MATH 5071 - Problem Set 7

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1) Solve for x:

i) $\log_8(4x-3) = \log_8(33)$

$8 \qquad 8$

$4x - 3 = 33$, $+3 \quad +3$ $\qquad \frac{4x}{4} = \frac{36}{4}$ $\qquad \boxed{x = 9}$

ii) $\log_4(x+2) = 2$

$4 \qquad 4$

$x + 2 = 4^2$ $\qquad x + 2 = 16$, $-2 \quad -2$ $\qquad \boxed{x = 14}$

iii) $4^x = 0.4$

$\log 4^x = \log 0.4 \Rightarrow \frac{x \log 4}{\log 4} = \frac{\log 0.4}{\log 4}$

$x = \frac{\log 0.4}{\log 4} = \boxed{-.661}$ OR $\log_4 4^x = \log_4 0.4$ $\quad x = \log_4 0.4$

$\boxed{x = -.661}$

iv) $4.6 \cdot 10^x = 7$

$10^x = \frac{7}{4.6}$ $\qquad \log 10^x = \log \frac{7}{4.6}$ $\qquad x = \log \frac{7}{4.6}$ $\qquad \boxed{x = 0.182}$

v) $\frac{9(4.5)^x}{9} = \frac{10}{9}$ $\qquad \log 4.5^x = \log\left(\frac{10}{9}\right)$ $\qquad \frac{x \log 4.5}{\log 4.5} = \frac{\log\left(\frac{10}{9}\right)}{\log 4.5}$ $\qquad \boxed{x = .0700}$ OR $x = \log_{4.5}\left(\frac{10}{9}\right)$

2) A particular satellite has a radio-isotope power supply. with power output given by $P = 60e^{-\frac{t}{300}}$. where t is the time (in days) and P is the power output (in Watts).

i) What is the power output at the end of the isotope's first year? $\quad t = 1$

$P = 60e^{-\frac{1}{300}} = 60 \cdot (.997) = \boxed{59.8 \text{ Watts}}$

ii) When the power output drops below 6 Watts, there will be insufficient power to operate the satellite. Find how long the satellite will remain operable.

$P = 6$ W

$\frac{6}{60} = \frac{60e^{-t/300}}{60}$ $\qquad \ln \frac{1}{10} = \ln e^{-\frac{t}{300}}$ $\qquad -300 \times \ln\left(\frac{1}{10}\right) = -\frac{t}{300} \times -300$

$t = -300 \ln\left(\frac{1}{10}\right) = \underline{690.8 \text{ years}}$

3) If money is invested at 6% interest compounded continuously, how long will it take for the money to double in value?

P = initial amount invested

2P = the initial amount doubled

$A = Pe^{rt}$ $\quad r = 6\% = .06$

$\frac{2P}{P} = \frac{Pe^{.06t}}{P}$ $\quad 2 = e^{.06t}$ $\quad \ln 2 = \ln e^{.06t}$ $\quad \frac{\ln 2}{.06} = \frac{.06t}{.06}$ $\quad t = \frac{\ln 2}{.06} =$ $\boxed{t = 11.6 \text{ yr}}$

4) Strontium-90 has a half-life of 25 years. How long will it take 10 grams on Strontium-90 to decay to 3 grams?

$A = A_0 b^{t/k}$

$b = 1/2$ (half-life) $\quad k = 25$ (25 years for the amount to decrease by half)

$A_0 = 10$ grams $\quad A = 3$ grams

$3 = 10\left(\frac{1}{2}\right)^{t/25}$ $\quad \log \frac{3}{10} = \log\left(\frac{1}{2}\right)^{t/25}$ $\quad 25 \cdot \log\left(\frac{3}{10}\right) = \frac{t}{25}\log\left(\frac{1}{2}\right) \cdot 2$

$\frac{25\log\left(\frac{3}{10}\right)}{\log\left(\frac{1}{2}\right)} = \frac{t\cancel{\log\left(\frac{1}{2}\right)}}{\cancel{\log\left(\frac{1}{2}\right)}}$

$t = \frac{25\log\left(\frac{3}{10}\right)}{\log\left(\frac{1}{2}\right)}$

$\boxed{t = 43.4 \text{ years}}$

or

$t = 25\log_{1/2}\left(\frac{3}{10}\right)$

5) The population of Iceland in 2008 was about 304,000, with an annual growth rate of 0.78%. Assume the growth rate is constant.

i) Give an equation for the population of Iceland n years after 2008.

$A_0 = 304{,}000$

$b = 1 + .0078 = 1.0078$

$k = 1$ year $\quad \boxed{A = 304{,}000 \cdot (1.0078)^t}$

ii) Predict the population in 2020.

$t = 12$ (12 years after 2008 for 2020)

$A = 304{,}000 \cdot (1.0078)^{12} = \boxed{333{,}707 \text{ people}}$

iii) Predict when the population will reach 400,000.

