<u>Complex Numbers</u>

Recall that when you try to take a square root of a negative number there is no real solution. Consider the following expression:

$$\sqrt{-1}$$

In the expression above we want to evaluate the square root of **-1**. In other words, we want to know what number multiplied by itself will equal **-1**. We know that a product of two positive numbers is positive and the product of two negative numbers is also positive so there is no real number, positive or negative, that you can multiply by itself to equal **-1**. However, even though the square root of **-1** is not positive or negative, we do know that it is some sort of number that does exist. We know this because if we square this quantity, we will get a real answer of **-1**:

$$\sqrt{-1}^2 = -1$$

Since the square root of **-1** exists as a number but it is not a real number, mathematicians had to come up with a new way of representing this number. So we say that the square root of **-1** is equal to ***i*** which stands for an imaginary unit. This does not mean that ***i*** is a fictitious number that exists only in our imaginations. Without ***i***, engineers would not be able to analyze electrical waves and physicists would not be able to calculate the fundamental forces that govern our universe via quantum mechanics. Let us now take a closer look at the number ***i***. We already know that:

$$i = \sqrt{-1}$$

Let us consider what happens when we raise ***i*** to the second power:

$$i^2 = \sqrt{-1}^2$$

So:

$$i^2 = -1$$

Now let us see what happens when ***i*** is raised to the third power:

$$\boldsymbol{i^3 = i^2 \cdot i}$$

And since $\boldsymbol{i^2 = -1}$, we get:

$$\boldsymbol{i^3 = i^2 \cdot i = -1 \cdot i = -i}$$

Similarly, using the fact that $\boldsymbol{i^2 = -1}$ we can see that when we raise $\boldsymbol{i}$ to the forth power we get:

$$\boldsymbol{i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1}$$

By using the results above we can easily figure out what happens when we raise $\boldsymbol{i}$ to any integer power. The last result is especially useful because we can generalize it in the following way:

$$\boldsymbol{i^{4n} = (i^4)^n}$$

And since $\boldsymbol{i^4 = 1}$:

$$\boldsymbol{i^{4n} = (i^4)^n = 1^n = 1}$$

This means that if you raise $\boldsymbol{i}$ to a power that is divisible by **4**, it will equal **1**. Now let us use the results above to evaluate the result of raising $\boldsymbol{i}$ to an integer power. Consider the expression below:

$$\boldsymbol{i^{83}}$$

The largest number that is less than or equal to **83** and divisible by **4** is **80**. So we can rewrite the expression above as:

$$\boldsymbol{i^{83} = i^{80} \cdot i^3 = 1 \cdot -i = -i}$$

Check yourself:

In exercises below evaluate the following expressions:

1) i^{13}

2) i^{47}

3) i^{1000}

Answers:

1) i

2) $-i$

3) 1

We can also use the fact that we know that $\boldsymbol{i} = \sqrt{-\mathbf{1}}$ to evaluate the square roots of negative numbers. Consider the following expression:

$$\sqrt{-\mathbf{49}}$$

We can break up **-49** into a product of **-1** and **49**:

$$\sqrt{-\mathbf{49}} = \sqrt{-\mathbf{1}} \cdot \sqrt{\mathbf{49}}$$

And since we know that $\sqrt{-\mathbf{1}} = \boldsymbol{i}$ and $\sqrt{\mathbf{49}} = \mathbf{7}$ we have:

$$\sqrt{-\mathbf{49}} = \sqrt{-\mathbf{1}} \cdot \sqrt{\mathbf{49}} = \boldsymbol{i} \cdot \mathbf{7} = \mathbf{7}\boldsymbol{i}$$

Check yourself:

In exercises below evaluate the following expressions:

4) $\sqrt{-4}$

5) $\sqrt{-121}$

6) $\sqrt{-31}$

Answers:

4) $2i$

5) $11i$

6) $i\sqrt{31}$

It is not possible to simplify the sum or a difference of a real number and an imaginary number. For example **3 + 6*i*** is a number that has both a real and an

imaginary part and cannot be simplified further. Any number that has a real part and imaginary part is called a complex number. Complex numbers actually include real numbers because any real number like **12** can be written as a complex number with the imaginary part being equal to zero **12 + 0*i***. Similarly, any imaginary number is also a complex number with the real part equal to zero, for example: **-2*i* = 0 – 2*i***. In this part of the section we will take a look at how to add, subtract, multiply and divide complex numbers.

