Factoring Quadratic Trinomials

In this section we are going to look at factoring quadratic trinomials. Quadratic trinomials are expressions of the form $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c}$ where x is a variable and **a**, **b**, and **c** are constants. This expression is quadratic because the highest power of x is two and it is a trinomial because there are three terms. An example of a quadratic trinomial would be: $4x^2 - 3x + 11$.

To factor a quadratic trinomial we will need to work backwards. We know that the two expressions that can be multiplied to result in a quadratic trinomial will have the form of $(\mathbf{p}x + \mathbf{q})$ and $(\mathbf{r}x + \mathbf{s})$ where x is a variable and **p**, **q**, **r**, and **s** are constants. For example the product of $(2x - 3)$ and $(x + 5)$ will be a quadratic trinomial $(2x^2 + 7x - 15)$.

When we multiply $(\mathbf{p}x + \mathbf{q})$ by $(\mathbf{r}x + \mathbf{s})$ we get:

$$(\mathbf{p}x + \mathbf{q})(\mathbf{r}x + \mathbf{s}) = \mathbf{pr}x^2 + \mathbf{ps}x + \mathbf{qr}x + \mathbf{qs} = \mathbf{pr}x^2 + (\mathbf{ps} + \mathbf{qr})x + \mathbf{qs}$$

This means that if a general quadratic trinomial $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c}$ can be factored into $(\mathbf{p}x + \mathbf{q})(\mathbf{r}x + \mathbf{s})$ then:

$$\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = \mathbf{pr}x^2 + (\mathbf{ps} + \mathbf{qr})x + \mathbf{qs}$$

And this means that $\mathbf{a} = \mathbf{pr}$ and $\mathbf{c} = \mathbf{qs}$. We can use this fact to factor quadratic trinomials. Let us consider the following example:

$$3x^2 + 5x + 2$$

We know that the factored form of this expression will be: $(\mathbf{p}x + \mathbf{q})(\mathbf{r}x + \mathbf{s})$. Since $\mathbf{a} = \mathbf{pr}$ and $\mathbf{c} = \mathbf{qs}$, we know that $3 = \mathbf{pr}$ and $2 = \mathbf{qs}$. Since the only product of natural numbers that equals **3** is $3 \cdot 1$, then $\mathbf{p} = 3$ and $\mathbf{r} = 1$. Similarly, since $2 = \mathbf{qs}$ then either:

$$\mathbf{q} = 1 \text{ and } \mathbf{s} = 2$$

$$\mathbf{q} = 2 \text{ and } \mathbf{s} = 1$$

$$\mathbf{q} = -1 \text{ and } \mathbf{s} = -2$$

$$\mathbf{q} = -2 \text{ and } \mathbf{s} = -1$$

This means that there are four possible solutions to this factoring problem:

$$(3x + 2)(x + 1)$$

$$(3x + 1)(x + 2)$$

$$(3x - 2)(x - 1)$$

$$(3x - 1)(x - 2)$$

These are the only three possible factorizations that result in a quadratic trinomial $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c}$ where $\mathbf{a} = \mathbf{3}$ and $\mathbf{c} = \mathbf{2}$. Only one of these three possible answers is correct and this correct answer will be the one that has $\mathbf{b} = \mathbf{5}$. To figure out which answer is correct we have to guess and check:

$$(3x + 2)(x + 1) = 3x^2 + 5x + 2$$

$$(3x + 1)(x + 2) = 3x^2 + 7x + 2$$

$$(3x - 2)(x - 1) = 3x^2 - 5x + 2$$

$$(3x - 1)(x - 2) = 3x^2 - 7x + 2$$

We see that the first option is the one that results in $\mathbf{b} = \mathbf{5}$ so the answer is:

$$3x^2 + 5x + 2 = (3x + 2)(x + 1)$$

This method of factoring quadratic trinomials is called guess and check, because it involves a lot of guessing at what the answer is and then checking if the answer is correct. With practice it becomes easier.

Check yourself:

In exercises below, factor the quadratic trinomials:

1) $x^2 + 8x + 15$

2) $2x^2 + 3x - 2$

3) $7x^2 - 17x + 6$

Answers:

1) $(x + 5)(x + 3)$

2) $(2x - 1)(x + 2)$

3) $(7x - 3)(x - 2)$

It is also possible to factor quadratic trinomials with two variables. These quadratic trinomials take the form of:

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2$$

Where x and y are variables and **a**, **b** , and **c** are constants. For example:

$$4x^2 + 7xy + 3y^2$$

is a quadratic trinomial with two variables. Just like we learned earlier in this section that

$$\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = (\mathbf{p}x + \mathbf{q})(\mathbf{r}x + \mathbf{s})$$

The factorization of $\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2$ is:

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 = (\mathbf{p}x + \mathbf{q}y)(\mathbf{r}x + \mathbf{s}y)$$

Consider the following expression:

$$10x^2 + 11xy - 6y^2$$

We know that the factored form of this expression will be: $(\mathbf{p}x + \mathbf{q}y)(\mathbf{r}x + \mathbf{s}y)$. Since **a = pr** and **c = qs**, we know that **10 = pr** and **-6 = qs**. This means that either

p = 10 and **r = 1**

p = 5 and **r = 2**.

Similarly:

q = 6 and **s = -1**

q = 3 and **s = -2**

q = 2 and **s = -3**

q = 1 and **s = -6**

q = -1 and **s = 6**

q = -2 and **s = 3**

q = -3 and **s = 2**

q = -6 and **s = 1**

After going through the guessing and checking we can see that:

$$\mathbf{10x^2 + 11xy - 6y^2 = (5x - 2y)(2x + 3y)}$$

Check yourself:

In exercises below, factor the quadratic trinomials:

4) $x^2 + 12xy + 20y^2$

5) $2x^2 - 7xy + 6y^2$

6) $12x^2 + 4xy - 5y^2$

Answers:

4) $(x + 10y)(x + 2y)$

5) $(2x - 3y)(x - 2y)$

6) $(6x + 5y)(2x - y)$