MATH 5071 - Exam 1 (100 pts)

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Name: Solutions

1) Let $f(x) = \frac{7}{x} - 3$ and $g(x) = x^2 + 3$.

a) What is the domain of $f(x)$? (1 pt)

not allowed to divide by zero so $x \neq 0$ $\quad \underline{(-\infty, 0) \cup (0, \infty)}$

b) Find $f^{-1}(x)$. (2 pts)

$f(x)$ $\quad y = \frac{7}{x} - 3 \quad$ $f^{-1}(x)$ $\quad x = \frac{7}{y} - 3 \quad x + 3 = \frac{7}{y} \quad y = \frac{7}{x+3} \quad \boxed{f^{-1}(x) = \frac{7}{x+3}}$

c) What is the domain of $f^{-1}(x)$? (1 pt)

not allowed to divide by zero so $x + 3 \neq 0 \quad x \neq -3 \quad \underline{(-\infty, -3) \cup (-3, \infty)}$

d) Using your answers from parts **a** and **c**, find the ranges of both $f(x)$ and $f^{-1}(x)$. Explain your reasoning. (2 pts)

range $f(x)$ = domain $f^{-1}(x) = (-\infty, -3) \cup (-3, \infty)$

range $f^{-1}(x)$ = domain $f(x) = (-\infty, 0) \cup (0, \infty)$

e) Evaluate $(f \circ g)(2)$. (2 pts)

$(f \circ g)(2) = f(g(2)) \quad g(2) = 2^2 + 3 = 7$

$f(g(2)) = f(7) = \frac{7}{7} - 3 = \underline{\underline{-2 = (f \circ g)(2)}}$

f) Find $f(g(x))$. (2 pts)

$f(g(x)) = \frac{7}{g(x)} - 3 = \boxed{\frac{7}{x^2+3} - 3 = f(g(x))}$

2) Let $Y = -4 + i$ and $X = 2 + 3i$.
Perform the following operations and report your answers in $a + bi$ form.

a) $2Y + X$ (3 pts)

$$= 2(-4+i) + (2+3i)$$
$$= -8 + 2i + 2 + 3i$$
$$= (-8+2) + (2+3)i$$
$$\boxed{= -6 + 5i}$$

b) $Y \cdot (2X)$ (3 pts)

$$= (-4+i)\left[2(2+3i)\right]$$
$$= (-4+i)(4+6i)$$
$$= -16 - 24i + 4i + 6i^2$$
$$= -16 - 20i - 6$$
$$\boxed{= -22 - 20i}$$

c) $\frac{2Y}{X}$ (4 pts)

$$= \frac{2(-4+i)}{(2+3i)} = \frac{(-8+2i)}{(2+3i)}\frac{(2-3i)}{(2-3i)} = \frac{-16+24i+4i-6i^2}{4-6i+6i-9i^2} = \frac{-16+28i+6}{4+9}$$
$$= \frac{-10+28i}{13} = \boxed{\frac{-10}{13} + \frac{28i}{13}}$$

3) Factor the following polynomials *completely*.

$a^2 - b^2 = (a+b)(a-b)$

a) $72x - 50x^3$ (3 pts)

$$72x - 50x^3 = 2x(36 - 25x^2)$$
$a^2 = 36$ $\quad b^2 = 25x^2$
$a = 6$ $\quad b = 5x$
$$\boxed{= 2x(6-5x)(6+5x)}$$

b) $x^3 - x^2 - 25x + 25$ (3 pts) grouping!

$$(x^3 - x^2) + (-25x + 25) = x^2(x-1) - 25(x-1)$$
$$= (x^2 - 25)(x-1) = \boxed{(x+5)(x-5)(x-1)}$$
$a^2 - b^2 = (a+b)(a-b)$
$a^2 = x^2$ $\quad b^2 = 25$
$a = x$ $\quad b = 5$

c) $18x^2 + 41x - 10$ (4 pts)

$18 \cdot -10 = -180 = m \cdot n$
$m + n = 41$
$m = 45$
$n = -4$

$= 18x^2 - 4x + 45x - 10$ grouping!
$= 2x(9x-2) + 5(9x-2)$
$$\boxed{= (9x-2)(2x+5)}$$

4) Simplify: $\frac{2x^2-9x-5}{3x^2-13x-10}$ (5 pts)

$$\frac{(2x+1)\cancel{(x-5)}}{(3x+2)\cancel{(x-5)}} = \boxed{\frac{2x+1}{3x+2}}$$

