Exponents

Recall that we use exponents to write products of repeated factors. For example:

2^5 is defined as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

In exercises below write the exponential expressions in product form:

1) 4^2

2) x^5

3) $(x - 2y)^4$

Answers:

1) $4 \cdot 4$

2) $x \cdot x \cdot x \cdot x \cdot x$

3) $(x - 2y) \cdot (x - 2y) \cdot (x - 2y) \cdot (x - 2y) \cdot (x - 2y)$

In exercises below convert the expressions from product form to exponential form:

4) $7 \cdot 7 \cdot 7 \cdot 7$

5) $r \cdot r \cdot r \cdot r \cdot r \cdot r \cdot r$

6) $(3u–5) \cdot (3u–5) \cdot (3u–5) \cdot (3u–5) \cdot (3u–5)$

Answers:

4) 7^4

5) r^7

6) $(3u - 5)^5$

Consider the following expression:

$$a^m \cdot a^n$$

We can rewrite this as:

$$(a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a)(a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a) = a^{n+m}$$

'a' appears as a factor 'm' times 'a' appears as a factor 'n' times

And so we have shown that if 'm' and 'n' are natural numbers and 'a' is a real number then:

$$\boxed{a^m \cdot a^n = a^{m+n} \qquad (1)}$$

This formula will work even if 'm' and 'n' are not natural numbers.

In the following exercises simplify the expressions below:

7) $c^5 \cdot c^7$

8) $h^3 \cdot h^4 \cdot h^2$

9) $(2w + 3t)^6 \cdot (2w + 3t)^8$

Answers:

7) c^{12}

8) h^9

9) $(2w + 3t)^{14}$

In case of a more complicated problem where more than one variable is involved we can still use formula (1).

Consider the following expression:

$$6d^3k^2 \cdot 4dk^7$$

Because the order in which we multiply does not matter we can regroup the factors so that factors with the same bases are together:

$$6d^3k^2 \cdot 4dk^7 = 6 \cdot 4 \cdot d^3 \cdot d \cdot k^2 \cdot k^7$$

And now we apply formula (1) to get:

$$24d^4k^9$$

Try the following exercises:

10) $3a^4b^3 \cdot 5a^7b$

11) $4u^2v^4 \cdot 2u^7w^7 \cdot 10vw^4$

12) $3m^2n^{7x} \cdot 8m^{4y}n^3$

Answers:

10) $15a^{11}b^4$

11) $80u^9v^5w^{11}$

12) $24m^{4y+2}n^{7x+3}$

Consider the following expression:

$$a^n \cdot a^0$$

According to formula (1):

$$a^n \cdot a^0 = a^{n+0}$$

$$a^n \cdot a^0 = a^n$$

If we now divide both sides by a^n:

$$\frac{\cancel{a^n} \cdot a^0}{\cancel{a^n}} = \frac{\cancel{a^n}}{\cancel{a^n}}$$

$$a^0 = 1$$

As a result we have our second rule:

If $a \neq 0$

$$\boxed{a^0 = 1 \qquad (2)}$$

VERY IMPORTANT: if a = 0 then a^0 is undefined. In higher mathematics 0^0 is called the indeterminate form.

Try the following exercises:

13) 4^0

14) $x^0 - 3^0$

15) $-(v - 2)^0 - w^0$

Answers:

13) 1

14) 0 (if $x \neq 0$)

15) -2 (if $v \neq 2$ and $w \neq 0$)

Consider the following expression:

$$a^n \cdot a^{-n}$$

By applying formula (1) we get:

$$a^n \cdot a^{-n} = a^0$$

And by applying formula (2) we have

$$a^n \cdot a^{-n} = 1$$

If we now divide both sides by a^n:

$$\frac{\cancel{a^n} \cdot a^{-n}}{\cancel{a^n}} = \frac{1}{a^n}$$

$$a^{-n} = \frac{1}{a^n}$$

As a result we have our third rule:

$$\boxed{a^{-n} = \frac{1}{a^n} \qquad (3)}$$

In the following exercises rewrite the expressions below by using positive integers:

16) x^{-2}

17) $s^{-5} + t^{-5}$

18) $(2cd + f)^{-7}$

Answers:

16) $\frac{1}{x^2}$

17) $\frac{1}{s^5} + \frac{1}{t^5}$

18) $\frac{1}{(2cd+f)^7}$

In the following exercises evaluate the expressions below:

19) 2^{-3}

20) $3^{-2} + 6^{-1}$

21) $(5^{-2} - 10^{-2})^{-1}$

Answers:

19) $\frac{1}{8}$

20) $\frac{5}{18}$

21) $\frac{100}{3}$

Consider the following expression:

$$\frac{a^m}{a^n}$$

We can rewrite it as:

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n}$$

And by formula (3):

$$a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n}$$

And by formula (1):

$$a^m \cdot a^{-n} = a^{n-m}$$

As a result we have our forth rule:

$$\boxed{\frac{a^m}{a^n} = a^{n-m} \quad (4)}$$

In the following exercises simplify the expressions below and rewrite them using positive exponents:

22) $\frac{k^5}{k^2}$

23) $\frac{r^6 \cdot r^{-3}}{r^7}$

24) $\frac{2x^6y^2 \cdot 6x^{-2}y^3}{4x^3 \cdot x^{-8}y^{13}}$

Answers:

22) k^3

23) $\frac{1}{r^4}$

24) $\frac{3x^9}{y^8}$

Consider the following expression:

$$(a^m)^n$$

We can rewrite this as:

$$a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot a^m \cdot \ldots \cdot a^m$$

'a^m' appears as a factor 'n' times

By formula (1) this equals:

$$a^{(\overbrace{m + m + m + m + m + m + m + m + \ldots + m}^{\text{('n' times)}})} = a^{mn}$$

As a result we have our fifth rule:

$$\boxed{(a^m)^n = a^{mn} \qquad (5)}$$

In the following exercises simplify the expressions below

25) $(p^4)^3$

26) $\dfrac{(x^5)^b}{x^8}$

27) $((y + 3)^5)^3$

Answers:

25) p^{12}

26) $x^{5b - 8}$

27) $(y + 3)^{15}$

When a product of more than one variable is raised to a power, the exponent will be distributed amongst all the factors which brings us to the sixth formula:

$$\boxed{(a \cdot b)^n = a^n b^n \qquad (6A)}$$

and in general:

$(a \cdot b \cdot c \cdot d...)^n = a^n b^n c^n d^n...$	(6B)

Consider the following expression:

$$(2u^4v^3)^5$$

By applying formula (6B) we get

$$(2u^4v^3)^5 = (2)^5 \cdot (u^4)^5 \cdot (v^3)^5$$

And now we apply formula (5) to get:

$$32u^{20}v^{15}$$

In the following exercises simplify the expressions below

28) $(2q^5)^3$

29) $\frac{(3c^5h)^4}{9c^{15}}$

30) $(3t^5(5y - 1)^{4w})^x$

Answers:

28) $8q^{15}$

29) $9c^5h^4$

30) $3^x t^{5x}(5y - 1)^{4wx}$