MATH 5071 - Exam 2 (100 pts)

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Name: Solutions

1) Find all solutions for x by **factoring**:

a) $x^2 - 24x + 144 = 0$ (4 pts)

$(x-12)(x-12) = 0$ $\quad x-12=0$ $\quad \boxed{x=12}$

b) $x^3 + 2x^2 = 15x$ (5 pts)

$x^3+2x^2-15x=0$

$x(x^2+2x-15)=0$

$x(x+5)(x-3)=0$

$\boxed{x=0}$ AND

$(x+5)=0$ $\quad \boxed{x=-5}$ AND

$x-3=0$ $\quad \boxed{x=3}$

c) $(3x+2)(x-3) = 7x - 1$ (6 pts)

FOIL

$3x^2-9x+2x-6=7x-1$

$3x^2-7x-6=7x-1$

$-7x+1 \quad -7x+1$

$3x^2-14x-5=0$

$3x^2-15x+x-5=0$

$3x(x-5)+1(x-5)=0$

$(3x+1)(x-5)=0$

$3x+1=0$ $\quad \frac{3x}{3}=\frac{-1}{3}$ $\quad \boxed{x=-\frac{1}{3}}$ AND

$-1 \quad -1$

$x-5=0$ $\quad \boxed{x=5}$

2) Find all solutions to the following systems of equations.

By Elimination:

a) $\begin{cases} 2x + y = 5 \\ -4x + 6y = -2 \end{cases}$ (5 pts)

$2 \times (2x + y = 5)$

$\cancel{4x} + 2y = 10$

$-\cancel{4x} + 6y = -2$

$\frac{8y}{8} = \frac{8}{8}$ $\boxed{y = 1}$

$2x + (1) = 5$

$\frac{2x}{2} = \frac{4}{2}$

$\boxed{x = 2}$

b) $\begin{cases} 2x + 3y = 7 \\ -y + x = 1 \end{cases}$ (5 pts)

By substitution

$x = 1 + y$ plug into eq. 1

$2(1+y) + 3y = 7$

$2 + 2y + 3y = 7$

$5y = 5$

$\boxed{y = 1}$

$x = 1 + 1$

$\boxed{x = 2}$

3) Simplify and report your answer in $a + bi$ form: $2i^{70} - 5i^{45} + 3i^{27} - 7i^{16}$ (2 pts)

$2i^{70} = 2i^2$ $5i^{45} = 5i$ $3i^{27} = 3i^3$ $7i^{16} = 7$

$i = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$2i^{70} - 5i^{45} + 3i^{27} - 7i^{16} = 2i^2 - 5i + 3i^3 - 7 = 2(-1) - 5i + 3(-i) - 7$

$= -2 - 5i - 3i - 7 = \boxed{-9 - 8i}$

4) Evaluate: $\log_4(8)$ (3 pts)

$\log_4(8) = x$

$4^x = 8$ $(2^2)^x = 2^3$ $2^{2x} = 2^3$ $\frac{\cancel{2}x}{\cancel{2}} = \frac{3}{2}$ $\boxed{x = \frac{3}{2}}$

5) Rachael invests $10,000 dollars into a savings account that earns 12.5% interest.

$P = 10{,}000$ $\quad r = 0.125$

a) How much money will be in the account after 2 years if the interest is compounded quarterly? (3 pts)

$t = 2$ $\quad n = 4$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 10{,}000\left(1 + \frac{.125}{4}\right)^{4\cdot(2)} = 10{,}000(1.03125)^8 = \boxed{\$12{,}791.21}$$

b) If she wants to use this invested money to buy a $25,000 car, how long will the money have to sit in the account (in years) if the interest is compounded continuously? (6 pts)

A

$$A = Pe^{rt}$$

$A = 25{,}000$
$P = 10{,}000$
$r = 0.125$

$$\frac{25{,}000}{10{,}000} = \frac{10{,}000e^{0.125\cdot t}}{10{,}000}$$

$$\ln(2.5) = \ln\left(e^{0.125t}\right)$$

$$\frac{\ln(2.5)}{0.125} = \frac{0.125t}{0.125}$$

$$\boxed{t = 7.33 \text{ years}}$$

c) If the interest is compounded once a year, instead, how much longer than part (b) will she have to wait (in years)? (6 pts)