5) Combine into a single fraction and simplify: $\frac{8}{x^2+2x-3} - \frac{6}{x^2+3x}$ (7 pts)

$$\frac{(x)}{(x)} \cdot \frac{8}{(x+3)(x-1)} - \frac{6}{x(x+3)} \cdot \frac{(x-1)}{(x-1)} = \frac{8x - 6(x-1)}{x(x+3)(x-1)}$$

$$= \frac{8x - 6x + 6}{x(x+3)(x-1)} = \frac{2x+6}{x(x+3)(x-1)}$$

$$= \frac{2\cancel{(x+3)}}{x\cancel{(x+3)}(x-1)} = \boxed{\frac{2}{x(x-1)}}$$

6) Find the quotient using polynomial long division and simplify: $(2x^3 - 9x^2 + 15) \div (2x - 5)$ (8 pts)

$$x^2 - 2x - 5 - \frac{10}{2x-5} \Leftarrow \text{Solution}$$

$$\begin{array}{r|l} 2x-5 & 2x^3 - 9x^2 + 0x + 15 \\ & -(2x^3 - 5x^2) \\ \hline & 0 - 4x^2 + 0x \\ & -(-4x^2 + 10x) \\ \hline & 0 - 10x + 15 \\ & -(-10x + 25) \\ \hline & 0 \quad -10 \end{array}$$

7) Simplify. Report your answers **WITHOUT** negative exponents.

a) $\frac{(x^{-3}y^{-4})^2}{(xy^3)^{-4}}$ (3 pts) $= \frac{x^{-6}y^{-8}}{x^{-4}y^{-12}} = x^{-6+4}y^{-8+12} = x^{-2}y^{4} = \boxed{\frac{y^4}{x^2}}$

b) $\left[5x^{\frac{2}{3}}y^{-\frac{1}{2}}\right]^{-3} \cdot \left[16xy^{-3}\right]^{\frac{1}{2}}$ (4 pts) $5^{-3}x^{-2}y^{\frac{3}{2}} \cdot 16^{\frac{1}{2}}x^{\frac{1}{2}}y^{-\frac{3}{2}} = \frac{4x^{-2+\frac{1}{2}}}{125}y^{\frac{3}{2}-\frac{3}{2}}$

$= \frac{4x^{-3/2}}{125} = \boxed{\frac{4}{125x^{3/2}} \text{ or } \frac{4}{125\sqrt{x^3}}}$

8) Simplify fully.

a) $3\sqrt{242x^5y^5} + 11xy\sqrt{2x^3y^3}$ (3 pts)

$= 3\sqrt{2\cdot 121\cdot x^4xy^4y} + 11xy\sqrt{2x^2xy^2y}$

$= 3\cdot 11\cdot x^2\cdot y^2\sqrt{2xy} + 11x^2y^2\sqrt{2xy}$

$= 33x^2y^2\sqrt{2xy} + 11x^2y^2\sqrt{2xy} = \boxed{44x^2y^2\sqrt{2xy}}$

b) $\sqrt[3]{54a^7b^4}$ (3 pts)

$= \sqrt[3]{27\cdot 2\cdot a^6\cdot a\cdot b^3\cdot b}$

$\boxed{= 3a^2b\sqrt[3]{2ab}}$

9) Let $X = 3 - \sqrt{45}$ and $Y = 2 + \sqrt{80}$. $3-\sqrt{45} = 3-\sqrt{5\cdot 9} = 3-3\sqrt{5}$ $\quad 2+\sqrt{80} = 2+\sqrt{16\cdot 5} = 2+4\sqrt{5}$

Perform the following operations and fully simplify your answers.

a) $X - \frac{Y}{2}$ (3 pts)

$$3-\sqrt{45} - \left[\frac{2+\sqrt{80}}{2}\right] = (3-3\sqrt{5}) - \left[\frac{2+4\sqrt{5}}{2}\right]$$
$$= 3-3\sqrt{5} - 1 - 2\sqrt{5}$$
$$= \boxed{2-5\sqrt{5}}$$

b) $\left(\frac{X}{2}\right)\cdot Y$ (3 pts)

$$\frac{X}{2}\cdot Y = X\cdot\frac{Y}{2} = (3-3\sqrt{5})(1+2\sqrt{5})$$
$$= 3+6\sqrt{5}-3\sqrt{5}-6(5)$$
$$= 3+3\sqrt{5}-30$$
$$\boxed{= -27+3\sqrt{5}}$$

c) $\frac{2X}{Y}$ (4 pts)

$$\frac{2(3-3\sqrt{5})}{(2+4\sqrt{5})} = \frac{6-6\sqrt{5}}{2+4\sqrt{5}}\cdot\frac{(2-4\sqrt{5})}{(2-4\sqrt{5})} = \frac{12-24\sqrt{5}-12\sqrt{5}+24(5)}{4-8\sqrt{5}+8\sqrt{5}-16(5)}$$
$$= \frac{12-36\sqrt{5}+120}{4-80} = \frac{\cancel{12}\,66-\cancel{36\sqrt{5}}\,18\sqrt{5}}{\cancel{-76}\,-38} = \boxed{\frac{-66}{38}+\frac{18\sqrt{5}}{38}} = \boxed{\frac{-33}{19}+\frac{9\sqrt{5}}{19}}$$

