MATH 5071 - Lecture 5

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Introduction to Logarithms

1. A **logarithm** is used to find the power to which a specified base is raised in order to recover a given number.

$\log_a y = x$ is an example of a logarithmic equation. This equation finds the power, x, to which we would need to raise the base, a, in order to equal y.

All logarithmic equations can also be written as exponential equations. The logarithmic equation $\log_a y = x$ is rewritten in exponential form as: $a^x = y$.

Example 1: Evaluate the following logarithms by first writing an equivalent exponential equation.

i) $y = \log_2(8)$

ii) $y = \log_5(625)$

iii) $y = \log_6(\frac{1}{36})$

iv) $y = \log_3(\sqrt[5]{9})$

Example 1 Solutions:

i) $2^y = 8$; $y = 3$

ii) $5^y = 625$; $y = 4$

iii) $6^y = \frac{1}{36}$; This can be rewritten as $6^y = 36^{-1} = (6^2)^{-1} = 6^{-2}$; $y = -2$

iv) $3^y = \sqrt[5]{9}$; This can be rewritten as $3^y = 9^{(\frac{1}{5})} = (3^2)^{(\frac{1}{5})} = 3^{(\frac{2}{5})}$; $y = \frac{2}{5}$

Example 2: Between which two consecutive integers must $\log_3(40)$ lie? Start by writing the equivalent exponential equation.

Example 2 Solution: $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$. Since $3^3 < 40 < 3^4$, this means $3 < \log_3(40) < 4$.

2. Every logarithm **must** have a specified base in order to be evaluated. You can assume that a logarithm that does not have a base explicitly specified is referring to the base of 10. In other words, $\log(x) = \log_{10}(x)$. The log button on your calculator is a good example of this.

Example 3: Use your calculator to evaluate the following logarithms. If possible, check your answer by writing the equivalent exponential equation.

i) $y = \log(100)$

ii) $y = \log(\frac{1}{1000})$

iii) $y = \log(\sqrt{10})$

iv) $y = \log(\frac{1}{729})$

Example 3 Solutions:

i) $10^y = 100$; $y = 2$

ii) $10^y = \frac{1}{1000}$; $y = -3$

iii) $10^y = \sqrt{10}$; $y = \frac{1}{2}$

iv) $10^y = \frac{1}{729}$; if you plug the original logarithmic expression into the calculator, you get $y = -2.86$.

The Natural Logarithm

3: Just as π is a constant commonly used for circular geometries, e is another mathematical constant. e is particularly used in modeling exponential growth and decay. Just in case you ever need to approximate it, $e \approx 2.72$. We will study e more in depth next week when we cover exponential equations.

4: I have briefly introduced you to e because you can also take logarithms with respect to the base e. This base-e logarithm is called the **natural logarithm**, ln, and is the other special logarithm (besides log) that you can find on your calculator. The natural logarithm is defined such that $\ln(x) = \log_e(x)$.

Example 4: Evaluate the following logarithms by writing the equivalent exponential equation.

i) $y = \ln(e^2)$

ii) $y = \ln(\frac{1}{e^5})$

iii) $y = \ln(\sqrt{e})$

iv) $y = \ln(\sqrt[3]{e^4})$

Example 4 Solutions:

i) $e^y = e^2$; $y = 2$

ii) $e^y \frac{1}{e^5}$; $y = -5$

iii) $e^y = \sqrt{e}$; $y = \frac{1}{2}$

iv) $e^y = \sqrt[3]{e^4} = e^{\frac{4}{3}}$; $y = \frac{4}{3}$

Properties of Logarithms

5: Remember that logarithms of different bases are all logarithms, so regardless of the base, they all exhibit the same properties. This includes log and ln!!!

6: Regardless of the base (a=0 is the exception), $\log_a(1) = 0$. This property is apparent when the logarithm is written in exponential form: $a^0 = 1$.

7: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

8: $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

9: $\log_a(x^y) = y \cdot \log_a(x)$

10: $\log_b(x) = \frac{log_a(x)}{\log_a(b)}$. This is the **change of base formula** and allows you to change from one base to another. b is your original base, and a is the base to which you want to change.

Example 5: Write $\log_3(5x)$ as the combination of two logarithms.

Example 5 Solution: $\log_3(5x) = \log_3(5) + \log_3(x)$

Example 6: Write an equivalent expression for $\log\left(\frac{x^2}{1000}\right)$.

Example 6 Solution: $\log\left(\frac{x^2}{1000}\right) = \log(x^2) - \log(10^3) = 2\log(x) - 3\log(10) = 2\log(x) - 3$

Example 7: Write an equivalent expression for $\log_3\left(\frac{\sqrt{5}}{27}\right)$.

Example 7 Solution: $\log_3\left(\frac{\sqrt{5}}{27}\right) = \log_3(\sqrt{5}) - \log_3(27) = \log_3\left(5^{\frac{1}{2}}\right) - \log_3(3^3) = \frac{1}{2}\log_3(5) - 3\log_3(3) = \frac{1}{2}\log_3(5) - 3$

Example 8: Write an equivalent expression for $\log_2\left(\sqrt{32x^7}\right)$.

Example 8 Solution: $\log_2\left(\sqrt{32x^7}\right) = \log_2\left((32x^7)^{\frac{1}{2}}\right) = \frac{1}{2}\log_2(32x^7) = \frac{1}{2}[\log_2(32)+\log_2(x^7)] = \frac{1}{2}[\log_2(2^5)+\log_2(x^7)] = \frac{1}{2}[5\log_2(2) + 7\log_2(x)] = \frac{5}{2} + \frac{7}{2}\log_2(x)$

Example 9: Simplify the following expression $\log_4(y^2 - 16) - \log_4(y + 4)$ assuming $y \neq -4$.

Example 9 Solution: $\log_4(y^2 - 16) - \log_4(y + 4) = \log_4\left(\frac{y^2-16}{y+4}\right) = \log_4\left(\frac{(y+4)(y-4)}{y+4}\right) = \log_4(y - 4)$

Example 10: Write an equivalent expression for $\ln\left(\frac{\sqrt{e}}{y^3}\right)$ assuming $y \neq 0$.

Example 10 Solution: $\ln\left(\frac{\sqrt{e}}{y^3}\right) = \ln(\sqrt{e}) - \ln(y^3) = \ln\left(e^{\frac{1}{2}}\right) - \ln(y^3) = \frac{1}{2}\ln(e) - 3\ln(y) = \frac{1}{2} - 3\ln(y)$

Example 11: Use the change of base formula to evaluate $\log_3(200)$ with your calculator.

Example 11 Solution: $\log_3(200) = \frac{\log(200)}{\log(3)}$. If you evaluate this expression on a calculator, you get $\frac{\log(200)}{\log(3)} = 4.82$

Example 12: Use the change of base formula to evaluate $\log_7(200)$ with your calculator.

Example 12 Solution: $\log_7(200) = \frac{\log(200)}{\log(7)} = 2.72$