$n = 1$

$A = 25{,}000$
$P = 10{,}000$
$r = 0.125$
$n = 1$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\frac{25{,}000}{10{,}000} = \frac{10{,}000\left(1 + \frac{0.125}{1}\right)^{1\cdot t}}{10{,}000}$$

$$\log(2.5) = \log(1.125)^t$$

$$\frac{\log(2.5)}{\log 1.125} = \frac{t\log 1.125}{\log 1.125}$$

$$t = \frac{\log 2.5}{\log 1.125}$$

$$t = 7.78 \text{ years}$$

OR

$$\log_{1.125}(2.5) = \log_{1.125} 1.125^t$$

$$\log_{1.125}(2.5) = t$$

which turns into this using the change of base formula

$$\Delta t = 7.78 - 7.33 = \boxed{.45 \text{ years longer}}$$

6) Find all solutions for x: $\frac{8}{x^2+2x-3} = \frac{4}{x^2+3x}$ (5 pts)

$$\cancel{(x+3)}\cancel{(x-1)} \cdot \frac{8}{\cancel{(x+3)}\cancel{(x-1)}} = \frac{4 \cdot \cancel{(x+3)}(x-1)}{x\,\cancel{(x+3)}}$$

get rid of the denominators by multiplying by them!

$$x \cdot 8 = \frac{4(x-1)}{\cancel{x}} \cdot \cancel{x}$$

$8x = 4(x-1)$

$8x = 4x - 4$

$4x = -4$

$\boxed{x = -1}$

NOTE

$x = -3$ CANNOT be a solution because neither function exists at $x = -3$

7) Solve by **completing the square:** $x^2 + 8x - 4 = 0$ (5 pts)

$x^2 + 8x - \cancel{4} = 0$

$\quad\quad\quad +\cancel{4} \quad +4$

$x^2 + 8x + \underline{16} = 4 + \underline{16}$

$2b = 8$

$b = 4$

$b^2 = 16$

$x^2 + 8x + 16 = 20$

$\sqrt{(x+4)^2} = \pm\sqrt{20}$

$x + 4 = \pm\sqrt{4 \cdot 5}$

$x + 4 = \pm 2\sqrt{5}$

$x = -4 \pm 2\sqrt{5}$

$\boxed{x = -4 + 2\sqrt{5} \quad \text{or} \quad x = -4 - 2\sqrt{5}}$

8) Let $f(x) = \frac{7}{x} - 3$.

a) What is the domain of $f(x)$? (1 pt)

$x \neq 0$ D: $\boxed{(-\infty, 0) \cup (0, \infty)}$

b) Find $f^{-1}(x)$. (1 pts)

$y = \frac{7}{x} - 3$

$x = \frac{7}{y} - 3$ $\frac{7}{y} = x + 3$ $\boxed{\frac{7}{x+3} = y}$

c) What is the domain of $f^{-1}(x)$? (1 pt)

$x \neq -3$ $\boxed{(-\infty, -3) \cup (-3, \infty)}$

d) Using your answers from parts **a** and **c**, find the ranges of both $f(x)$ and $f^{-1}(x)$. Explain your reasoning. (2 pts)

domain $f(x)$ = range $f^{-1}(x)$ = $(-\infty, 0) \cup (0, \infty)$

domain $f^{-1}(x)$ = range $f(x)$ = $(-\infty, -3) \cup (-3, \infty)$

9) Find the solution set for the following inequality. First, draw your solution set on a number line. Then, report your solution set in interval notation: $x^2 + 5x - 20 \geq 3x - 5$ (5 pts)

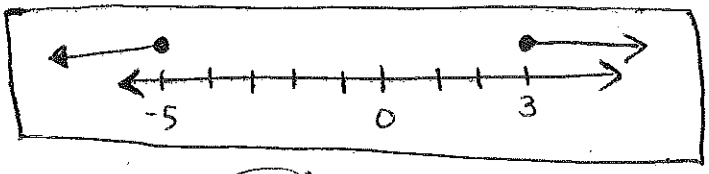
$x^2 + 5x - 20 = 3x - 5$

$-3x + 5 \quad -3x + 5$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x = -5 \quad x = 3$



test points: $x = -6$

$(-6)^2 + 5(-6) - 20 \geq 3(-6) - 5$

$36 - 30 - 20 \geq -18 - 5$

$-14 \geq -23$ true!