Addition:

When adding complex numbers we simply add the real part and imaginary parts separately. Consider the following example:

$$(4 + 8i) + (3 - 2i)$$

By grouping the real parts and imaginary parts we get:

$$(4 + 3) + (8i - 2i) = 7 + 6i$$

Subtraction:

When subtracting complex numbers we simply subtract the real part and imaginary parts separately. Consider the following example:

$$(4 + 8i) - (3 - 2i)$$

By grouping the real parts and imaginary parts we get:

$$(4 - 3) + (8i - (-2i)) = 1 + 10i$$

Check yourself:

In exercises below evaluate the following expressions:

7) $3 - (4 + i)$

8) $(2 - 5i) + (6 - 8i)$

9) $(7 + 2i) - (11 - 9i)$

Answers:

7) $-1 - i$

8) $8 - 13i$

9) $-4 + 11i$

Multiplication:

When multiplying complex numbers we multiply in the same way we perform polynomial multiplication. Consider the following example:

$$(4 + 8i)(3 - 2i)$$

We can now multiply every term in the first parenthesis by every term in the second parenthesis to get:

$$(4 + 8i)(3 - 2i) = 4 \cdot 3 + 4 \cdot (-2i) + 8i \cdot 3 + 8i \cdot (-2i)$$

Which equals:

$$12 - 8i + 24i - 16i^2$$

Since $i^2 = -1$ we can rewrite this expression as:

$$12 - 8i + 24i - 16(-1)$$

And finally after combining like terms we get:

$$28 + 16i$$

Check yourself:

In exercises below evaluate the following expressions:

10) $(a + bi)(a - bi)$

11) $(2 - 5i)(6 - 8i)$

12) $(7 + 2i)(11 - 9i)$

Answers:

10) $a^2 + b^2$

11) $-28 - 46i$

12) $95 - 41i$

Division:

To divide complex numbers, we use something called a **complex conjugate**. A complex conjugate of a number is a number that has the same real part and opposite imaginary part. For example, the complex conjugate of $\mathbf{4 - 3i}$ is $\mathbf{4 + 3i}$. When dividing one complex number by another, our goal is for the denominator to not have an imaginary part. To accomplish this we multiply the fraction by **1** in the form of another fraction where the numerator and denominator are the complex conjugate of the denominator. Consider the following expression:

$$\frac{\mathbf{4 + 8i}}{\mathbf{3 - 2i}}$$

We first need to find the complex conjugate of the denominator. Since the denominator equals $\mathbf{3 - 2i}$ its complex conjugate is $\mathbf{3 + 2i}$. By multiplying the fraction by **1** in the form of another fraction where the numerator and denominator are the complex conjugate of the denominator we get:

$$\frac{\mathbf{4 + 8i}}{\mathbf{3 - 2i}} \cdot \frac{\mathbf{3 + 2i}}{\mathbf{3 + 2i}} = \frac{\mathbf{4 \cdot 3 + 4 \cdot 2i + 8i \cdot 3 + 8i \cdot 2i}}{\mathbf{3 \cdot 3 + 3 \cdot 2i + (-2i) \cdot 3 + (-2i) \cdot 2i}}$$

Which equals:

$$\frac{\mathbf{12 + 8i + 24i + 16i^2}}{\mathbf{9 + 6i - 6i - 4i^2}}$$

After converting $\boldsymbol{i^2}$ to **-1** and combining like terms all imaginary parts will disappear from the denominator to give us:

$$\frac{\mathbf{12 + 8i + 24i + 16i^2}}{\mathbf{9 + 6i - 6i - 4i^2}} = \frac{\mathbf{12 + 32i - 16}}{\mathbf{9 + 4}} = \frac{\mathbf{-4 + 32i}}{\mathbf{13}}$$

We can now distribute the denominator into the real and imaginary part of the numerator to give us:

$$\frac{-4+32i}{13} = \frac{-4}{13} + \frac{32}{13}i$$

Check yourself:

In exercises below evaluate the following expressions:

13) $\frac{3-5i}{2+9i}$

14) $\frac{2+8i}{5-4i}$

15) $\frac{11-6i}{3+i}$

Answers:

13) $\frac{-39}{85} - \frac{37}{85}i$

14) ~~$\frac{-11}{18} + \frac{4}{3}i$~~ $\frac{-22 + 48i}{41}$

15) $\frac{27}{10} - \frac{29}{10}i$