10) Simplify and report your answer in $a+bi$ form: $2i^{70} - 5i^{45} + 3i^{27} - 7i^{16}$ (2 pts)

$i^{70} = i^2$

$i^{45} = i$

$i^{27} = i^3$

$i^{16} = i^4$

$i = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$$= 2i^2 - 5i + 3i^3 - 7i^4$$
$$= 2(-1) - 5i + 3(-i) - 7(1)$$
$$= -2 - 5i - 3i - 7$$
$$\boxed{= -9-8i}$$

11) Evaluate:

a) $\log_3\left(\sqrt[5]{9}\right)$ (2 pts)

$$3^x = \sqrt[5]{9} = \sqrt[5]{3^2} = \left(3^2\right)^{1/5} = 3^{2/5}$$

$$\boxed{x = \frac{2}{5}}$$

b) $\log_4(8)$ (3 pts)

$$4^x = 8 = 2^3 = \left(\sqrt{4}\right)^3 = \left(4^{1/2}\right)^3 = 4^{3/2}$$

$$\boxed{x = \frac{3}{2}}$$

12) Between which two consecutive integers must $\log_4(7342)$ fall? (2 pts)

$4^1 = 4;\ 4^2 = 16;\ 4^3 = 64;\ 4^4 = 256;\ 4^5 = 1024;\ 4^6 = 4096;\ 4^7 = 16{,}384$

$$\underset{(4096)}{4^6} < 7342 < \underset{(16{,}384)}{4^7} \quad \text{so} \quad \boxed{6 < \log_4(7342) < 7}$$

13) Rewrite as a single logarithm: $3\log(\sqrt{x}) - \frac{1}{2}\log\left(y^4\right) - 5\log\left(\frac{1}{z}\right)$. (3 pts)

$$= 3\log\left(x^{1/2}\right) - \frac{1}{2}\log\left(y^4\right) - 5\log\left(z^{-1}\right)$$

$$= \log\left(x^{3/2}\right) - \log\left(y^2\right) + \log\left(z^5\right) = \boxed{\log\left(\frac{x^{3/2}\, z^5}{y^2}\right)}$$

14) If $a = \log_2(x)$, then rewrite $\log_2(8x^6)$ as a function of a only (no x). (4 pts)

$$\log_2\left(8x^6\right) = \log_2 8 + \log_2\left(x^6\right)$$

$$= \cancel{\log_2 2^3} + 6\underbrace{\log_2(x)}_{a}$$

$$\boxed{= 3 + 6a}$$

15) Expand the products and simplify:

a) $(x+2)(x-2)(x+4)(x-4)$ (3 pts)

$(x+2)(x-2)$

$(a+b)(a-b) = a^2 - b^2$

$a = x \quad b = 2$

$(x+2)(x-2) = x^2 - 4$

$(x+4)(x-4)$

$a = x \quad b = 4$

$(x+4)(x-4) = x^2 - 16$

$(x+2)(x-2)(x+4)(x-4) =$

$= (x^2-4)(x^2-16) =$

$= x^4 - 16x^2 - 4x^2 + 64$

$= x^4 - 20x^2 + 64$

b) $(2x-1)^3$ (3 pts)

$= (2x-1)(2x-1)(2x-1)$

$= (2x-1)^2(2x-1)$

$(a-b)^2 = a^2 - 2ab + b^2$

$a = 2x \quad b = 1$

$(2x-1)^2 = 4x^2 - 4x + 1$

$(2x-1)^2(2x-1) = (4x^2-4x+1)(2x-1)$

	$4x^2$	$-4x$	1
$2x$	$8x^3$	$-8x^2$	$2x$
-1	$-4x^2$	$4x$	-1

$\boxed{= 8x^3 - 12x^2 + 6x - 1}$

16) Express as a single, simplified fraction: $\frac{\frac{1}{8}-\frac{1}{2x}}{\frac{1}{x}-\frac{x}{16}}$. (5 pts)

$\frac{(x)}{(x)} \cdot \frac{1}{8} - \frac{1(4)}{2x(4)} = \frac{x}{8x} - \frac{4}{8x} = \frac{x-4}{8x}$ (numerator)

$\frac{(16)}{(16)} \cdot \frac{1}{x} - \frac{x \cdot (x)}{16 \cdot (x)} = \frac{16}{16x} - \frac{x^2}{16x} = \frac{16-x^2}{16x}$ (denominator)

$$= \frac{\frac{x-4}{8x}}{\frac{16-x^2}{16x}} = \frac{x-4}{\cancel{8x}} \cdot \frac{\overset{2}{\cancel{16}}\,\cancel{x}}{16-x^2} = \frac{2(x-4)}{(4+x)(4-x)} = \frac{2\cancel{(x-4)}}{-(4+x)\cancel{(x-4)}}$$

$(a^2-b^2) = (a+b)(a-b)$

$a^2 = 16 \quad b^2 = x^2$

$a = 4 \quad b = x$

$$= \boxed{\frac{-2}{4+x}}$$