$x = 0$

$0^2 + 5(0) - 20 \geq 3(0) - 5$

$-20 \geq -5$ false!

$x = 4$

$4^2 + 5(4) - 20 \geq 3(4) - 5$

$16 + 20 - 20 \geq 12 - 5$

$16 > 7$ true!

interval notation:

$(-\infty, -5] \cup [3, \infty)$

10) The loudness of a sound is given by the following formula:

$$\mathrm{dB} = 10\log(\mathrm{I}), \tag{1}$$

where dB is the loudness of a sound measured in decibels and I is the intensity of that sound. Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. What is the maximum intensity of sound that you can be exposured to for extended periods of time without any danger of hearing damage? (5 pts)

$$\frac{85}{10} = \frac{\cancel{10}\log(I)}{\cancel{10}}$$

$$10^{8.5} = {}_{10}\log(I)$$

$$\boxed{I = 10^{8.5} = 316{,}227{,}766}$$

11) Solve for all values of x using the **quadratic equation**: $18 - 10x + x^2 = 0$ (5 pts)

$$x^2 - 10x + 18 = 0 \qquad a = 1 \quad b = -10 \quad c = 18$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(18)}}{2(1)} = \frac{10 \pm \sqrt{100 - 72}}{2}$$

$$x = \frac{10 \pm \sqrt{28}}{2} = \frac{10 \pm 2\sqrt{7}}{2} = 5 \pm \sqrt{7}$$

$$\boxed{x = 5 + \sqrt{7} \text{ AND } x = 5 - \sqrt{7}}$$

12) Solve for all values of x: $x - 3 = \sqrt{30 - 2x}$ (5 pts)

$$(x-3)^2 = \left(\sqrt{30 - 2x}\right)^2$$

$$x^2 - 6x + 9 = 30 - 2x$$
$$+2x \quad -30 \quad -30 \quad +2x$$

$$x^2 - 4x - 21 = 0$$

$$(x-7)(x+3) = 0$$

$$x = 7$$
$$\cancel{x = -3}$$ > test both solutions!

$$\boxed{x = 7}$$

$x = 7$:

$$7 - 3 = \sqrt{30 - 2 \cdot 7}$$
$$4 = \sqrt{30 - 14}$$
$$4 = \sqrt{16}$$
$$4 = 4 \checkmark$$

$$\cancel{x = -3}$$

$$-3 - 3 = \sqrt{30 - 2(-3)}$$
$$-6 = \sqrt{30 + 6}$$
$$-6 \neq 6$$

13) The half-life of Carbon-14 is 5,730 years. An archeologist finds a bone containing 1.2 grams of Carbon-14. If the bone originally contained 20 grams of Carbon-14, how old is the bone (in years)? (10 pts)

$A = A_0 \, b^{t/k}$

$k = 5730$

$b = \frac{1}{2}$ (for half-life)

$A = 1.2 \text{ g}$

$A_0 = 20 \text{ g}$

$\frac{1.2}{20} = \frac{20\left(\frac{1}{2}\right)^{t/5730}}{20}$

$\log_{1/2} \frac{1.2}{20} = \log_{1/2} \frac{1}{2}^{t/5730}$

$\log_{1/2}\left(\frac{1.2}{20}\right) = \frac{t}{5730}$

$t = 5730 \cdot \log_{1/2}\left(\frac{1.2}{20}\right) = 5730 \frac{\log\left(\frac{1.2}{20}\right)}{\log\left(\frac{1}{2}\right)} = \boxed{23{,}257.5 \text{ years}}$