$A = 400{,}000$

$\frac{400{,}000}{304{,}000} = \frac{304{,}000(1.0078)^t}{304{,}000}$

$\log 1.316 = \log(1.0078)^t$

$\frac{\log 1.316}{\log 1.0078} = \frac{t\cancel{\log 1.0078}}{\cancel{\log 1.0078}}$

$t = \frac{\log 1.316}{\log 1.0078} = \boxed{35.3 \text{ yrs}}$

or

$t = \log_{1.0078} 1.316$

6) Solve the following inequalities. Give both inequality, interval notation, and number line forms of the solutions.

i) $-2(x-3) < 5(x+1) - 12$

$-2x + 6 < 5x + 5 - 12$

$-2x + 6 < 5x - 7$

$-2x + 7 \quad +2x \quad +7$

$\frac{13}{7} < \frac{7x}{7}$ $\quad x > \frac{13}{7}$ $\quad \frac{13}{7}$ $\quad (\frac{13}{7}, \infty)$

ii) $-14 < -7(3x+2) \leq 1$

$-14 < -21x - 14 \leq 1$

$-14 < -21x - 14$ $\quad 0 < -21x$ $\quad x < 0$

$-21x - 14 \leq 1$

$\frac{-21x}{-21} \leq \frac{15}{-21}$ $\quad x \geq \frac{-15}{21}$ $\quad x \geq -\frac{5}{7}$

$-\frac{5}{7}$ $\quad 0$ $\quad [-\frac{5}{7}, 0)$

iii) $x^2 - 10 \leq 3x$

$x^2 - 3x - 10 \leq 0$ $\quad (x-5)(x+2) \leq 0$

$x = -2 \quad x = 5$ $\quad -2 \quad 5$ $\quad [-2, 5]$ or $-2 \leq x \leq 5$

pick points:

$-3: (-3)^2 - 10 \leq 3(-3)$

$-1 \leq -9$ false

$0: 0^2 - 10 \leq 0$ true

$6: 6^2 - 10 \leq 3(6)$

$36 - 10 \leq 18$ false

iv) $x^4 + 4x^3 - 12x^2 \leq 0$

$x^2(x^2 + 4x - 12) \leq 0$ $\quad x^2(x+6)(x-2) \leq 0$

$x = 0 \quad x = -6 \quad x = 2$ $\quad -6 \quad 0 \quad 2$

$-7: (1)(-)(-) \leq 0$ false

$-5: (1)(+)(-) \leq 0$ true

$1: (+)(+)(-) \leq 0$ true

$3: (+)(+)(+) \leq 0$ false

$[-6, 2]$ or $-6 \leq x \leq 2$

v) $\frac{3x+1}{x+4} > 1$

$3x + 1 > x + 4$

$-x \quad 1 \quad -x \quad -1$

$2x > 3$

$\frac{2x}{2} > \frac{3}{2}$

$x > \frac{3}{2}$ $\quad 3/2$ $\quad (\frac{3}{2}, \infty)$

vi) $\frac{x-8}{x} > 3 - x$

$x - 8 > 3x - x^2$

$x^2 - 2x - 8 > 0$

$(x-4)(x+2) > 0$

$x = 4 \quad x = -2$ $\quad -2 \quad 4$

$-3: (-)(-) > 0$ ✓

$0: (-)(+) > 0$ ✗

$5: (+)(+) > 0$ ✓

$(-\infty, -2) \cup (4, \infty)$

$x < -2$ or $x > 4$

7) Find the quadratic equation the passes through the points (2,37), (5,10), and (9,2) by solving a system of equations. Use matrix multiplication to check your answers.

$y = ax^2 + bx + c$

$37 = a(2)^2 + b(2) + c$

$10 = a(5)^2 + b(5) + c$

$2 = a(9)^2 + b(9) + c$

$\begin{cases} 37 = 4a + 2b + c & (1) \\ 10 = 25a + 5b + c & (2) \\ 2 = 81a + 9b + c & (3) \end{cases}$

(eq 1) + −1×(eq 2) $\quad 37 = 4a + 2b + \cancel{c}$

$-10 = -25a - 5b - \cancel{c}$

eq 4 $\quad 27 = -21a - 3b$

−1×(eq 2) + eq 3 $\quad -10 = -25a - 5b - \cancel{c}$

$2 = 81a + 9b + \cancel{c}$

eq 5 $\quad -8 = 56a + 4b$

4×(eq 4) + 3×(eq 5): $\quad 108 = -84a - \cancel{12b}$

$-24 = 168a + \cancel{12b}$

$84 = 84a$

$a = 1$

eq 4: $\quad 27 = -21(1) - 3b$

$27 = -21 - 3b$

$+21 \quad +21$

$\frac{48}{-3} = \frac{-3b}{-3}$

$b = -16$

eq 1: $\quad 37 = 4(1) + 2(-16) + c$

$37 = 4 - 32 + c$

$37 = -28 + c$

$+28 \quad +28$

$65 = c$

$\boxed{y = x^2 - 16x + 65}$

Matrix Mult.

$$\begin{bmatrix} 4 & 2 & 1 \\ 25 & 5 & 1 \\ 81 & 9 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 37 \\ 10 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 25 & 5 & 1 \\ 81 & 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 37 \\ 10 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -16 \\ 65 \end{bmatrix}$$