of exponents we learned in Section 1.3 we see that:

$$a^{\left(\frac{1}{n}\cdot n\right)} = a$$

Also from the beginning of this section we know that:

$$\left(\sqrt[n]{a}\right)^{n} = a$$

This means that:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Or in more general terms we can say that:

$$x^{\frac{a}{b}} = \sqrt[b]{x}^{a}$$

This formula will allow us to use roots to evaluate expressions with fractional coefficients. Consider the following example:

$$\mathbf{16^{\frac{3}{2}}}$$

We can evaluate this expression once we rewrite it in radical form:

$$\mathbf{16^{\frac{3}{2}} = \sqrt{16}^{3} = 4^{3} = 64}$$

Check yourself:

In exercises below, evaluate the following exponential expressions:

4) $4^{\frac{5}{2}}$

5) $16^{\frac{3}{4}}$

6) $81^{\frac{3}{2}}$

Answers:

4) 32

5) 8

6) 729

We can also use this formula to simplify radical expressions. Consider the following expression:

$$\sqrt[3]{x^{12}}$$

By using the formula above we see that:

$$\sqrt[3]{x^{12}} = x^{\frac{12}{3}} = x^{4}$$

Now let us take a closer look at simplifying radical expressions. The actual action of simplifying a radical expression is not necessarily getting rid of the radical like in the previous example. Sometimes this is impossible and so we must settle for

taking out as much as we can out of the radical. For this we make use of the fact that as shown above, radicals are exponents in disguise and all rules for exponents that we learned in Section 1.3 also apply to radicals. Specifically, since:

$$(\boldsymbol{ab})^{\boldsymbol{n}} = \boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{n}}$$

then:

$$\sqrt[\boldsymbol{n}]{\boldsymbol{ab}} = \sqrt[\boldsymbol{n}]{\boldsymbol{a}}\sqrt[\boldsymbol{n}]{\boldsymbol{b}}$$

Consider the following expression:

$$\sqrt{\mathbf{20}}$$

20 is not a perfect square, but **20** is divisible by **4** which does happen to be a perfect square. This means that:

$$\sqrt{\mathbf{20}} = \sqrt{\mathbf{4} \cdot \mathbf{5}}$$

By using the formula above we get:

$$\sqrt{\mathbf{4} \cdot \mathbf{5}} = \sqrt{\mathbf{4}}\sqrt{\mathbf{5}}$$

And since $\sqrt{\mathbf{4}} = \mathbf{2}$ we get:

$$\sqrt{\mathbf{4}}\sqrt{\mathbf{5}} = \mathbf{2}\sqrt{\mathbf{5}}$$

Let us consider another expression:

$$\sqrt{\boldsymbol{x}^{\mathbf{7}}}$$

The largest perfect square factor of $\boldsymbol{x}^{\mathbf{7}}$ is $\boldsymbol{x}^{\mathbf{6}}$ so we have:

$$\sqrt{\boldsymbol{x}^{\mathbf{7}}} = \sqrt{\boldsymbol{x}^{\mathbf{6}}\boldsymbol{x}} = \sqrt{\boldsymbol{x}^{\mathbf{6}}}\sqrt{\boldsymbol{x}} = \boldsymbol{x}^{\mathbf{3}}\sqrt{\boldsymbol{x}}$$

Let us consider another example:

$$\sqrt{\mathbf{50}\boldsymbol{x}^{\mathbf{3}}\boldsymbol{y}^{\mathbf{6}}}$$

Since the larges perfect square that divides **50** is **25**, largest perfect square that divides $\boldsymbol{x^3}$ is $\boldsymbol{x^2}$, and the largest perfect square that divides $\boldsymbol{y^6}$ is $\boldsymbol{y^6}$, we can rewrite the expression above as:

$$\sqrt{50x^3y^6} = \sqrt{25x^2y^6 \cdot 2x} = \sqrt{25} \cdot \sqrt{x^2} \cdot \sqrt{y^6} \cdot \sqrt{2x} = 5xy^3\sqrt{2x}$$

We can use a similar method to simplify radical expressions of higher degree than two. Consider the following expression:

$$\sqrt[3]{24x^{14}}$$

In this case we are dealing with cube roots as opposed to square roots and so we must look to take out the perfect cube factors. Since the larges perfect cube that divides **24** is **8** and the largest perfect cube that divides $\boldsymbol{x^{14}}$ is $\boldsymbol{x^{12}}$, we can rewrite the expression above as:

$$\sqrt[3]{24x^{14}} = \sqrt[3]{8x^{12} \cdot 3x^2} = \sqrt[3]{8} \cdot \sqrt[3]{x^{12}} \cdot \sqrt[3]{3x^2} = 2x^4\sqrt[3]{3x^2}$$

Check yourself:

In exercises below, simplify the following radical expressions:

7) $\sqrt{75}$

8) $\sqrt{54x^4y^3}$

9) $\sqrt[3]{16x^7y^{11}}$

Answers:

7) $5\sqrt{3}$

8) $3x^2y\sqrt{6y}$

9) $2x^2y^3\sqrt[3]{2xy^2}$