14) Find the quadratic equation (in standard form) that passes through the following points **(NO CALCULATOR)**: (0,7), (3,-8), (6,13) (10 pts)

$y = ax^2 + bx + c$

$7 = a(0)^2 + b(0) + c$ $\quad \boxed{7 = c}$

$-8 = a(3)^2 + b(3) + c$ $\quad -8 = 9a + 3b + 7$ (1)

$13 = a(6)^2 + b(6) + c$ $\quad 13 = 36a + 6b + 7$ (2)

multiply equation 1 by -2

$16 = -18a - 6b - 14$ (1)×-2

$+ \quad 13 = 36a + 6b + 7$ (2)

$29 = 18a - 7$

$+7 \qquad +7$

$\frac{36}{18} = \frac{18a}{18}$

$\boxed{a = 2}$

plug in equation (2)

$13 = 36(2) + 6b + 7$

$13 = 72 + 6b + 7$

$13 = 79 + 6b$

$-79 \quad -79$

$\frac{-66}{6} = \frac{6b}{6}$

$\boxed{b = -11}$

$\boxed{y = 2x^2 - 11x + 7}$

Extra Credit 1: Express as a single, simplified fraction: $\frac{\frac{1}{8}-\frac{1}{2x}}{\frac{1}{x}-\frac{x}{16}}$. (5 pts)

$$\frac{\frac{x\cdot 1}{x\cdot 8}-\frac{1\cdot 4}{2x\cdot 4}}{\frac{16\cdot 1}{16\cdot x}-\frac{x\cdot x}{16\cdot x}} = \frac{\frac{x}{8x}-\frac{4}{8x}}{\frac{16}{16x}-\frac{x^2}{16x}} = \frac{\frac{x-4}{8x}}{\frac{16-x^2}{16x}} = \frac{\frac{x-4}{8x}}{\frac{(4-x)(4+x)}{16x}}$$

$$= \frac{\cancel{x-4}}{\cancel{8x}}\cdot\frac{\overset{2}{\cancel{16x}}}{(4+x)(-1)\cancel{(x-4)}} = \boxed{\frac{-2}{4+x}}$$

Extra Credit 2: The current population of Baconburg is 20,000. Since the people of Baconburg have such terrible heart problems, the population decreases by 15% each year. In how many years from now will the population of Baconburg be 1? (5 pts)

$A = A_0 b^{t/k}$

$k = 1$ year

15% decrease = 85% of original value

$b = (.85)$

$A_0 = 20{,}000$

$A = 1$

$1 = 20{,}000(.85)^{t/1}$

$\log_{.85}\frac{1}{20000} = \log_{.85}(.85)^t$

$t = \log_{.85}\left(\frac{1}{20{,}000}\right)$

or

$t = \frac{\log\left(\frac{1}{20{,}000}\right)}{\log(.85)}$

$\boxed{t = 60.9 \text{ years}}$

Extra Credit 3: Solve by **completing the square**: $2p^2 + 20 = 6p$ (5 pts)

$2p^2 + 20 = 6p$

$2p^2 - 6p + 20 = 0$
$\quad\quad -20 \quad -20$

$\frac{2p^2}{2} - \frac{6p}{2} = \frac{-20}{2}$

$p^2 - 3p + \frac{9}{4} = -10 + \frac{9}{4}$

$2b = -3$

$b = \frac{-3}{2}$

$b^2 = \frac{9}{4}$

$p^2 - 3p + \frac{9}{4} = \frac{-40}{4} + \frac{9}{4}$

$\sqrt{\left(p - \frac{3}{2}\right)^2} = \sqrt{\frac{-31}{4}}$

$p - \frac{3}{2} = \pm\frac{i}{2}\sqrt{31}$

$$\boxed{\begin{aligned} p &= \frac{3}{2} + \frac{i}{2}\sqrt{31} \\ p &= \frac{3}{2} - \frac{i}{2}\sqrt{31} \end{aligned}}$$